A Distributed Primal-Dual Interior-Point Method for Loosely Coupled Problems Using ADMM

Mariette Annergren (KTH), Sina Khoshfetrat Pakazad (LiU), Anders Hansson (LiU), Bo Wahlberg (KTH).

Objective

Solve optimization problem and take advantage of the loose coupling.

- Primal-dual interior point method on main problem.
- ADMM on subproblems.

Related methods

First order methods:

- (Sub-) gradient.
- Proximal point.

Related methods

Second order methods:

- Interior point methods:
  - Primal-dual.

Problem formulation

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_1(x) + \cdots + f_N(x), \\
\text{subject to} \quad & G^i(x) \preccurlyeq 0, \qquad i = 1, \dots, N, \\
& A^i x = b^i, \qquad i = 1, \dots, N.
\end{aligned}
$$

number of subsystems

Assume loosely coupled

$$\begin{aligned}
\underset{S,\,x}{\text{minimize}} \quad & \bar{f}_1(s^1) + \cdots + \bar{f}_N(s^N), \\
\text{subject to} \quad & \bar{G}^i(s^i) \preccurlyeq 0, && i = 1, \dots, N, \\
& \bar{A}^i s^i = b^i, && i = 1, \dots, N, \\
& \bar{E}x = S.
\end{aligned}$$

links subsystems

$S = (s^1 \dots s^N)$.

$\bar{E}$ is the consistency matrix.

Primal-dual interior point method

Solving optimization problem.

$$\Leftrightarrow$$

Solving the KKT conditions.

Primal-dual interior point method

- KKT: $A_{KKT}(x) = 0$
- Perturbed KKT: $A_{KKT}(x) = \delta$
- Linearized perturbed KKT:

$$\nabla A_{KKT}(x^*)\Delta \mathrm{x} = -A_{KKT}(x^*) + \delta$$

"KKT conditions"

Primal-dual interior point method

Repeat until convergence:

1. Solve ”KKT conditions” → step direction.
2. Perform line search → step length.
3. Take step.

”KKT conditions”

step direction

$$\begin{pmatrix} H^{(l)} & 0 & \bar{A}^T & I \\ 0 & 0 & 0 & -\bar{E}^T \\ \bar{A} & 0 & 0 & 0 \\ I & -\bar{E} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta S \\ \Delta x \\ \Delta v \\ \Delta v_c \end{pmatrix} = \begin{pmatrix} r \\ -\bar{E}^T v_c{}^{(l)} \\ r_{primal} \\ r_c \end{pmatrix}$$

$l =$ iteration number of primal-dual method

Corresponding optimization problem

$$\begin{aligned} \underset{\Delta S,\, \Delta x}{\text{minimize}} \quad & F_1{}^{(l)}(\Delta S) + F_2{}^{(l)}(\Delta x), \\ \text{subject to} \quad & A\Delta S + B\Delta x = c^{(l)}, \end{aligned}$$

with dual variables Δv and Δv_c.

$$\left.\begin{array}{l} \textbf{separable} \\ \textbf{structured} \end{array}\right\} \textbf{use ADMM}$$

ADMM

Augmented Lagrangian

$$L_\rho(\Delta S, \Delta x, \Delta w) = F_1{}^{(l)}(\Delta S) + F_2{}^{(l)}(\Delta x) +$$

$$\frac{\rho}{2}\left\|A\Delta S + B\Delta x - c^{(l)} + \Delta w\right\|_2^2$$

scaled dual variable

ADMM

Repeat until convergence:

1. $\Delta S^{k+1} = \underset{\Delta S}{\operatorname{argmin}}\, L_\rho{}^{(l)}\left(\Delta S, \Delta x^k, \Delta w^k\right).$

2. $\Delta x^{k+1} = \underset{\Delta x}{\operatorname{argmin}}\, L_\rho{}^{(l)}\left(\Delta S^{k+1}, \Delta x, \Delta w^k\right).$

3. $\Delta \mathrm{w}^{k+1} = \Delta w^k + A\Delta S^{k+1} + B\Delta x^{k+1} - c^{(l)}.$

ADMM

Highly parallellizable:

1. $\Delta S^{k+1} \leftarrow \Delta S^{(i),k+1}$ separately.
2. $\Delta x^{k+1} \leftarrow \Delta x_{N(i)}{}^{k+1}$ separately.
3. $\Delta w^{k+1} \leftarrow \Delta w^{(i),k+1}$ separately.

