An ADMM Algorithm for Solving l_1 Regularized MPC

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Contribution

An ADMM algorithm for efficiently solving optimal control problems for dynamical systems with an l_1 regularized least-squares cost function

Specifically, l_1 regularized MPC

Outline

- l_1 regularized MPC
- ADMM
- Simulation examples
- Future research topics

MPC optimization problem

$$\underset{u(t+i),i=0,\ldots,N}{\text{minimize}} \quad V(t)$$

subject to dynamics (model)

input constraints

output constraints

l_1 regularized MPC optimization problem

$$\underset{u(t+i),i=0,\ldots,N}{\text{minimize}} \quad V(t) + \lambda \|u(t+i) - u(t+i-1)\|_1$$

$$\text{subject to} \quad \text{dynamics (model)}$$

l_1 regularization promotes few steps in u

$u(t+i) = u(t+i-1)$ for most values of i

(M. Gallieri and J. M. Maciejowski, 2012)

Why use Alternating Direction Method of Multipliers (ADMM)?

- Easy to implement
- Converges fast to moderate accuracy
- Enables separation of optimization problem into sub-problems that can be solved in parallel

(useful for distributed control design)

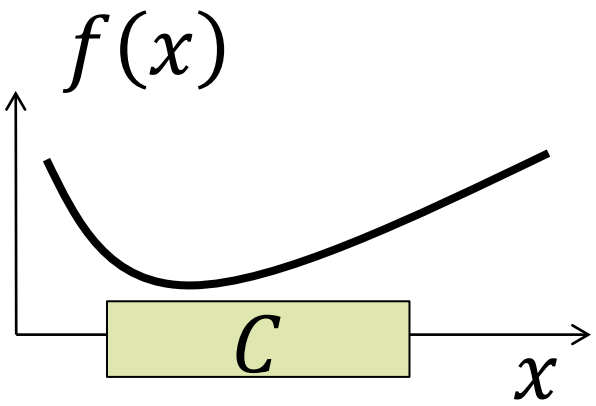
ADMM: Key ideas

- Introduce additional variables
- Solve original problem iteratively via sub-problems

ADMM: Considered optimization problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in C \end{aligned}$$

