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Two Aspects of Viscous Shocks:
Existence of a Solution and Numerical Errors

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Abstract

We consider two aspects of viscous shocks.

First, we develop a technique to prove existence of solutions of non-linear ODEs, based on that an approximate numerical solution exists. The aim is to show existence of stationary viscous shock-wave solutions of hyperbolic conservation laws. The technique is applied to viscous Burgers' equation.

Next, we consider the phenomenon that numerical solutions of hyperbolic conservation laws obtained by formally higher order methods are, in many cases, only first order accurate downstream of shocks. We use matched asymptotic expansions to analyze the phenomenon for one dimensional time dependent hyperbolic systems. We also show how to design the artificial viscosity term in order to avoid the first order error. Numerical computations verify that second order accurate solutions are obtained.

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Preface

This thesis consists of an introduction and two papers.

Paper 1: Gunilla Kreiss, Malin Siklosi. *Proving Existence of Solutions of Nonlinear Differential Equations Using Numerical Approximations*. Accepted for publication in the proceedings of the “8th International Conference on Hyperbolic Problems. Theory, Numerics, Applications.”, Magdeburg, Germany, 2000.

Paper 2: Malin Siklosi, Gunilla Kreiss. *Elimination of First Order Errors in Time Dependent Shock Calculations*. Submitted to SIAM Journal of Numerical Analysis.

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Introduction

Hyperbolic conservation laws are challenging equations, from many points of view. They are difficult to study analytically and difficult to solve numerically. In both cases, it is mainly the property that discontinuities develop in the solution which is causing the difficulties. There are different kinds of discontinuities. In this thesis we only consider so called shocks.

In one space dimension hyperbolic conservation laws can be described by the system of partial differential equations

$$\frac{\partial}{\partial t}\mathbf{u}(x, t) + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}(x, t)) = 0. \quad (1)$$

The system is hyperbolic if the Jacobian matrix $\mathbf{f}'(\mathbf{u})$ have real eigenvalues and the Jacobian matrix has a complete set of linearly independent eigenvectors. A very nice introduction to hyperbolic conservation laws is given in [11], where both theoretical and numerical aspects are considered. For a comprehensive survey of the theory we recommended [13].

The corresponding slightly viscous problem

$$\frac{\partial}{\partial t}\mathbf{u}(x, t) + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}(x, t)) = \frac{\partial}{\partial x}\left(\varepsilon \frac{\partial}{\partial x}\mathbf{u}(x, t)\right). \quad (2)$$

is related to (1) in many ways. Hyperbolic conservation laws are used as models of physical phenomena where the viscous effects are so small that it is assumed that they can be neglected. However, often the solution of (1) is not unique. This is a result of that some physical effects, as viscous effects, have been neglected. The standard way to define a unique solution of (1) is to pick the solution which is obtained in the limit as the coefficient ε of the viscous term in (2) goes to zero (the vanishing viscosity method). However, there is no guarantee that the limit exists. Note that the viscous term in (2) is the simplest one could think of. The physical viscosity term has generally not this simple form, which may effect the vanishing viscosity limit.

General theory concerning existence of solutions both for hyperbolic equations and for slightly viscous equations is lacking. Global existence of a weak solution of a strictly hyperbolic system (1) with initial data with small total variation was

proven in [5]. For such a system, it can also be shown, see [7], that there exists a family of viscous solutions converging to the inviscid solution as $\varepsilon \rightarrow 0$. Also for some other special cases, as e.g. a scalar conservation law (see [10]), existence of an inviscid solution as the limit of a family of viscous solutions can be shown. In other cases, the question is open.

In Paper 1, we consider the existence of viscous shock-wave solutions of hyperbolic conservation laws. The work is inspired by the observation that though generally existence of a solution can not be proven, often a numerical solution can be found. We develop a technique to prove existence of a solution based on the fact that an approximate numerical solution exists. The final goal is to show existence of stationary viscous shock-wave solutions of systems of conservation laws. Here we only apply the technique to the scalar viscous Burger's equation. As mentioned above, in this case existence can be proven by other methods.

