Proving Existence of Solutions of Nonlinear Differential Equations Using Numerical Approximations

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Abstract. A technique to prove existence of solutions of non-linear ODEs is presented. It is based on the fact that an approximate numerical solution is the solution of a near-by problem. The aim is to show existence of stationary viscous shock-wave solutions of hyperbolic conservation laws. The technique is applied to viscous Burgers' equation. Equations for the difference between the exact and the approximate solution are constructed. Sufficient conditions for existence of a solution of these equations are derived using a fixed point argument. Estimates of the solution of the linearized ODE are needed to derive the conditions. If the truncation error of the approximate solution is small enough, the sufficient conditions are satisfied, and hence the existence of an exact solution is proven.

1. Introduction

For many PDEs, approximate numerical solutions can be found, though existence of a solution has not been proven. We are interested in viscous shock-wave solutions of systems of conservation laws. Existence can only be shown in some special cases, see for example Smoller [8], Freistühler and Szmolyan [1] or Freistühler and Rohde [2]. We want to investigate whether it is possible to develop a technique to prove existence using the fact that a computed approximate solution exists. In this paper we only consider a stationary shock-wave solution of a scalar problem.

In such a technique, sufficient conditions for existence can be derived for a class of problems, but to evaluate the sufficient conditions, computations are needed for each single case. An existence proof of this form will consist of two main parts:

- Derivation of sufficient conditions on the approximation solution to guarantee the existence of an exact solution.
- Computation of an approximate solution that satisfies the conditions.

The first part consists of classical mathematical work. The evaluation of the conditions in the second part of the proof is done numerically, since the approximate solution is known numerically. To base a rigorous existence proof on numerically computed quantities, we must use some kind of rigorous numerics. By rigorous numerics we mean numerical approximations with rigorous error bounds, see for example [7]. In this paper, we have concentrated on the first part of the proof. For the numerical evaluation of the conditions we have used standard methods. Since we have no exact control of the errors we can not rigorously show that the sufficient conditions for existence are satisfied.

We use the following method for the first part of the proof: Denote the exact stationary solution by $u(x)$ and the known approximate solution by $v(x)$. The approximate solution is constructed by a finite difference method followed by interpolation to obtain a continuous function. We chose an interpolation method which yields an interpolant with a sufficient number of continuous derivatives. How many derivatives that are needed depends on the problem. Next, substitute $v(x)$ into the differential equation of the stationary problem to get the truncation error, which we call $\delta q(x)$, where δ is positive constant. If the discrete method used to compute $v(x)$ has order of accuracy p we chose $\delta = h^p$. We expect $q(x)$ to be essentially independent of h. We prove existence of $u(x)$ by proving existence of $w(x)$, where $u = v + \delta w$. The equations for w will be of the form

$$\mathcal{L}(x, v)w = q(x) - \delta p(x, v, w), \tag{1}$$

where $\mathcal{L}$ is a linear differential operator and p is a nonlinear differential operator. The operator $\mathcal{L}$ will be the linearization at v of the ODE. We construct a fixed point iteration for w and derive sufficient conditions for the iteration to converge to a solution of (1). Hence, the sufficient conditions for convergence of the fixed point iteration will be sufficient conditions for the existence of $u(x)$. The conditions will be satisfied if $||\delta q||$ and $||\delta q_x||$ are sufficiently small.

We are mainly interested in proving existence of stationary solutions of viscous conservation laws

$$u_t + g(u)_x = u_{xx} + F(x).$$

In cases where

$$\frac{d}{dt}\int u(x,t)dx = 0,$$

for the time dependent problem, the quantity $\int udx$ is determined by the initial conditions. Hence, to make the solution of the stationary problem unique, we add the condition

$$\int udx = C.$$

In this paper we consider periodic Burgers' equation, in the special case $C = 0$, i.e.

$$\left(\frac{u^2}{2}\right)_x = u_{xx} + F(x), \quad 0 \le x \le L, \quad L > 1, \tag{2a}$$

$$\int_0^L udx = 0, \tag{2b}$$

where $F(x) = \frac{d}{dx}g(x)$ and $g(x)$ is a periodic C^2-function. Note that for this problem, existence is known. For scalar conservations laws in general, methods for proving existence are fully developed. For systems, however, it is only known how to prove existence in some special cases. We aim at applying this technique to systems, and as a first test we apply it to the scalar problem (2).