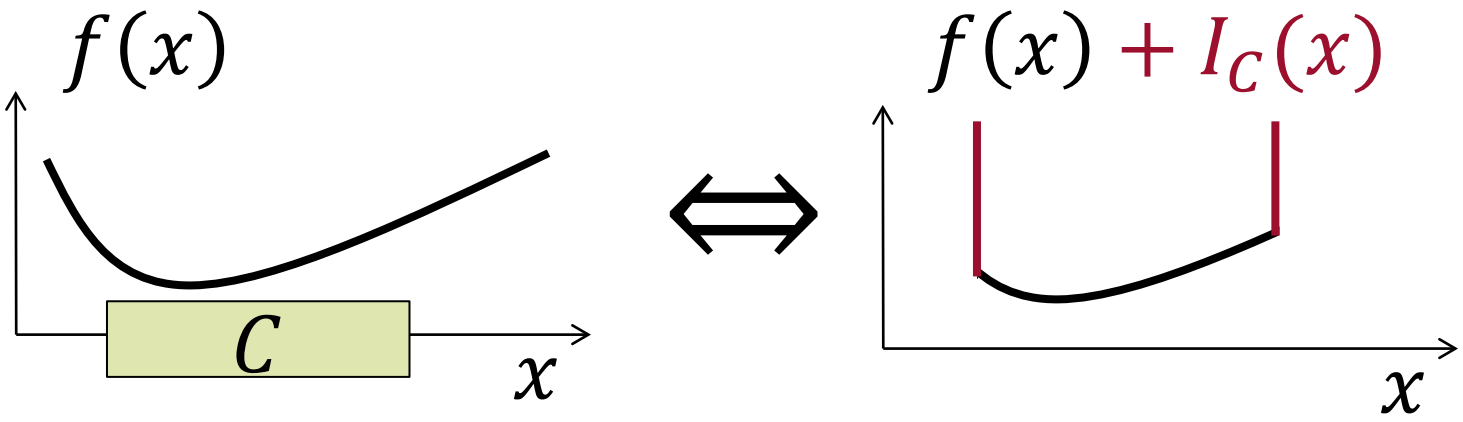
$f(x)$: convex function
C: convex set



ADMM: Considered optimization problem

$$\underset{x}{\text{minimize}} \quad f(x) + I_C(x)$$

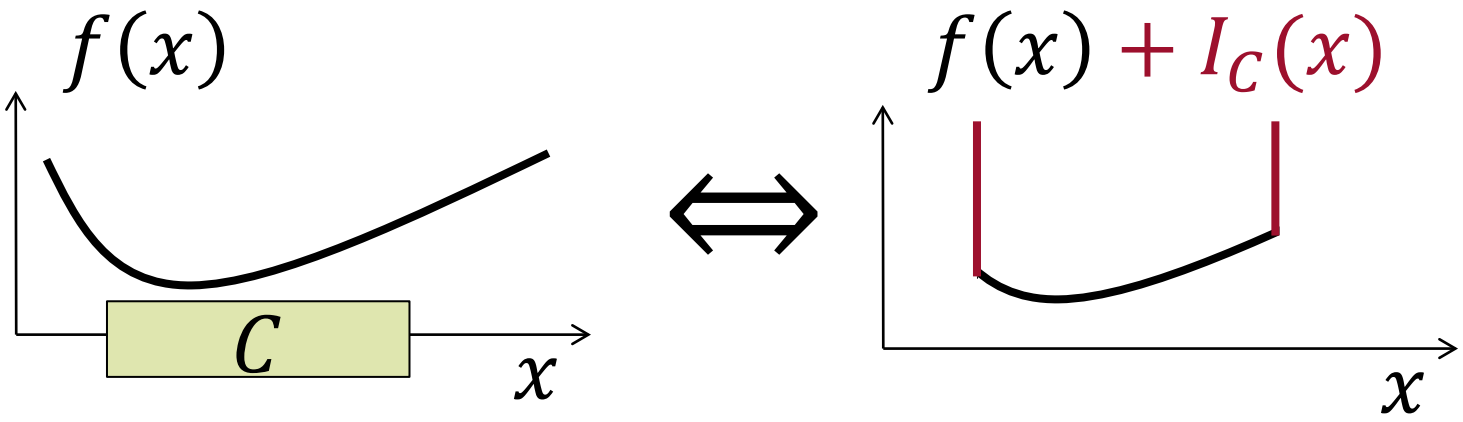
where $I_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C \end{cases}$



ADMM: Considered optimization problem

$$\underset{x, x_c}{\text{minimize}} \quad f(x) + I_C(x_c)$$
$$\text{subject to} \quad x = x_c$$

where $I_C(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C \end{cases}$



ADMM: Sub-problems

- Lagrangian

$$L(x, x_c, x_d) = \underbrace{f(x) + I_C(x_c)}_{\text{objective}} + \underbrace{x_d^T(x - x_c)}_{\text{constraint}}$$

- Augmented Lagrangian

$$L_\rho(x, x_c, x_d) =$$
$$f(x) + I_C(x_c) + x_d^T(x - x_c) + 1/\rho\|x - x_c\|_2^2$$

ADMM: Sub-problems

Scaled augmented Lagrangian

$$L_\rho(x, x_c, x_s) =$$
$$f(x) + I_C(x_c) + \frac{\rho}{2}\|x - x_c + x_s\|_2^2 + \text{constant}$$

where $x_s = x_d/\rho$

ρ is the only tuning parameter

ADMM: Algorithm

The three steps at iteration k

Step 1 $\rightarrow$ $$\boldsymbol{x}^{k+1} = \underset{\boldsymbol{x}}{\operatorname{argmin}}\{L_\rho\left(\boldsymbol{x}, x_c^k, x_s^k\right)\}$$

Step 2 $\rightarrow$ $$\boldsymbol{x}_c^{k+1} = \underset{\boldsymbol{x}_c}{\operatorname{argmin}}\{L_\rho\left(x^{k+1}, \boldsymbol{x}_c, x_s^k\right)\}$$

Step 3 $\rightarrow$ $$\boldsymbol{x}_s^{k+1} = \boldsymbol{x}_s^k + (x^{k+1} - x_c^{k+1})$$

ADMM: Algorithm

The three steps at iteration k

Step 1 $\longrightarrow$ $$\boldsymbol{x}^{k+1} = \underset{\boldsymbol{x}}{\operatorname{argmin}}\{L_\rho\left(\boldsymbol{x}, x_c^k, x_s^k\right)\}$$

$$\boldsymbol{x}^{k+1} = \underset{\boldsymbol{x}}{\operatorname{argmin}}\{\boldsymbol{f}(\boldsymbol{x}) + \frac{\boldsymbol{\rho}}{\mathbf{2}}\left\|\boldsymbol{x} - \boldsymbol{x}_c^k + \boldsymbol{x}_s^k\right\|_2^2\}$$

ADMM: Algorithm

The three steps at iteration k

Step 1 $\longrightarrow$ $$\boldsymbol{x}^{k+1} = \underset{\boldsymbol{x}}{\operatorname{argmin}}\{L_\rho(\boldsymbol{x}, x_c^k, x_s^k)\}$$

If $f(x) = f_1(x_1) + \cdots + f_n(x_n)$ then

$$x_1^{k+1} = \underset{x_1}{\operatorname{argmin}}\{f_1(x_1) + \rho/2\left\|x_1 - x_{c,1}^k + x_{s,1}^k\right\|_2^2\}$$

$$\vdots$$

$$x_n^{k+1} = \underset{x_n}{\operatorname{argmin}}\{f_n(x_n) + \rho/2\left\|x_n - x_{c,n}^k + x_{s,n}^k\right\|_2^2\}$$

