Large deviations and ruin probabilities for multivariate heavy-tailed risk processes

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based on joint work with H. Hult, T. Mikosch and G. Samorodnitsky
A classical univariate ruin problem

Let \((Z_k)\) be an iid sequence of claims and let

\[
C_t = \sum_{k=1}^{N_t} Z_k
\]

be the claim amount process, where \(N_t = \#\{n \geq 1 : T_n \leq t\}\) is a renewal process independent of \((Z_k)\) with renewal sequence

\[
T_0 = 0, \quad T_n = W_1 + \cdots + W_n, \quad (W_k) \text{ iid sequence}.
\]

Set \(U_t = u + pt - C_t\). Ruin occurs if \(U_t < 0\) and the ruin probability is

\[
\psi_u = P \left( \inf_{t>0} U_t < 0 \right).
\]
We have (suppose $E(Z_1), E(W_1)$ exist):

\[
\left\{ \inf_{t>0} U_t < 0 \right\} = \left\{ \inf_{n \geq 1} U_{T_n} < 0 \right\} = \left\{ \inf_{n \geq 1} \left[ u + pT_n - \sum_{k=1}^{n} Z_k \right] < 0 \right\}
\]

\[
= \left\{ \inf_{n \geq 1} \left[ u - \sum_{k=1}^{n} (Z_k - pW_k) \right] < 0 \right\} = \left\{ \sup_{n \geq 1} \tilde{S}_n > u \right\}
\]

\[
= \left\{ \sup_{n \geq 1} \left[ \tilde{S}_n - E(\tilde{S}_n) + E(\tilde{S}_n) \right] > u \right\} = \left\{ \sup_{n \geq 1} (S_n - cn) > u \right\},
\]

where $S_n = X_1 + \cdots + X_n$, $(X_k)$ iid sequence with $E(X_1) = 0$.

Hence, the ruin probability is the probability that a random walk with drift exceeds the level $u$. 
Regular variation

A nonnegative random variable $Z$ is said to be **regularly varying** with index $\alpha$ if

$$\lim_{u \to \infty} \frac{P(Z > xu)}{P(Z > u)} = x^{-\alpha}, \quad x > 0.$$  

**Canonical example** Pareto($\alpha$)-distribution: $P(Z > x) = x^{-\alpha}$, $x > 1$.

**Moments** If $Z$ is regularly varying with index $\alpha$, then

$$E(Z^\beta) = \infty \text{ if } \beta > \alpha, \quad E(Z^\beta) < \infty \text{ if } \beta < \alpha.$$  

**Theorem** (Embrechts and Veraverbeke 1982) If $Z_1$ is regularly varying with index $\alpha > 1$, then for $c = E(pW_1 - Z_1) > 0$,

$$\lim_{u \to \infty} \frac{\psi_u}{u P(Z_1 > u)} = \lim_{u \to \infty} \frac{P(\sup_{n \geq 1} (S_n - cn) > u)}{u P(X_1 > u)} = \frac{1}{(\alpha - 1)c}.$$  


$S_n - n > 500$ for some $n \leq 5000$, $t_2$-distributed steps
$S_n - n/5 > 500$ for some $n \leq 5000$, $t_2$-distributed steps
Multivariate regular variation

An \( \mathbb{R}^d \)-valued random vector \( X \) is \textbf{regularly varying} if there exists a nonzero measure \( \mu \) such that for every \( A \in \mathcal{B}(\mathbb{R}^d) \) bounded away from 0 with \( \mu(\partial A) = 0 \),

\[
\lim_{u \to \infty} \frac{P(X \in uA)}{P(|X| > u)} = \mu(A).
\]

It follows that \( \mu(x \cdot) = x^{-\alpha} \mu(\cdot) \) for some \( \alpha \).

One \textbf{equivalent definition} of multivariate regular variation: \( X \) is regularly varying if there exist a nonzero measure \( \mu \) and a sequence \( a_n \uparrow \infty \) such that for every \( A \in \mathcal{B}(\mathbb{R}^d) \) bounded away from 0 with \( \mu(\partial A) = 0 \),

\[
\lim_{n \to \infty} n P(X \in a_n A) = \mu(A)
\]

We write \( X \in \text{RV}(\alpha, \mu) \).
• Multivariate regular variation serves as domain of attraction condition for partial sums of iid random vectors (Rvačeva 1962) and as maximum domain of attraction condition for component-wise maxima (Resnick 1987).

• Under general conditions, the solution $Y_\infty$ to a stochastic recurrence equation $Y_t = A_t Y_{t-1} + B_t$ is regularly varying (Kesten 1973).