An existence proof of this form will consist of two main parts:

- Derivation of sufficient conditions on the approximation solution to guarantee the existence of an exact solution.
- Computation of an approximate solution that satisfies the conditions.

The first part consists of classical mathematical work. The evaluation of the conditions in the second part of the proof is done numerically, since the approximate solution is known numerically. Hence, sufficient conditions for existence can be derived for a class of problems, while the numerical evaluation must be carried out for each specific equation separately.

To base a rigorous existence proof on numerically computed quantities, we must use some kind of rigorous numerics. By rigorous numerics we mean numerical approximations with rigorous error bounds. In [12] a survey of theory and software for rigorous numerics is presented, as well as examples of computer aided proofs in analysis. In Paper 1, we have concentrated on the first part of the proof. For the numerical evaluation of the conditions we have used standard methods. Since we have no exact control of the errors we can not rigorously show that the sufficient conditions for existence are satisfied.

The idea to use a numerically computed solution in a rigorous existence proof for viscous shocks is, as far as the author knows, new. In [8], a technique to prove the existence of discrete shock profiles using a numerically determined approximate discrete shock profile is developed. The technique is applied to a scalar hyperbolic conservation law.

The hyperbolic system (1) and the slightly viscous system (2) are also related from a numerical point of view, since numerical solutions of hyperbolic equations often also can be seen as solutions of a slightly viscous problem. In Paper 2 we study a numerical phenomenon concerning viscous shock waves. The order of accuracy of a numerical method is defined as the power with which the local truncation error decays. In the literature, the standard way to validate the performance of the numerical method has been to solve a problem with a smooth solution and

check that it converges as indicated by the formal order of accuracy. Problems with discontinuous solutions have mostly been used only to illustrate the ability of the scheme to obtain a stable solution also when discontinuities are present in the solution. However, recently it has been reported (see e.g. [1], [4] and [3]) that numerical solutions of hyperbolic conservation laws obtained by higher order shock capturing schemes often are only first order accurate downstream of shocks.

Intuitively, this is not surprising if one just consider two well known facts concerning hyperbolic equations and their numerical solutions. First, hyperbolic equations has the property that information follows characteristics. Second, in the shock layer the error in the numerical solution is pointwise $\mathcal{O}(1)$. It therefore seems reasonable that, if there are characteristics going out of the shock, the error in the shock region might follow the outgoing characteristics and pollute the numerical solution on the downstream side.

The phenomenon was studied in [4], where analytic examples are constructed where the numerical solution is only first order accurate downstream of a shock, although the numerical scheme is formally second order. It is also shown that a converging numerical method will yield solutions having the formal order of accuracy in domains where no characteristics have passed through a shock. In [1], the phenomenon is studied for a specific ENO scheme using a scalar, linear model equation. For this specific method, the first order downstream error can be avoided if the shock always is located on a cell interface. After such a modification of the method, the numerical solutions obtain the formal order of accuracy both upstream and downstream of the shock.

A useful technique for studying the behavior of numerical solutions is to model the discrete numerical solution by a slightly viscous equation, a so called modified equation (see e.g. [11] or [6]), which is characteristic for the numerical method. In [2], asymptotic analysis of the modified equation was used to analyze the first order downstream error for a one dimensional steady state solution of a hyperbolic systems with a source term. Based on that analysis, it is shown in [9] how to design the artificial viscosity term in order to avoid the first order error. Also, numerical computations are presented that verify that second order accurate solutions are obtained.

In Paper 2, we use asymptotic analysis of the modified equation to analyze the first order downstream error for one dimensional time dependent hyperbolic systems. We show how to design the artificial viscosity term in order to avoid the first order error. We also present numerical computations that verify that second order accurate solutions are obtained.

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