ADMM: Algorithm

The three steps at iteration k

Step 2 $\longrightarrow$ $$\boldsymbol{x_c^{k+1}} = \underset{\boldsymbol{x_c}}{\operatorname{argmin}}\{L_\rho\left(x^{k+1}, \boldsymbol{x_c}, x_s^k\right)\}$$

$$x_c^{k+1} = \underset{x_c}{\operatorname{argmin}}\{I_C(x_c) + \frac{\rho}{2}\left\|x^{k+1} - x_c + x_s^k\right\|_2^2\}$$

$\Leftrightarrow$ **Projection onto C:** $\boldsymbol{x_c^{k+1} = \Pi_C(x^{k+1} + x_s^k)}$

Possible to perform projections in a distributed way if dynamics weakly coupled

ADMM: Algorithm

The three steps at iteration k

Step 3 $\longrightarrow$ $\boldsymbol{x_s}^{k+1} = \boldsymbol{x_s^k} + (x^{k+1} - x_c^{k+1})$

Simple update

ADMM: Convergence

- Primal residual

$$e_{\text{primal}}^{k} = x^{k} - x_{c}^{k}$$

- Dual residual

$$e_{\text{dual}}^{k} = -\rho(x_{c}^{k} - x_{c}^{k-1})$$

ADMM: Convergence

As $k \to \infty$

- $e_{\text{primal}}^{k} \to 0$
- $f\left(x^{k}\right) + I_{C}\left(x_{c}^{k}\right) \to$ optimal objective value
- $x_{s}^{k} \to$ optimal scaled dual variable

ADMM: Algorithm for l_1 regularized MPC

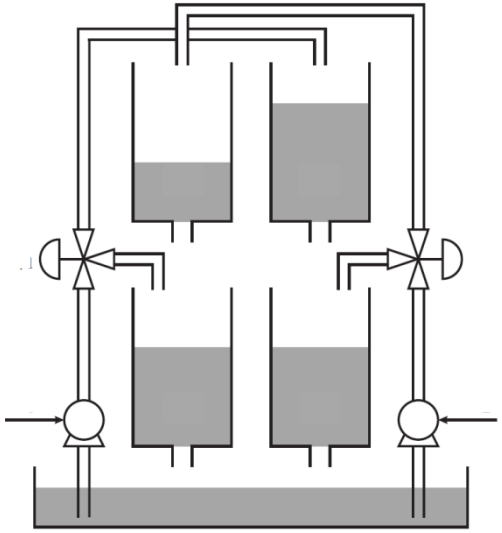
Step 1 → **Analytical solutions**

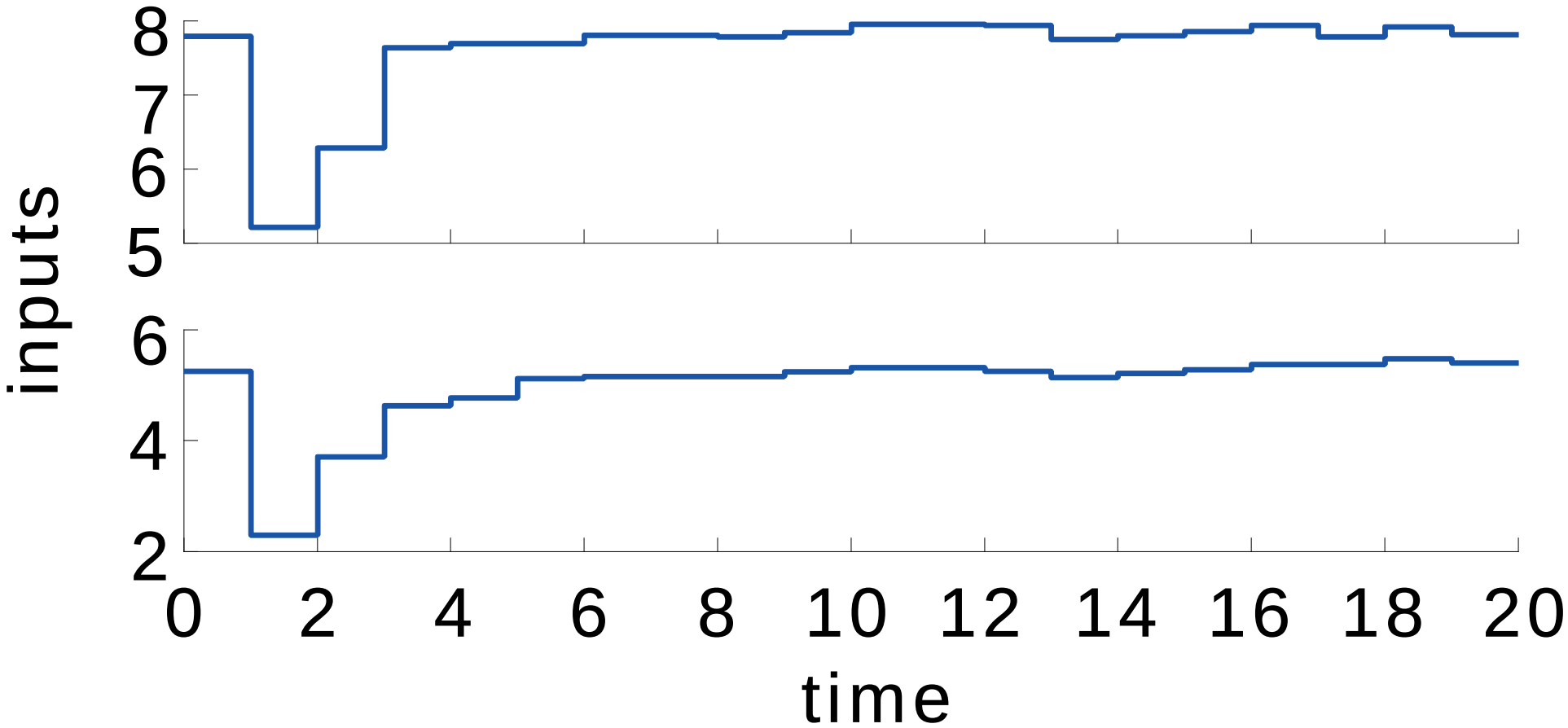
Step 2 → **Projection using Riccati recursion**

