Elimination of First Order Errors in Time Dependent Shock Calculations

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Abstract. First order errors downstream of shocks have been detected in computations with higher order shock capturing schemes in one and two dimensions. We use matched asymptotic expansions to analyze the phenomenon for one dimensional time dependent hyperbolic systems and show how to design the artificial viscosity term in order to avoid the first order error. Numerical computations verify that second order accurate solutions are obtained.

1 Introduction

In many cases, solutions of hyperbolic conservation laws obtained by formally higher order methods are only first order accurate downstream of shocks, see e.g. [1], [4] and [3]. Basically, errors from the shock region follow outgoing characteristics and pollute the solution downstream. Examples in one space dimension where this effect can be seen are steady state calculations for systems with a source term and time dependent calculations for systems with non-constant solution. The effect can not be seen in one dimensional Riemann problems, because the exact global conservation determines the post shock states.

The degeneration in accuracy is troublesome, even though the first order term for reasonable mesh-sizes seems to be small in many cases. In some applications, as e.g. aeroacoustics where small amplitude waves need to be computed accurately, it is however important to achieve very high accuracy. It is also important to understand the phenomenon more deeply in order to be able to design new methods which do not suffer from this deficiency.

The aim of this paper is to show that the first order error can be understood by matched asymptotic analysis of the modified equation and that the analysis can be used to construct methods that yield second order accurate solutions.

We consider the case of a system with time dependent solution. We assume that the numerical solution can be modeled by a slightly viscous equation, a so called modified equation. In the shock layer, the coefficient of the viscous term is $\mathcal{O}(h)$, where h is the grid size. We analyze the solution of the modified equation using matched asymptotic expansions. It is assumed that an inner solution is valid in the shock region, and an outer solution is valid elsewhere. The two solutions are matched in a so called matching zone. From the analysis, we see that generally, the outer solution contains a term of $\mathcal{O}(h)$ downstream of the shock. We also see that if the inner solution satisfies a certain condition, the $\mathcal{O}(h)$ term would be eliminated. Based on this observation, we design a matrix

valued viscosity coefficient which gives the inner solution the right shape to eliminate the $\mathcal{O}(h)$ downstream term. We construct a numerical scheme, using this matrix viscosity coefficient, and show in numerical experiments that the first order downstream error really is eliminated. However, we do not claim to have constructed an efficient and robust numerical method which can be used in realistic computations.

Similar analysis and construction of a matrix viscosity coefficient is done for the case of a steady state solution of a system with a source term in [6]. In [2], matched asymptotic expansions for a problem which is very similar to the problem we study in this paper is analyzed for other purposes. The phenomenon has also been studied by other methods in [4] and [1].

2 Model of the First Order Down-Stream Error

We will now introduce a model that explains how the first order down-stream error arises. Exactly how this error behaves depends both on which numerical method that is used, and on what mathematical problem that is solved.

2.1 The Inviscid Problem

The mathematical problem under consideration is

$$\begin{aligned} &\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \quad 0 \le x \le x_{\text{end}}, \\ &\mathbf{u}(x,0) = \mathbf{g}(x), \end{aligned} \tag{1}$$

where $\mathbf{u}(x,t) \in \mathbf{R}^n$. We assume that the eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{u})$, denoted $\lambda_i(\mathbf{u}), i = 1,2,\ldots,n$, are real and ordered in increasing order and that the eigenvectors span $\mathbf{R}^n$. The initial and boundary conditions are chosen such that a shock forms at some inner point $s(t)$. At the shock the solution satisfies the Rankine-Hugoniot condition. We assume that the shock is a classical Lax 1-shock i.e. $\dot{s} < \lambda_1^-$, $\lambda_1^+ < \dot{s} < \lambda_2^+$, where $\lambda_1^\pm = \lim_{\delta \to 0^+} \lambda(\mathbf{u}(s(t) \pm \delta, t))$. Corresponding notation for other quantities will also be used. The problem is closed by suitable boundary conditions at $x = 0$ and $x = x_{\text{end}}$, e.g. prescribing ingoing characteristic variables.