One example is the **GARCH-process** (Basrak, Davis, Mikosch 2002):

\[ X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = (\alpha_1 Z_{t-1}^2 + \beta_1) \sigma_{t-1}^2 + \alpha_0, \]

\[ Y_t = \begin{pmatrix} X_t^2 \\ \sigma_t^2 \end{pmatrix}, \quad A_t = \begin{pmatrix} \alpha_1 Z_t^2 & \beta_1 Z_t^2 \\ \alpha_1 & \beta_1 \end{pmatrix}, \quad B_t = \begin{pmatrix} \alpha_0 Z_t^2 \\ \alpha_0 \end{pmatrix}. \]
Fix $n \geq 1$. For every $A \in \mathcal{B}(\mathbb{R}^d)$ bounded away from 0 with $\mu(\partial A) = 0$,

$$\lim_{u \to \infty} \frac{P(S_n \in uA)}{nP(|X_1| > u)} = \mu(A).$$

**Idea: only one term contributes to large $S_n$.**

It is therefore plausible (and well-know in the univariate case: A. Nagaev 1969, S. Nagaev 1979) that

$$\lim_{n \to \infty} \frac{P(S_n \in \lambda_n A)}{nP(|X_1| > \lambda_n)} = \mu(A)$$

if $\lambda_n \uparrow \infty$ fast enough.

**Example** For a suitable sequence $(\lambda_n)$, $S_n/\lambda_n \xrightarrow{p} 0$ and $P(S_n/\lambda_n \in xA)$ decays like $x^{-\alpha} \mu(A)n P(|X_1| > \lambda_n)$. 
Let \((X_k)\) be an iid sequence of \(\mathbb{R}^d\)-valued random vectors, let \(S_n = X_1 + \cdots + X_n\), \(n \geq 1\), and write \(S^n = (S_{[nt]})_{t \in [0,1]}\) for the càdlàg embedding of \((S_n)\).

**Theorem** Suppose \(X_1 \in \text{RV}(\alpha, \mu)\) and consider \((\lambda_n)\) such that

\[
\lambda_n^{-1} S_n \xrightarrow{P} 0, \quad \alpha < 2
\]
\[
\lambda_n^{-1} S_n \xrightarrow{P} 0, \lambda_n/\sqrt{n^{1+\gamma}} \to \infty \text{ some } \gamma > 0 \quad \alpha = 2
\]
\[
\lambda_n^{-1} S_n \xrightarrow{P} 0, \lambda_n/\sqrt{n \log n} \to \infty \quad \alpha > 2,
\]

hold. Then

\[
\lim_{n \to \infty} \frac{P(S^n \in \lambda_n B)}{nP(|X_1| > \lambda_n)} = m(B)
\]

for every \(B \in \mathcal{B}(\mathbb{D})\) (\(\mathbb{D} = \{\text{càdlàg functions}\}\)) bounded away from 0 satisfying \(m(\partial B) = 0\). (Example: \(B = \{x \in \mathbb{D} : \sup_{t \in [0,1]} |x_t| > 1\}\))
The measure $m$ satisfies $m(\mathcal{V}^c) = 0$, where

$$\mathcal{V} = \{ x \in \mathbb{D} : x = y1_{[v, 1]}, v \in [0, 1), y \in \mathbb{R}^d \setminus \{0\}\},$$

i.e. $\mathcal{V}$ is the family of step functions with one step.

Moreover, the marginal distributions satisfy $m_t = t\mu$.

In particular, for $A \in \mathcal{B}(\mathbb{R}^d)$ bounded away from 0 satisfying $\mu(\partial A) = 0$,

$$\lim_{n \to \infty} \frac{P(S_{[nt]} \in \lambda_n A)}{nP(|X_1| > \lambda_n)} = t\mu(A), \quad t > 0.$$

**Remark** If $\alpha > 1$ and $E(X_1) = 0$, then we can always choose $\lambda_n = n$. 
**Question** Why do we look at large deviations in a function space?

**Answer** For many interesting applications (e.g. ruin problems) knowing the sample path behavior is essential.

**Ruin probabilities for multivariate random walk with drift**

Consider the ruin probability

\[ \psi_u(A) = P \left( \sum_{k=1}^{N_t} Z_k - pt \in uA \text{ for some } t > 0 \right), \]

where \((Z_k)\) is an iid sequence of claims in \(d\) lines of business and \(p\) is a vector of premium rates. Then, as in the univariate case, we can write

\[ \psi_u(A) = P(S_n - cn \in uA \text{ for some } n \geq 1), \]

where \(S_n = X_1 + \cdots + X_n\) is a multivariate random walk with \(E(X_1) = 0\).
Let \( (S^n) \) be given by \( S^n_t = S_{nt} \) for \( t \in [0, 1] \). We know that

\[
\lim_{n \to \infty} \frac{P(S^n \in \lambda_n B)}{nP(|X_1| > \lambda_n)} = m(B),
\]

where \( m \) concentrates on the set of step functions with one step. This means that, for large \( n \), \( S_{nt} \) reaches a set \( nA \) bounded away from \( 0 \) for some \( t \) by making one large jump to that set.

**Remark** This holds also for Lévy processes \((X_s)\) with \( X^n_t = X_{nt} \) for \( t \in [0, 1] \):

\[
\lim_{n \to \infty} \frac{P(X^n \in \lambda_n B)}{nP(|X_1| > \lambda_n)} = m(B).
\]
Now consider the random walk with drift, $S_{[nt]} - c_{[nt]}$, and a set $nA$ such that the process cannot reach $nA$ by simply drifting in direction $-c$.