Step 3 → **Simple update**

Simulation

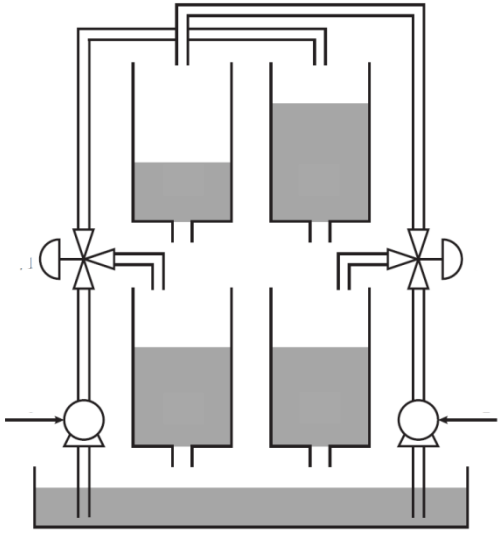
$$\lambda = 0.05$$

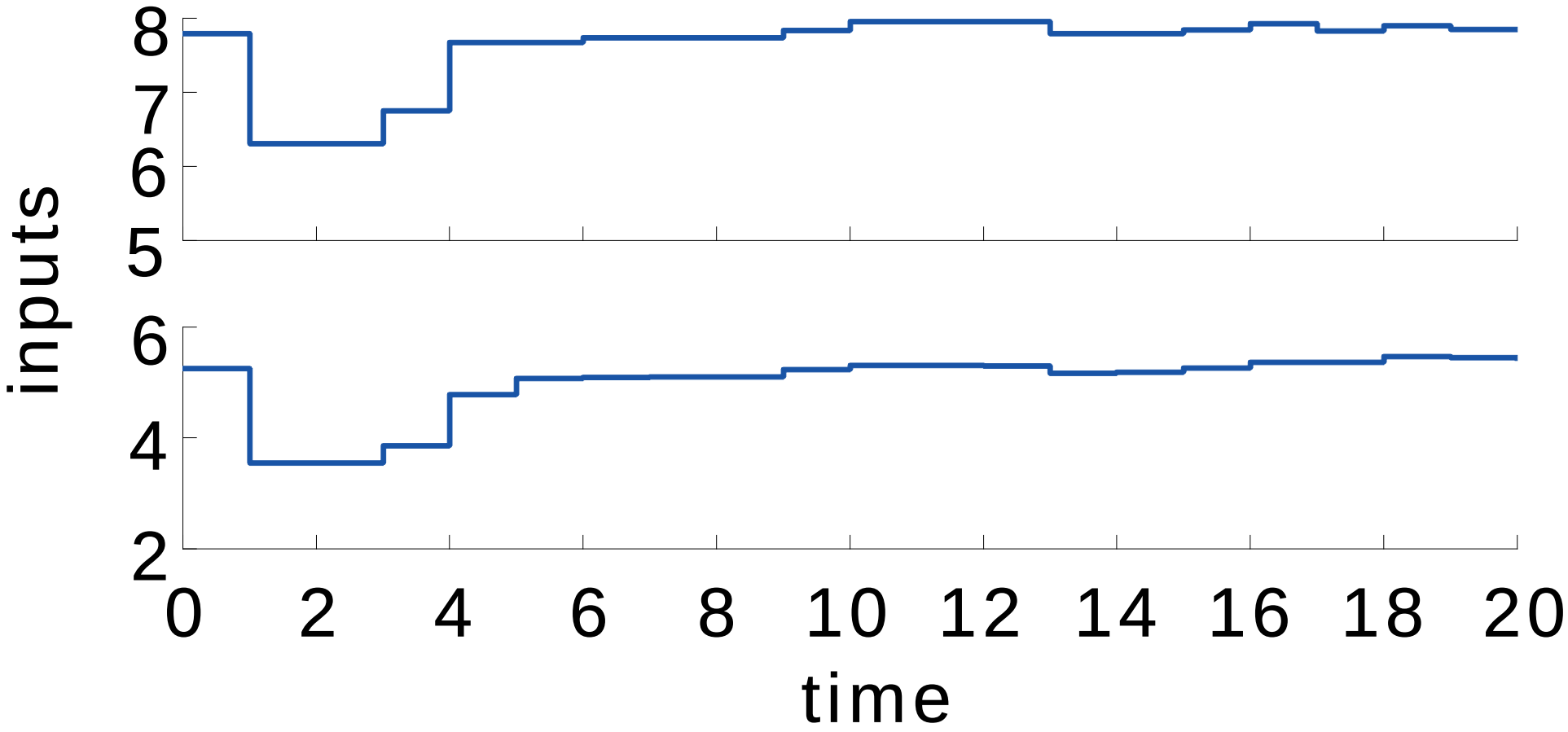




Simulation

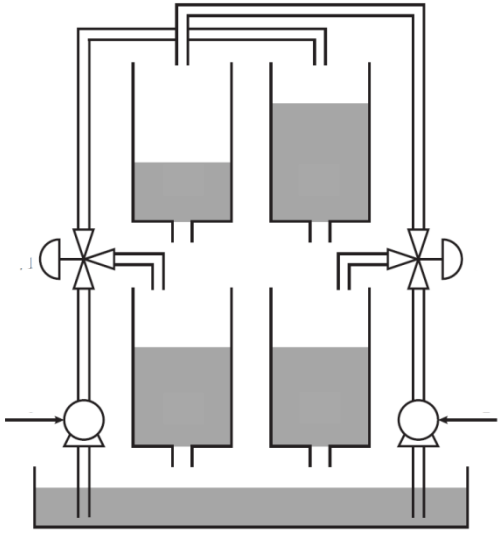