2.2 The Slightly Viscous Model

We intend to study the behavior of numerical solutions of (1), i.e. we want to study the behavior of discrete functions which are the solutions of difference equations. A useful technique for studying the behavior of solutions to difference equations is to model the difference equation by a differential equation. Such a differential equation is often called a modified equation, see e.g. [7], [5]. In this paper we consider methods which can be modeled by

$$\mathbf{u}_t^\varepsilon + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon(\phi \mathbf{u}_x^\varepsilon)_x + c_2 \varepsilon^2 \mathbf{u}_{xx}^\varepsilon, \tag{2}$$

where $\varepsilon = c_1 h$ and c_1 and c_2 a scalar constants. Here ϕ is a smooth function of $(x - s(t))/\varepsilon$ which is equal to one in the vicinity of the shock and zero away from the shock. We consider the same boundary conditions for $\mathbf{u}^\varepsilon$ as for $\mathbf{u}$ augmented by numerical boundary condition, e.g. extrapolation of outgoing characteristic variables.

2.3 Asymptotic Expansions

We assume the following: The solution of (2) can be described by an inner solution, valid in the shock layer, and an outer solution, valid elsewhere. These solutions can be expanded in powers of ε and matched in a region of overlap. Also the position of the shock layer can be expanded in ε. To leading order, the outer solution is equal to the solution of the corresponding inviscid problem.

The inner solution is expressed using the variables $(\tilde{x}, \tilde{t})$ where

$$\tilde{x} = \frac{x - s(t)}{\varepsilon}, \quad \tilde{t} = t.$$

Thus we have expansions of the form

$$\begin{aligned}
&\text{Outer:} && \mathbf{u}^\varepsilon \sim \mathbf{u}(x,t) + \varepsilon \mathbf{u}_1(x,t) + \varepsilon^2 \mathbf{u}_2(x,t) + \cdots, \\
&\text{Inner:} && \mathbf{u}^\varepsilon \sim \mathbf{U}_0(\tilde{x},\tilde{t}) + \varepsilon \mathbf{U}_1(\tilde{x},\tilde{t}) + \varepsilon^2 \mathbf{U}_2(\tilde{x},\tilde{t}) + \cdots, \\
&\text{Position:} && x^\varepsilon \sim s(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots.
\end{aligned}$$

The viscous problem (2) models a method which is a second order accurate approximation of (1) away from the shock region. We claim that the solution will be second order accurate upstream of the shock but in general only first order downstream. Hence we must show that $\mathbf{u}_1 = 0$ upstream and $\mathbf{u}_1 \neq 0$ downstream. We obtain equations for $\mathbf{u}_1$ by substituting the expansions into (2) and collect terms multiplying the same power of ε. In the upstream region it is clear that $\mathbf{u}_1 = 0$, since equation, initial data and boundary condition for $\mathbf{u}_1$ are homogeneous. Also in the downstream region equation and initial data and boundary conditions at $x = x_{\text{end}}$ are homogeneous. However, in order to close the problem, one boundary condition is needed at the downstream side of the shock. Such a boundary condition can be derived by integrating (2) over the shock region. We can conclude that in the general case $\mathbf{u}_1 \neq 0$ in the downstream region, since the boundary condition in general not is homogeneous. The term which make the boundary condition non-homogeneous is $\frac{\partial}{\partial t} I_3(\tilde{t})$, where

$$I_3(\tilde{t}) = \int_{-\infty}^{0} (\mathbf{U}_0(\tilde{x},\tilde{t}) - \mathbf{u}^-)\, d\tilde{x} + \int_{0}^{\infty} (\mathbf{U}_0(\tilde{x},\tilde{t}) - \mathbf{u}^+)\, d\tilde{x}.$$

We see that the first order downstream error is related to the shape of the solution in the shock layer.

We also consider a method which has the modified equation

$$\mathbf{u}_t^\varepsilon + \mathbf{f}(\mathbf{u}^\varepsilon)_x = \varepsilon(\phi \mathbf{E}(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon)_x + c_2 \varepsilon^2 \mathbf{u}_{xx}^\varepsilon, \tag{3}$$

where $\mathbf{E}(\mathbf{u}^\varepsilon)$ is a matrix valued function. Analysis as above reveals that it is possible to chose $\mathbf{E}(\mathbf{u}^\varepsilon)$ such that the first order downstream error is eliminated.

3 Numerical Experiments

In this section we consider the Euler equations with Riemann initial data connected by a 1-shock. In Figure 1 we see the solution at $t = 1.25$.