Tuning for better convergence

”KKT conditions”:

- Preconditioners

$$\boldsymbol{M}^{-1}\nabla A_{KKT}(x^*)\Delta x = \boldsymbol{M}^{-1}(-A_{KKT}(x^*) + \delta).$$

Tuning for better convergence

Interior point method:

- Perturbation of KKT conditions.
- Tuning of search for step length.

Tuning for better convergence

ADMM:

- Preconditioners.
- Choice of penalty parameter ρ.
- Stopping condition.

Numerical example

Number of:

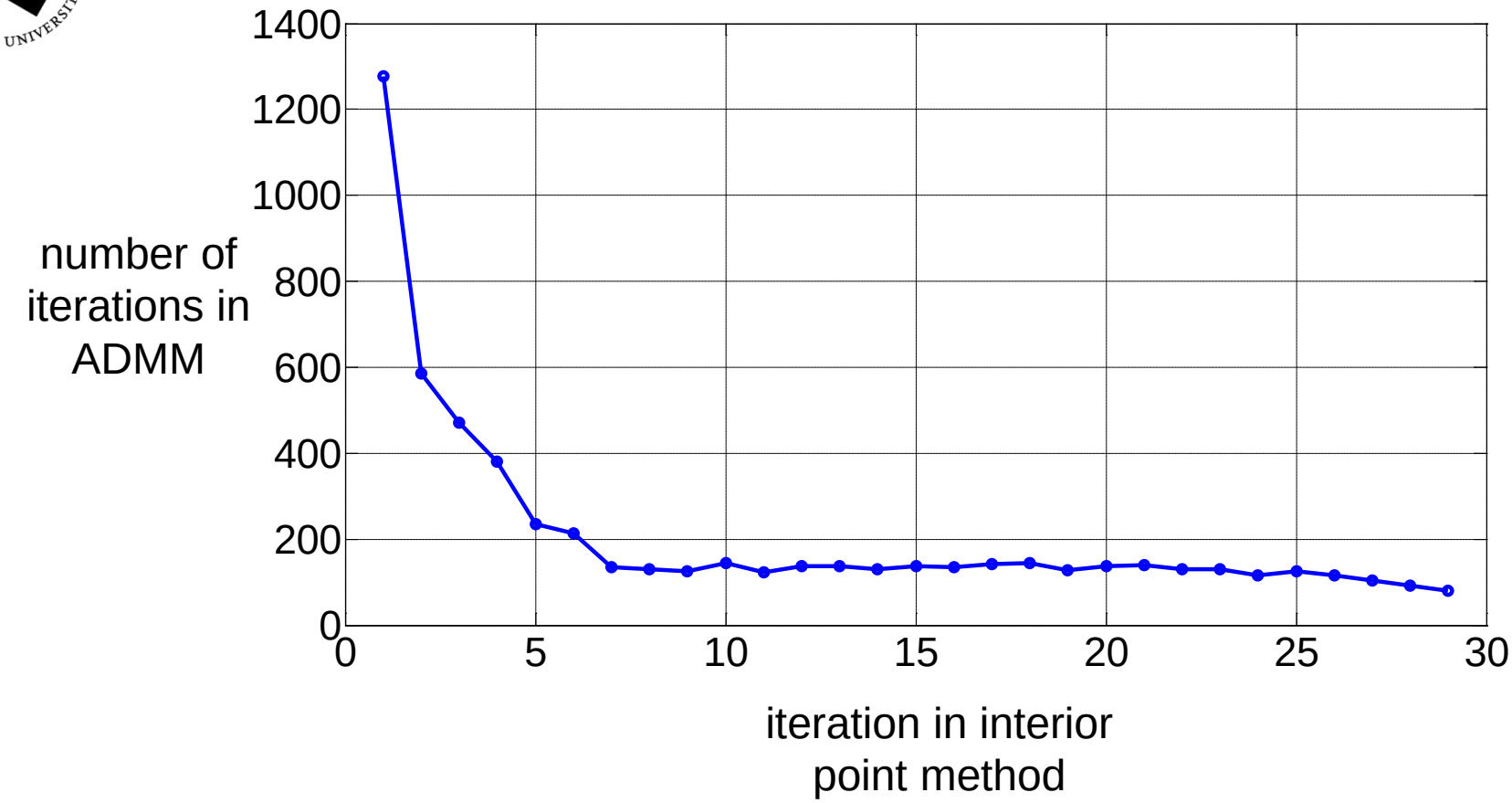
subsystems = 5,

local variables ~ $U(55\ 65)$,

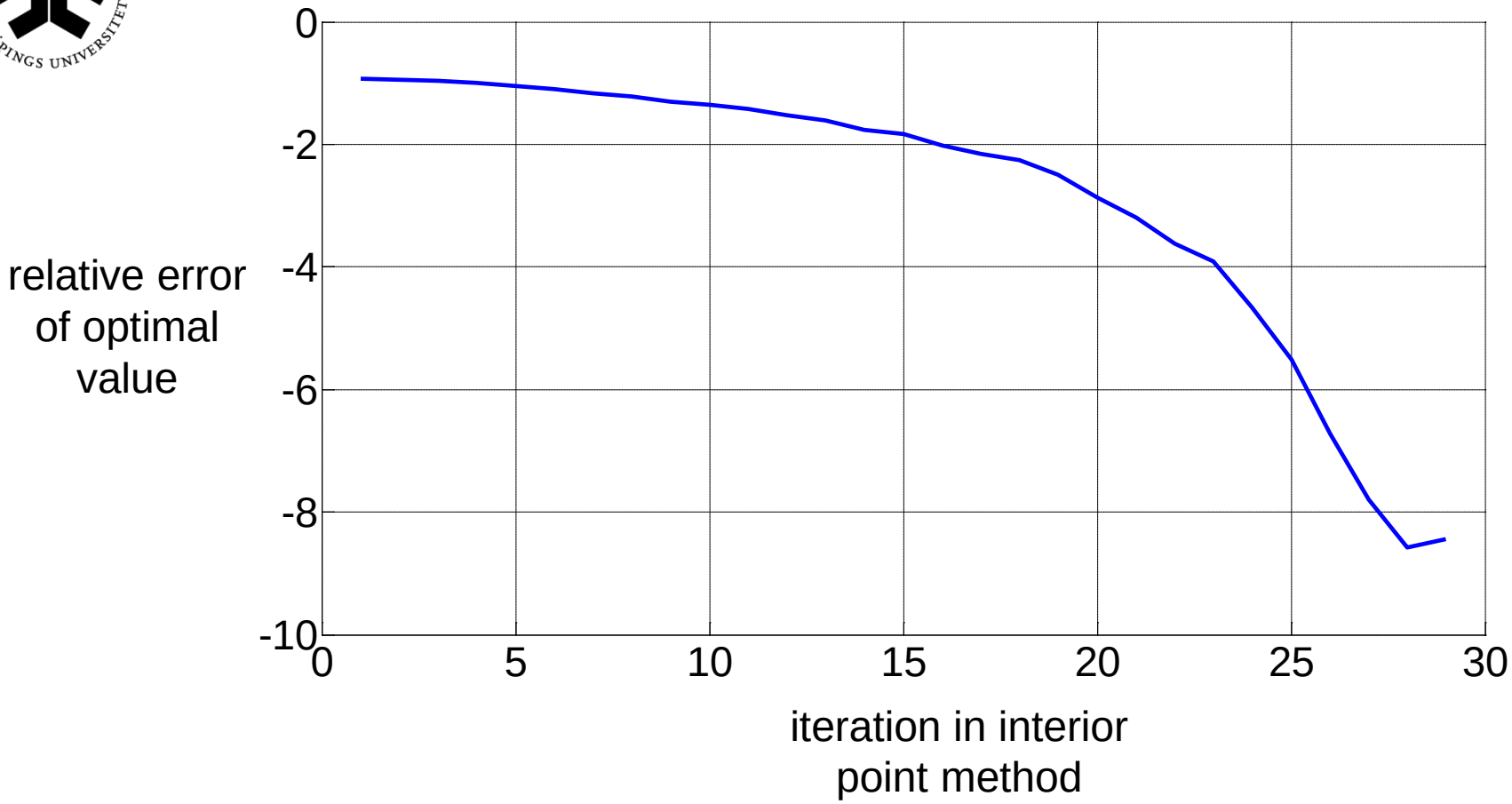
local equalities ~ $U(27\ 33)$,

local inequalities ~ $U(27\ 33)$.

Numerical example



Numerical example



Conclusions

- Primal-dual interior point method with ADMM.
- Highly distributed.
- Simple to implement.

Future work

- Convergence properties.
- Tuning.
- Other proximal point methods.
- Implementation of MPC setting.
- Context.