In a recent paper Jiang and Yu [4] use a similar strategy for proving the existence of discrete stationary shock profiles for conservative finite difference schemes which approximate scalar conservation laws

$$u_t + f(u)_x = 0. \tag{3}$$

They use a computed numerical solution and a fixed point argument. Similarly in Liu and Yu [6] existence of discrete weak profiles for systems is proved using a fixed point argument.

In section 2 we prove sufficient conditions that an approximate solution must satisfy to guarantee the existence of a solution of (2). In section 3 we compute an approximate solution of (2) for $F(x) = 0.1\sin(2\pi x/L), L = 30$ which satisfies the sufficient conditions. Note that we not have used rigorous numerics.

2. Sufficient Conditions for Existence

Let $v(x)$ be an approximate solution of (2), i.e. $v(x)$ is a known periodic function that satisfies

$$\left(\frac{v^2}{2}\right)_x = v_{xx} + F(x) - \delta q(x), \quad 0 \le x \le L, \quad L > 1, \tag{4a}$$

$$\int_0^L v dx = 0, \tag{4b}$$

for a scalar $\delta > 0$. Define $w(x)$ by $u = v + \delta w$. We have

$$\frac{(v+\delta w)^2}{2} = \frac{v^2}{2} + \delta v w + \frac{\delta^2 w^2}{2}. \tag{5}$$

Define the linear differential operator $\mathcal{L}$ by $\mathcal{L}w = -w_{xx} + (vw)_x$. From (2), (4) and (5) we see that w must satisfy

$$\mathcal{L}w = q - \delta(\frac{w^2}{2})_x, \tag{6a}$$

$$\int_0^L w dx = 0, \quad w \text{ periodic}. \tag{6b}$$

In this section we will prove a theorem that states conditions on v, δ and q that guarantee the existence of a solution of (6), and hence the existence of a solution (2). Note that the conditions are sufficient but not necessary. First, in subsection 2.1 we make an exponential scaling of the problem, to transform the problem to a form where the linear operator is self-adjoint. In subsection 2.2 we prove a few auxiliary lemmas. Finally, in subsection 2.3 we derive criteria for the existence of a solution of the transformed problem by constructing a sequence of functions, $\{\tilde{w}_n\}, n = 0, 1, 2, \ldots$, and investigating under what conditions the sequence will converge to the solution of the transformed problem.

2.1. Exponential Scaling

We define the inner-product and norm of two L-periodic functions $u = u(x)$ and $v = v(x)$ by

$$(u, v) = \int_0^L u(x)v(x)dx, \quad ||u|| = (u, u)^{1/2}.$$

We also use the norms

$$||u||^2_{H^p} = \sum_{j=0}^{p} ||\frac{d^j u}{dx^j}||^2,$$

and

$$|u|^2_{\infty,p} = \sum_{j=0}^{p} |\frac{d^j u}{dx^j}|^2_\infty,$$

The operator $\mathcal{L}$ is not self-adjoint, but it can be transformed into a self-adjoint operator. Generally, the transformation makes the transformed problem non-periodic. This is avoided in the special case $\int_0^L v dx = 0$. Since better energy estimates can be found for the transformed problem, we transform (6) and show existence of a solution of the transformed problem.

Define

$$f(x) = exp\left(\frac{1}{2}\int_a^x v(\xi)d\xi\right),$$

where a is an arbitrary constant which doesn't effect the convergence criteria. We chose a such that $|f|_\infty = 1$.

Also define $\tilde{w}(x)$ by $w(x) = f(x)\tilde{w}(x)$. Then (6) is equivalent to

$$\tilde{\mathcal{L}}\tilde{w} = \tilde{q} - \frac{\delta}{f}\left(\frac{f^2\tilde{w}^2}{2}\right)_x, \tag{7a}$$

$$\int_0^L f\tilde{w} = 0, \quad \tilde{w} \text{ periodic}, \tag{7b}$$

where

$$\begin{aligned} \tilde{\mathcal{L}}\tilde{w} &= -\tilde{w}_{xx} + c(x)\tilde{w}, \\ c(x) &= \frac{1}{2}\frac{d}{dx}v(x) + \frac{1}{4}v(x)^2, \end{aligned}$$

and $\tilde{q} = q/f$. The corresponding eigenvalue problem

$$\tilde{\mathcal{L}}\phi_i = \lambda_i\phi_i, \quad i = 0, 1, 2, \ldots \tag{8a}$$

$$\phi_i \text{ periodic}, \tag{8b}$$

is a periodic Sturm-Liouville system. Thus the eigenvalues are real, and the eigenfunctions form a complete set of orthogonal functions in L_2, see e.g. [3]. Note that $\phi_0 = f$. Also, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, see [5].

