A Distributed Primal-Dual Interior-Point Method for Loosely Coupled Problems Using ADMM

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Objective

Solve optimization problem and take advantage of the loose coupling.

• Primal-dual interior point method on main problem.

• ADMM on subproblems.
Related methods

First order methods:

• (Sub-) gradient.
• Proximal point.
Related methods

Second order methods:

• Interior point methods:
  – Primal-dual.
Problem formulation

\[
\begin{align*}
\text{minimize} \quad & f_1(x) + \cdots + f_N(x), \\
\text{subject to} \quad & G^i(x) \leq 0, \quad i = 1, \ldots, N, \\
& A^i x = b^i, \quad i = 1, \ldots, N.
\end{align*}
\]
Assume loosely coupled

minimize \[ \bar{f}_1(s^1) + \cdots + \bar{f}_N(s^N), \]

subject to \[ \bar{G}^i(s^i) \leq 0, \quad i = 1, \ldots, N, \]
[ \bar{A}^i s^i = b^i, \quad i = 1, \ldots, N, \]
[ \bar{E} x = S. \]

\[ S = (s^1 \ldots s^N). \]

\[ \bar{E} \] is the consistency matrix.

links subsystems
Primal-dual interior point method

Solving optimization problem.

⇔

Solving the KKT conditions.
Primal-dual interior point method

- KKT: $A_{KKT}(x) = 0$
- Perturbed KKT: $A_{KKT}(x) = \delta$
- Linearized perturbed KKT:
  \[ \nabla A_{KKT}(x^*) \Delta x = -A_{KKT}(x^*) + \delta \]

"KKT conditions"
Primal-dual interior point method

Repeat until convergence:

1. Solve "KKT conditions" → step direction.
2. Perform line search → step length.
3. Take step.
"KKT conditions"

\[
\begin{pmatrix}
H^{(l)} & 0 & \bar{A}^{T} & I \\
0 & 0 & 0 & -\bar{E}^{T} \\
\bar{A} & 0 & 0 & 0 \\
I & -\bar{E} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\Delta S \\
\Delta x \\
\Delta \nu \\
\Delta \nu_c
\end{pmatrix}
= 
\begin{pmatrix}
r \\
-\bar{E}^{T} \nu_c^{(l)} \\
r_{\text{primal}} \\
r_c
\end{pmatrix}
\]

\( l = \text{iteration number of primal-dual method} \)
Corresponding optimization problem

\[
\begin{align*}
\text{minimize} & \quad F_1^{(l)}(\Delta S) + F_2^{(l)}(\Delta x), \\
\text{subject to} & \quad A\Delta S + B\Delta x = c^{(l)},
\end{align*}
\]

with dual variables $\Delta \nu$ and $\Delta \nu_c$.

separable structured \[\rightarrow\] use ADMM
ADMM

Augmented Lagrangian

\[ L_\rho (\Delta S, \Delta x, \Delta w) = F_1^{(l)} (\Delta S) + F_2^{(l)} (\Delta x) + \frac{\rho}{2} \| A\Delta S + B\Delta x - c^{(l)} + \Delta w \|_2^2 \]
ADMM

Repeat until convergence:

1. \( \Delta S^{k+1} = \arg\min_{\Delta S} L^{(l)}_\rho (\Delta S, \Delta x^k, \Delta w^k) \).

2. \( \Delta x^{k+1} = \arg\min_{\Delta x} L^{(l)}_\rho (\Delta S^{k+1}, \Delta x, \Delta w^k) \).

3. \( \Delta w^{k+1} = \Delta w^k + A \Delta S^{k+1} + B \Delta x^{k+1} - c^{(l)} \).
ADMM

Highly parallellizable:

1. $\Delta S^{k+1} \leftarrow \Delta S^{(i),k+1}$ separately.

2. $\Delta x^{k+1} \leftarrow \Delta x_{N(i)}^{k+1}$ separately.

3. $\Delta w^{k+1} \leftarrow \Delta w^{(i),k+1}$ separately.
Tuning for better convergence

"KKT conditions":

• Preconditioners

\[ M^{-1} \nabla A_{KKT}(x^*) \Delta x = M^{-1}(-A_{KKT}(x^*) + \delta). \]
Tuning for better convergence

Interior point method:

• Perturbation of KKT conditions.
• Tuning of search for step length.
Tuning for better convergence

ADMM:

- Preconditioners.
- Choice of penalty parameter $\rho$.
- Stopping condition.
Numerical example

Number of:

subsystems = 5,

local variables ~ \( U(55 \ 65) \),

local equalities ~ \( U(27 \ 33) \),

local inequalities ~ \( U(27 \ 33) \).
Numerical example

The graph shows the number of iterations in ADMM (Alternating Direction Method of Multipliers) as a function of the iteration in the interior point method. The number of iterations decreases rapidly with increasing iteration number, indicating effective convergence of the algorithm.
Numerical example

Relative error of optimal value

Iteration in interior point method
Conclusions

• Primal-dual interior point method with ADMM.
• Highly distributed.
• Simple to implement.
Future work

• Convergence properties.
• Tuning.
• Other proximal point methods.
• Implementation of MPC setting.
• Context.