The process first **drifts** in direction $-c$. Then, at some time $v$, it takes a **large jump** $y$ from $-c[nv]$ to $-c[nv] + y$ and then **continues to drift** in direction $-c$.

Hence, for $S_{[nt]} - c_{[nt]}$ to reach $nA$ for some $t$ we must have

$$y \in c[nv] + \{x + cl, x \in nA, l \geq 0\}.$$  

**Example:** If $d = 1$, $A = [a, b) \subset (0, \infty)$ and $c > 0$, then $y \in c[nv] + [na, \infty)$. Hence here the jump only needs to be large enough, the process will eventually drift back into the set.
Given \( c \in \mathbb{R}^d \setminus \{0\} \), let \( \delta > 0 \) be such that the set

\[
K_c^\delta = \left\{ x \in \mathbb{R}^d \setminus \{0\} : \left| \frac{x}{|x|} + \frac{c}{|c|} \right| < \delta \right\}
\]

satisfies \( \mu((\partial K_c^\delta) \setminus \{0\}) = 0 \). We can choose \( \delta > 0 \) arbitrarily small.

**Theorem** For every \( A \in \mathcal{B}(\mathbb{R}^d \setminus K_c^\delta) \) bounded away from 0,

\[
\mu^*(A^\circ) \leq \liminf_{u \to \infty} \frac{\psi_u(A)}{u \, P(|X_1| > u)} \leq \limsup_{u \to \infty} \frac{\psi_u(A)}{u \, P(|X_1| > u)} \leq \mu^*(\overline{A}),
\]

where for any \( B \in \mathcal{B}(\mathbb{R}^d \setminus K_c^\delta) \)

\[
\mu^*(B) = \int_0^\infty \mu(cv + B_c)dv, \quad B_c = \{x + cl, x \in B, l \geq 0\}.
\]
Consider the case \( d = 1, \ c > 0 \) and \( \delta \in (0, 1) \).

\[
K^\delta_c = \left\{ x \neq 0 : \left| \frac{x}{|x|} + \frac{c}{|c|} \right| < \delta \right\} = \left\{ x \neq 0 : \left| \frac{x}{|x|} + 1 \right| < \delta \right\} = (-\infty, 0).
\]

Take \( B = (a, b) \), \( 0 < a < b \). Then \( B_c = \{ x + cl, x \in B, l \geq 0 \} = (a, \infty) \) and

\[
\mu^*(B) = \int_0^\infty \mu(cv + B_c)dv, = \int_0^\infty \mu((a + cv)(1, \infty))dv
\]

\[
= \mu((1, \infty)) \int_0^\infty (a + cv)^{-\alpha}dv = \mu((1, \infty)) \frac{1}{c} \int_a^\infty u^{-\alpha}du
\]

\[
= \frac{a^{-\alpha+1}}{(\alpha - 1)c} \mu((1, \infty)).
\]

With \( a = 1 \) we get the result of Embrechts and Veraverbeke (1982).
Sketch of main idea in the proof

Ruin can be early or late:

\[ \psi_u(A) = \psi_u^{(1)}(A) + \psi_u^{(2)}(A) \]

\[ = P(S_n - cn \in uA \text{ some } n \leq uM) + P(S_n - cn \in uA \text{ some } n > uM) . \]

One shows that late ruin is essentially impossible:

\[ \lim_{M \to \infty} \lim_{u \to \infty} \frac{\psi_u^{(2)}(A)}{u P(|X_1| > u)} = 0. \]

The probability of early ruin can be expressed as

\[ \psi_u^{(1)}(A) \approx P(S_{[Mut]} \in cMut + uA \text{ some } t \in [0, 1]) \]
\[ = P((Mu)^{-1}S_{[Mut]} \in ct + M^{-1}A \text{ some } t \in [0, 1]). \]
\[
\lim_{u \to \infty} \frac{\psi^{(1)}_{u}(A)}{u \mathbb{P}(|X_1| > u)} = \lim_{u \to \infty} \frac{M u \mathbb{P}(|X_1| > Mu)}{u \mathbb{P}(|X_1| > u)} \lim_{u \to \infty} \frac{\psi^{(1)}_{u}(A)}{M u \mathbb{P}(|X_1| > Mu)}
\]

\[
= \{ \text{set } v = Mu \} \\
= M^{1-\alpha} \lim_{v \to \infty} \frac{\mathbb{P}(v^{-1}S[v_{[v,t]}] \in ct + M^{-1}A \text{ some } t \in [0, 1])}{v \mathbb{P}(|X_1| > v)}
\]

\[
= M^{1-\alpha} M^{\alpha-1} \int_0^M \mu(y : y \in cl + A \text{ some } l \in [v, M])dv.
\]

We arrive at

\[
\lim_{M \to \infty} \lim_{u \to \infty} \frac{\psi^{(1)}_{u}(A)}{u \mathbb{P}(|X_1| > u)} = \lim_{M \to \infty} \int_0^M \mu(y : y \in cl + A \text{ some } l \in [v, M])dv
\]

\[
= \mu^*(A).
\]
References