2.2. Auxiliary Lemmas

Lemma 2.1. *Consider the linear problem*

$$\tilde{\mathcal{L}}\tilde{w} = H$$
$$(\phi_0, \tilde{w}) = 0, \quad \tilde{w} \text{ periodic},$$

where $||H||_{H^1} < \infty$*. Then there is a unique solution* $\tilde{w} \in H^3$ *such that* $||\tilde{w}||^2_{H^3} \leq \alpha||H||^2 + \beta||H_x||^2$*, where*

$$\begin{aligned}
\alpha &= \sum_{i=1}^{4} K_i^2, \\
\beta &= 3, \\
K_1^2 &= \frac{1}{\lambda_1^2}, \\
K_2^2 &= K_1 + |c|_\infty K_1^2, \\
K_3^2 &= 2(|c|^2_\infty K_1^2 + 1), \\
K_4^2 &= 3(|c_x|^2_\infty K_1^2 + |c|^2_\infty K_2^2).
\end{aligned}$$

Proof. Using the eigenfunctions of (8) and the first condition in (7b), we express the solution as

$$\tilde{w} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}(H, \phi_n)\phi_n.$$

Thus

$$||\tilde{\mathcal{L}}^{-1}||^2 \leq \frac{1}{\lambda_1^2},$$

giving $||\tilde{w}||^2 \leq K_1^2||H||^2$. The identity

$$(\tilde{w}, H) = ||\tilde{w}_x||^2 + (\tilde{w}, c\tilde{w})$$

yields $||\tilde{w}_x||^2 \leq K_2^2||H||^2$. From

$$\tilde{w}_{xx} = c\tilde{w} - H \tag{10}$$

we get $||\tilde{w}_{xx}||^2 \leq K_3^2||H||^2$. Differentiation of (10) gives $||\tilde{w}_{xxx}||^2 \leq K_4^2||H||^2 + \beta||H_x||^2$. The proof of the lemma is complete. □

By straightforward energy estimates we prove the following two lemmas.

Lemma 2.2. *The function*

$$G(\tilde{w}(x)) = \tilde{q}(x) - \frac{\delta}{f(x)}\left(\frac{f(x)^2\tilde{w}(x)^2}{2}\right)_x \tag{11}$$

satisfies $||G||^2 \leq c_0||\tilde{q}||^2+\delta^2K_5^2||\tilde{w}||^4_{H^3}$ *and* $||G_x||^2 \leq c_1||\tilde{q}_x||^2+\delta^2K_6^2||\tilde{w}||^4_{H^3}$, *where*

$$\begin{aligned} c_0 &= 3, \\ c_1 &= 5, \\ K_5^2 &= \frac{3}{2}|v|^2_\infty + 3, \\ K_6^2 &= 10|c|^2_\infty + \frac{45}{4}|v|^2_\infty + 15. \end{aligned}$$

Lemma 2.3. *Assume that* $||\tilde{w}_j||^2_{H^3} \leq z$ *for* $j = m, n$. *Then* $\tilde{G}(\tilde{w}_m, \tilde{w}_n) = G(\tilde{w}_m) - G(\tilde{w}_n)$ *satisfies* $||\tilde{G}||^2 \leq \delta^2 R_1||\tilde{w}_m - \tilde{w}_n||^2_{H^3}$ *and* $||\tilde{G}_x||^2 \leq \delta^2 R_2||\tilde{w}_m - \tilde{w}_n||^2_{H^3}$, *where* $G(\tilde{w})$ *is defined in (11) and*

$$\begin{aligned} R_1 &= (4|v|^2_\infty + 8)z, \\ R_2 &= (24|c|_\infty + 27|v|^2_\infty + 48)z. \end{aligned}$$

Lemma 2.4. *Consider the sequence*

$$z_{n+1} = a + bz_n^2, \quad z_0 = a, \quad a, b > 0.$$

If $4ab < 1$, *the sequence converge to* z,

$$z = \frac{1-\sqrt{1-4ab}}{2b}.$$

Also, $z_n \leq z$.