$$\lambda = 0.1$$

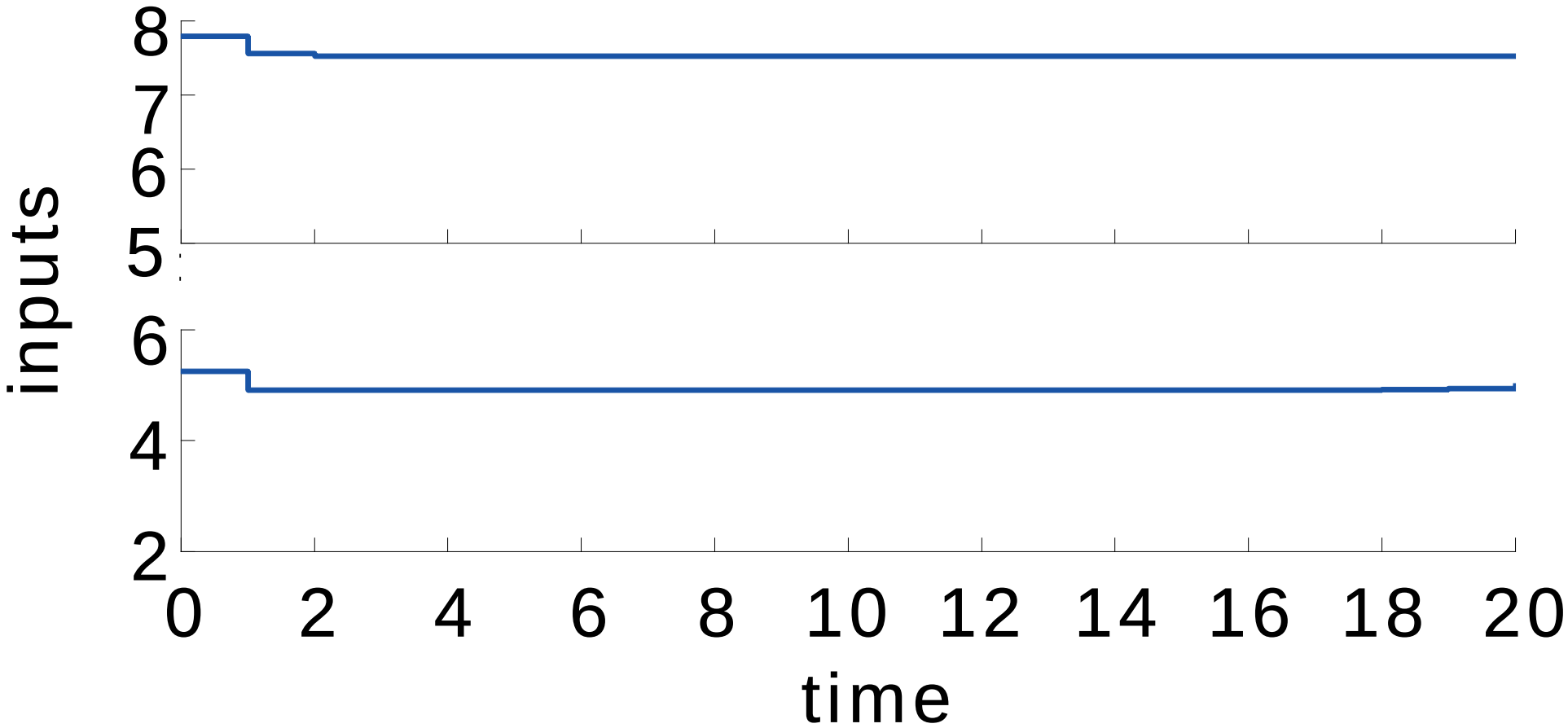




Simulation

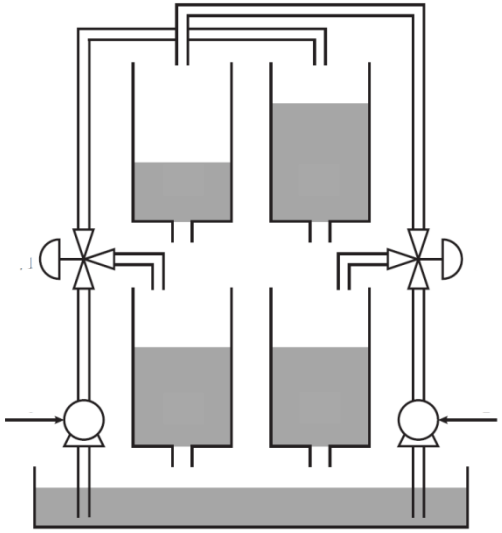
$$\lambda = 2$$

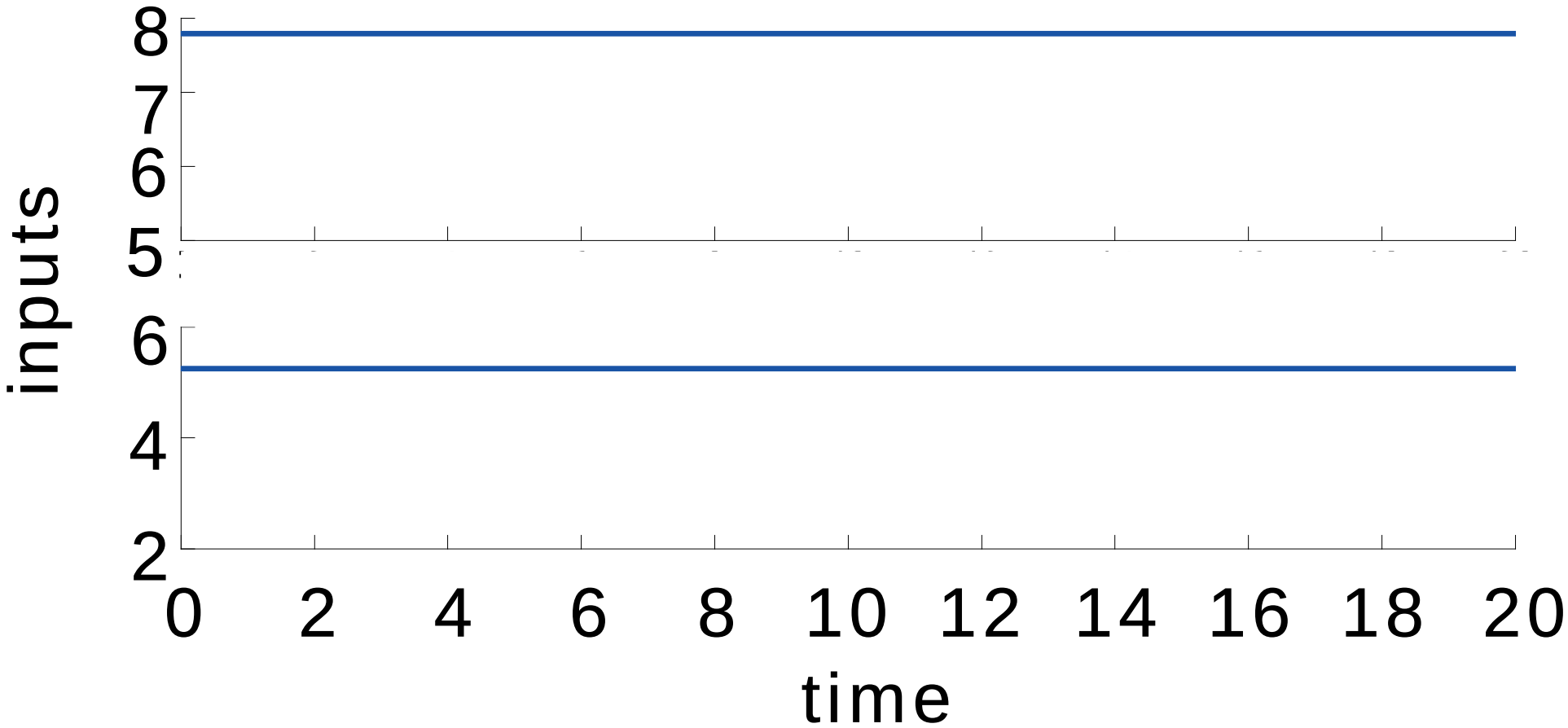