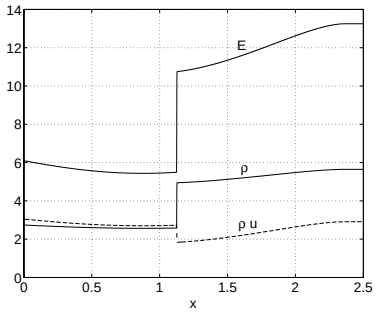
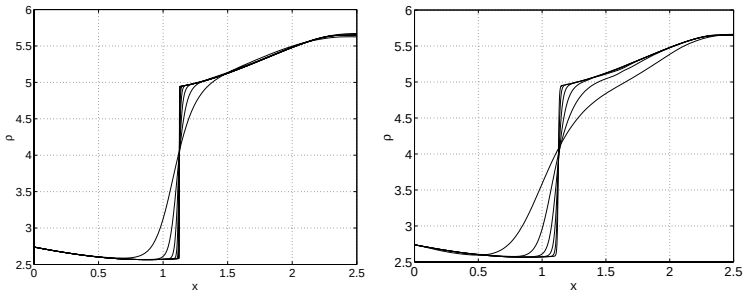


Fig. 1. The solution at $t = 1.25$ computed numerically by the standard method.

We solve the problem using two different methods. The methods are obtained by discretizing (2) and (3) using second order finite differences in space and fourth order Runge-Kutta in time. We refer to them as the standard method and the matrix viscosity method, respectively. The shock position $s(t)$ is numerically determined in both methods. To determine $\mathbf{E}(\mathbf{u}^\varepsilon)$ also the quantities $\dot{s}$, $\mathbf{u}^-$ and $\mathbf{u}^+$ must be numerically determined.

We have numerically investigated the speed of convergence of the standard method and the matrix viscosity method by solving the test problem with successively refined space and time step. In Figure 2 we see how the solutions converge. In Table 1 and 2 we have estimated the convergence order for the



(a) Standard method (b) Matrix viscosity method

Fig. 2. The ρ component of the solution computed successively using successively halved space and time step step.

standard method and the matrix method respectively. It is clear that the solutions computed by the standard method is second order accurate upstream, but only first order accurate downstream. The solutions computed by the matrix viscosity method is however second order accurate both upstream and downstream.

Table 1. Estimates of the order of accuracy (r) and the L_2-norm of the absolute error ($||e||$) for the ρu component of the solution for the standard method.

	Upstream		Downstream	
h	r	$\|\|e\|\|$	r	$\|\|e\|\|$
$1 \cdot 10^{-2}$		$1.5 \cdot 10^{-3}$		$3.1 \cdot 10^{-2}$
$5 \cdot 10^{-3}$	3.92	$1.0 \cdot 10^{-4}$	2.17	$6.9 \cdot 10^{-3}$
$2.5 \cdot 10^{-3}$	2.38	$1.9 \cdot 10^{-5}$	2.15	$1.6 \cdot 10^{-3}$
$1.25 \cdot 10^{-3}$	2.00	$4.6 \cdot 10^{-6}$	1.46	$5.7 \cdot 10^{-4}$
$6.25 \cdot 10^{-4}$	2.00	$1.2 \cdot 10^{-6}$	1.20	$2.5 \cdot 10^{-4}$
$3.125 \cdot 10^{-4}$	2.00	$2.9 \cdot 10^{-7}$	1.08	$1.2 \cdot 10^{-4}$
$1.5625 \cdot 10^{-4}$	2.00	$7.2 \cdot 10^{-8}$	1.03	$5.7 \cdot 10^{-5}$

Table 2. Estimates of the order of accuracy (r) and the L_2-norm of the absolute error ($||e||$) for the ρu component of the solution for the matrix viscosity method.

	Upstream		Downstream	
h	r	$\|\|e\|\|$	r	$\|\|e\|\|$
$1 \cdot 10^{-2}$		$6.3 \cdot 10^{-3}$		$1.8 \cdot 10^{-2}$
$5 \cdot 10^{-3}$	4.71	$2.4 \cdot 10^{-4}$	1.93	$4.8 \cdot 10^{-3}$
$2.5 \cdot 10^{-3}$	3.68	$1.9 \cdot 10^{-5}$	1.96	$1.2 \cdot 10^{-3}$
$1.25 \cdot 10^{-3}$	2.01	$4.6 \cdot 10^{-6}$	2.00	$3.1 \cdot 10^{-4}$
$6.25 \cdot 10^{-4}$	2.00	$1.2 \cdot 10^{-6}$	1.99	$7.7 \cdot 10^{-5}$

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