The proof is simple and is omitted.

2.3. The Main Theorem

Theorem 2.5. *Assume there exists a* $v(x)$ *that satisfies (4). Let*

$$\begin{aligned} z &= \frac{1-\sqrt{1-4ab\delta^2}}{2b\delta^2} \\ a &= \alpha c_0||\tilde{q}||^2 + \beta c_1||\tilde{q}_x||^2, \\ b &= \alpha K_5^2 + \beta K_6^2, \\ \kappa &= \alpha R_1 + \beta R_2, \end{aligned}$$

where $c_0, c_1, \alpha, \beta, K_5^2, K_6^2, R_1$ *and* R_2 *are defined in the auxiliary lemmas. If*

$$4ab\delta^2 < 1 \quad \text{and} \quad \delta^2\kappa < 1, \tag{12}$$

then there exists a solution of (2).

Proof. Consider the sequence of functions $\tilde{w}_n, n = 0, 1, 2, \ldots$ where

$$\begin{aligned} \tilde{w}_0 &= \tilde{\mathcal{L}}^{-1}\tilde{q}, \\ \tilde{w}_{n+1} &= \tilde{\mathcal{L}}^{-1}\left(\tilde{q} - \frac{\delta}{f}(\frac{\tilde{w}_n^2}{2})_x\right), \end{aligned}$$

and $\tilde{w}_n$ satisfies (7b). We will use the result of the auxiliary lemmas to prove that the sequence converges uniformly to the solution of (7). First, Lemma 2.1, 2.2

and 2.4 are used to show that $||\tilde{w}_n||_{H^3}$ is bounded. Then, Lemma 2.3 can be used to show the uniform convergence.

Lemma 2.1 and 2.2 gives $||\tilde{w}_{n+1}||^2_{H^3} \leq a + \delta^2 b||\tilde{w}_n||^4_{H^3}$. Applying Lemma 2.4 we get $\lim_{n\to\infty} ||\tilde{w}_n||^2_{H^3} \leq z$ and $||\tilde{w}_n||^2_{H^3} \leq z$.

Consider the difference $y_n = \tilde{w}_{n+p} - \tilde{w}_n$, where $p \geq 0$ is some arbitrary integer. The sequence $y_n, n = 0, 1, 2, \ldots$ satisfies

$$\begin{aligned} y_0 &= \tilde{\mathcal{L}}^{-1}(G(\tilde{w}_p) - G(\tilde{w}_0)), \\ y_{n+1} &= \tilde{\mathcal{L}}^{-1}(G(\tilde{w}_{n+p}) - G(\tilde{w}_n)) = \tilde{\mathcal{L}}^{-1}\tilde{G}(\tilde{w}_{n+p}, \tilde{w}_n), \end{aligned}$$

where G and $\tilde{G}$ are defined in lemma 2.2 and 2.3. Since $||\tilde{w}_n||_{H^3}$ is bounded, we can apply Lemma 2.3. We get $||y_{n+1}||^2_{H^3} \leq \delta^2\kappa||y_n||^2_{H^3}$, hence $||y_n||^2_{H^3} \leq (\delta^2\kappa)^n||y_0||^2_{H^3}$. If $\delta^2\kappa < 1$, then $||y_n||^2_{H^3}$ is arbitrary small if n is large enough. Sobolev inequalities give $|y_n|^2_{\infty,2} \leq 3||y_n||_{H^3}$. According to the Cauchy criterion, $\{\tilde{w}_n\}$ and its two first derivatives converges uniformly to a solution $\tilde{w}$ of (7). By construction, $u = v + \delta f\tilde{w}$ satisfies (2). This completes the proof of Theorem 2.5. □

3. Numerical Results

We have performed computations to investigate whether it is possible to obtain approximate solutions which satisfies the conditions of Theorem 2.5, and hence prove existence of a solution of (2).

We computed the approximate solutions by a solving the time dependent problem by method of lines to steady state. We used backward Euler in time and discretized with constant step-size h in space, approximating $\frac{d}{dx}$ by D_0 and $\frac{d^2}{dx^2}$ by D_+D_-. To obtain a continuous function we interpolated the discrete solution using Fourier interpolation. Next, we substituted the approximate solution into the differential equation and obtained the truncation error δq. The eigenvalue λ_1 was computed by solving the discrete eigenvalue problem, again using D_0 and D_+D_- to approximate derivatives with respect to x. Also the norms of q and v were computed numerically.