Simulation

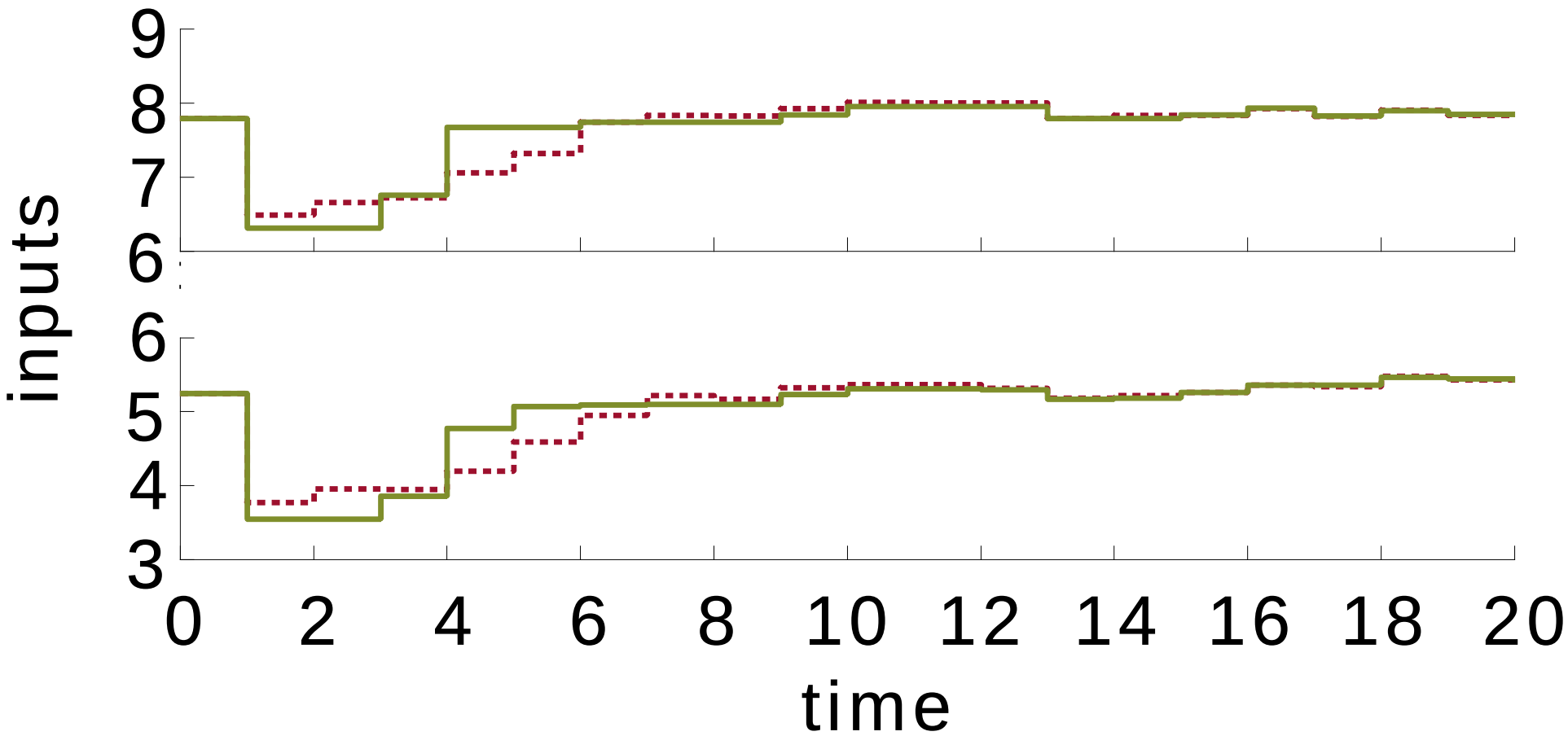
$$\lambda = 5$$





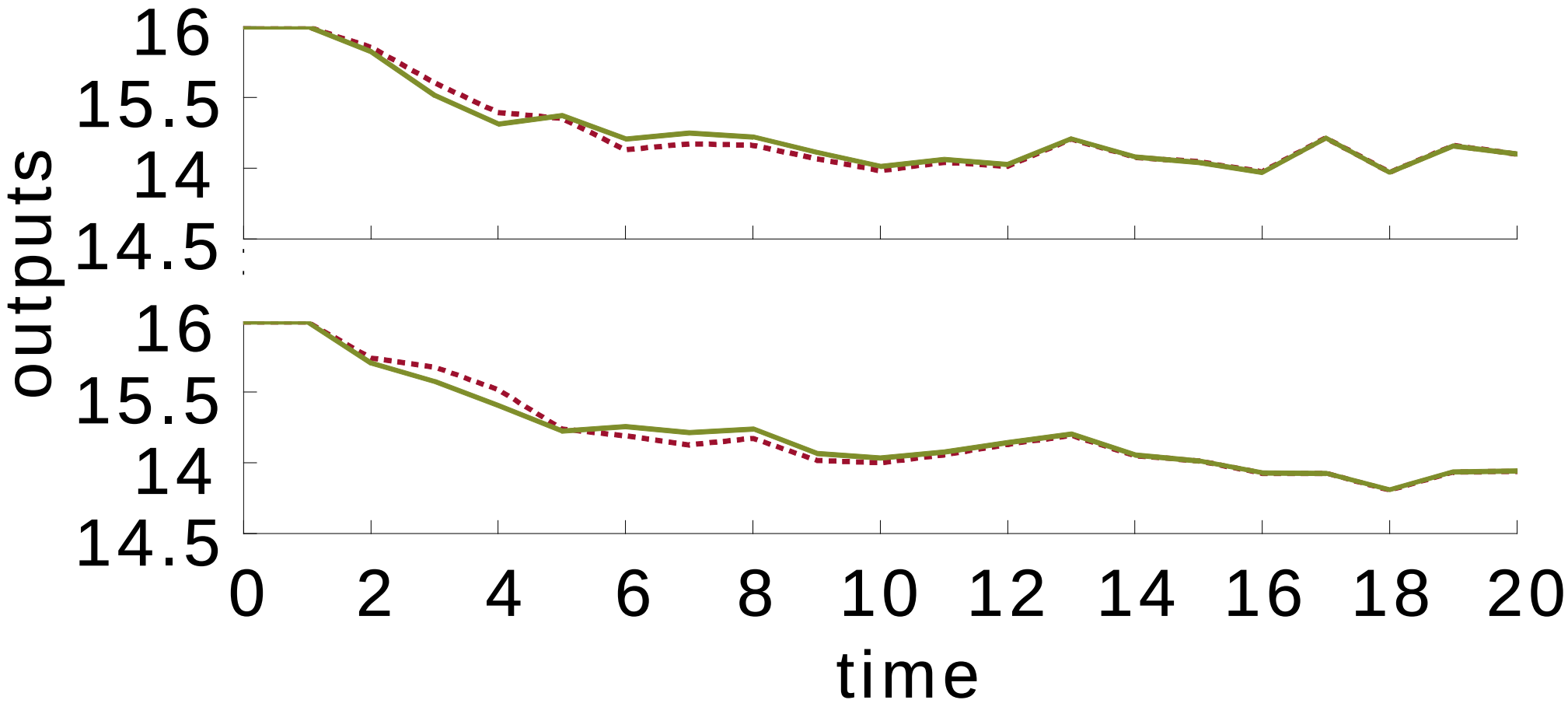
Simulation

$\lambda = 0.1$

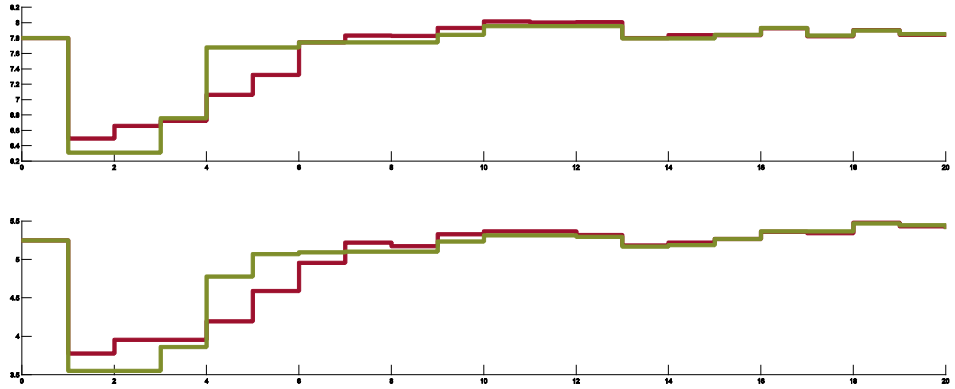


Simulation

$\lambda = 0.1$



Simulation



- Number of states of the system is 4
- Time horizon in MPC is 5
- 10 ADMM iterations allow **MPC updates at 100 Hz**

Future research topics

- Tuning parameter ρ
- Initialization of ADMM algorithm
- Inequality constraints in MPC
- State estimation
  (Ohlsson et al., 2012)

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Thank you!