In Figure 1 we show the result of computations for the problem (2) with $F(x) = 0.1sin(2\pi x/L)$ and $L = 30$. When 32 point/length unit where used in the discrete solution the conditions (12) where satisfied ($4ab\delta^2 = 0.52$ and $\delta^2\kappa = 0.41$), and hence Theorem 2.5 applies.

Remark 3.1. *Since we not have used rigorous numerics we have no exact control of the errors, (due to e.g. rounding and truncation in the computing process) and hence we have not rigorously shown that Theorem 2.5 applies.*

Due to the exponential scaling, a large part of the mass of $\tilde{q}(x)$ and $\tilde{q}_x(x)$ is situated away from the shock. The mass away from the shock will grow exponentially as L increases. Hence, it will require an exponentially growing computational effort to obtain an approximate solution which can prove the existence of a solution

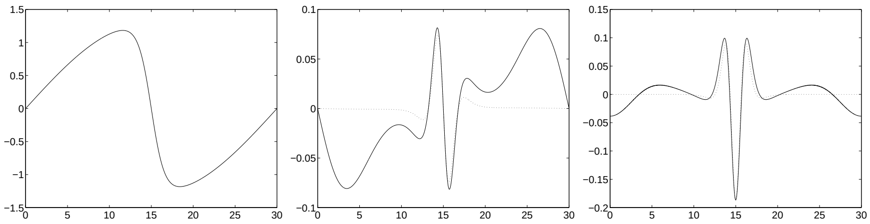

FIGURE 1. Approximate solution of (2) for $F(x) = 0.1sin(2\pi x/L)$ and $L = 30$. Step size $h = 3.125 \cdot 10^{-2}$ in the finite difference method gives $4ab\delta^2 = 0.52$ and $\delta^2\kappa = 0.41$. Left: $v(x)$. Middle: $\tilde{q}(x)$ (solid line), $q(x)$ (dotted line). Right: $\tilde{q}_x(x)$ (solid line), $q_x(x)$ (dotted line).

of (2). The exponential behavior arises since an exponential scaling is needed to transform the operator to self-adjoint form. Possibly the effect of the exponential scaling could be damped by using more than one eigenvalue in the estimate of $||\tilde{\mathcal{L}}||^{-1}$.

We have used Fourier interpolation to construct a continuous approximate solution $v(x)$. The interpolant constructed by Fourier interpolation becomes oscillatory when many points are used. When the number of interpolation points is large, $\tilde{q}$ and $\tilde{q}_x$ will be totally dominated by the oscillations, and the norm of $\tilde{q}$ and $\tilde{q}$ will increase as the space step h is decreased. Obviously, to obtain useful results for small h some other interpolation method must be used.

References

[1] H. Freistühler and P. Szmolyan, *Existence and Bifurcation of Viscous Profiles for all Intermediate Magnetohydrodynamic Shock Waves,* SIAM Journal of Mathematical Analysis, **26** (1995), 112–128.

[2] Heinrich Freistühler and Christian Rohde, *Numerical Methods for Viscous Profiles of Non-classical Shock Waves,* Hyperbolic problems:theory, numerics, applications, volume 2 (1999).

[3] D. H. Griffel, *Applied Functional Analysis* Ellis Horwood Limited, (1981).

[4] Guang-Shan Jiang and Shih-Hsien Yu, *Discrete Shocks for Finite Difference Approximations to Scalar Conservation Laws,* SIAM Journal of Numerical Analysis, **35** (1998), 749–772.

[5] Gunilla Kreiss and Heinz-Otto Kreiss, *Convergence to Steady State of Solutions of Burgers' Equation,* Applied Numerical Mathematics, **2** (1986), 161–179.

[6] Tai-Ping Liu and Shih-Hsien Yu, *Continuum Shock Profiles for Discrete Conservation Laws I: Construction,* Communications on Pure and Applied Mathematics, **52** (1999), 85–127.

[7] Kenneth R. Meyer and Dieter S. Schmidt, editors., *Computer Aided Proofs in Analysis,* The IMA Volumes in Mathematics and its Applications, **28**, Springer-Verlag (1991).

[8] Joel Smoller, *Shock Waves and Reaction-Diffusion Equations,* McGraw-Hill, Inc. (1985).

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