SZEGÖ TYPE LIMIT THEOREMS

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0. Introduction

Let S^1 be a unit circle with the standard measure, P_k be the orthogonal projection in $L_2(S^1)$ on the subspace spanned by e^{ijx}, $j = 0, \pm 1, \dots, \pm k$, and B be the operator of multiplication by a smooth function b in $L_2(S^1)$. The classical Szegö limit theorem states that under some assumptions on b

$$\operatorname{Tr} \log(P_k B P_k) \;=\; \frac{k}{\pi} \int_0^{2\pi} \log b(x)\, dx + O(1)\,, \qquad k \to +\infty\,. \tag{0.1}$$

The operators P_k coincide with the spectral projections P_λ of the selfadjoint operator $(-\,d^2/dx^2)^{1/2}$ in $L_2(S^1)$ corresponding to the intervals $[0,\lambda)$ with $k < \lambda \leqslant k+1$. Following V. Guillemin [G], we obtain a generalization of this theorem for P_λ being the spectral projections of an elliptic selfadjoint (pseudo)differential operator A on a manifold without boundary. We also study the case where A is the operator of an elliptic boundary value problem.

Moreover, we consider an arbitrary sufficiently smooth function ψ instead of the logarithm. In other words, we obtain asymptotics of the functional

$$\rho_\lambda(\psi) \;=\; \operatorname{Tr} P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda \;=\; \sum_k \psi(\mu_k(\lambda)), \tag{0.2}$$

where $\mu_k(\lambda)$ are the eigenvalues of the operator $P_\lambda B P_\lambda$. The functional ρ_λ is the sum of δ-functions located at the eigenvalues of the operator $P_\lambda B P_\lambda$. Obviously, ρ_λ contains full information about the spectrum of $P_\lambda B P_\lambda$. This means that we study the asymptotic behaviour of the spectra of operators $P_\lambda B P_\lambda$.

This idea is not new, and similar results have been obtained for various classes of differential and pseudodifferential operators [G], [H], [R], [Z]. However, in these papers the asymptotic formulae were proved without any remainder estimates. More precise results have been obtained only in some special cases [Wi1], [Wi2], [J], [Ok].

In this paper we suggest a new method, which is applicable to a wide class of operators and allows one to obtain asymptotics with the same remainder estimate as in the classical spectral asymptotic formulae (in many cases this estimate proves to

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The research was supported by the British Engineering and Physical Sciences Research Council, Grants GR/K39394 and B/93/AF/1559 and by the Swedish Natural Sciences Research Council.

be precise). The method is based on some results from the abstract operator theory. Namely, we use the combination of the Berezin–Lieb inequality (see Appendix) and the technique of double operator integrals developed by M. Birman and M. Solomyak (see Section 1).

Further on we often assume the function ψ in (0.2) to be from the Sobolev class W^2_∞, and then we always mean that ψ is continuous.

1. Some results from the abstract operator theory

1. In this subsection we recall some results from the theory of double operator integrals developed in [BS1] (see also [Ya]).

Let H be a Hilbert space and I be the identity operator in H. The class of Hilbert–Schmidt operators in H is denoted by $\mathfrak{S}_2$. The class $\mathfrak{S}_2$ itself is a Hilbert space with the inner product

$$(T_1, T_2)_{\mathfrak{S}_2} \;=\; \mathrm{Tr}(T_1\, T_2^*)\,, \qquad T_1, T_2 \in \mathfrak{S}_2\,.$$

Let E_1 and E_2 be spectral measures on $\mathbb{R}^1$. This means that the measures E_1 and E_2 are defined on the Borel subsets of $\mathbb{R}^1$, their values are orthogonal projections in H, and $E_1(\mathbb{R}^1) = E_2(\mathbb{R}^1) = I$. We consider integrals of the form

$$\iint f(t,s)\, dE_1(t)\, T\, dE_2(s)\,, \tag{1.1}$$

where f is a measurable complex-valued function on $\mathbb{R}^2$ and T is a bounded operator in H. The integral (1.1) is said to be a *double operator integral*. Under suitable assumptions it is well-defined and determines an operator in H.

To justify the integral (1.1) one needs certain restrictions on f and T. We shall use the simplest version of the theory, assuming $T \in \mathfrak{S}_2$. Then for any measurable sets $\omega_1, \omega_2 \in \mathbb{R}^1$, the map

$$\mathfrak{S}_2 \ni T \;\to\; E_1(\omega_1)\, T\, E_2(\omega_2) \in \mathfrak{S}_2$$

is an orthogonal projection in $\mathfrak{S}_2$. Let $\mathcal{E}$ be the spectral measure on $\mathbb{R}^2$ with values in the space of projections in $\mathfrak{S}_2$ such that

$$\mathcal{E}(\omega_1 \times \omega_2)\, T \;=\; E_1(\omega_1)\, T\, E_2(\omega_2)\,, \qquad \forall\, T \in \mathfrak{S}_2\,.$$

For measurable functions f on $\mathbb{R}^2$ we now define

$$\iint f(t,s)\, dE(t)\, T\, dE(s) \;=\; \int f(t,s)\, d\mathcal{E}(t,s)\, T\,,$$

where the right hand side is understood as an integral with respect to the spectral measure $\mathcal{E}$ in the Hilbert space $\mathfrak{S}_2$ (see, for example, [BS2, Ch.5]). From the previous definition it follows that the double operator integral (1.1) linearly depends on f and T, and that

$$\| \iint f(t,s)\, dE_1(t)\, T\, dE_2(s)\, \|_{\mathfrak{S}_2} \;\leqslant\; \sup |f|\; \|T\|_{\mathfrak{S}_2}\,. \tag{1.2}$$

We need the following obvious result.

Proposition 1.1. *Let g_1 and g_2 be bounded measurable functions on $\mathbb{R}^1$, and let*

$$G_1 = \int g_1(t)\, dE_1(t)\,, \qquad G_2 = \int g_2(s)\, dE_2(s)$$

be the corresponding normal operators. Then for all $T \in \mathfrak{S}_2$ and all measurable functions f on $\mathbb{R}^2$ we have

$$\iint g_1(t)\, f(t,s)\, g_2(s)\, dE_1(t)\, T\, dE_2(s) \;=\; \iint f(t,s)\, dE_1(t)\, G_1\, T\, G_2\, dE_2(s)\,.$$

2. Let B be a selfadjoint operator and P be an orthogonal projection in H. We shall now consider the operator PBP. The operator B is allowed to be unbounded, and then we must assume that PBP is well-defined. Moreover, we shall assume that the operator PB is from the Hilbert–Schmidt class $\mathfrak{S}_2$. Denote

$$K \;=\; \bigcup_{0\leqslant t\leqslant 1} t\,\sigma(B) \;\subset\; \mathbb{R}^1\,, \tag{1.3}$$

where $\sigma(B)$ is the spectrum of B. Then the spectra of operators B and PBP lie in K.

Let $\mathfrak{S}_1 \subset \mathfrak{S}_2$ be the trace class. From the Berezin-Lieb inequality one can deduce the following result.

Theorem 1.2. *Let $PB \in \mathfrak{S}_2$. Then for any function $\psi \in W^2_\infty(K)$*

$$P\psi(B)P - P\psi(PBP)P \in \mathfrak{S}_1$$

and

$$|\ \mathrm{Tr}\Big(P\psi(B)P - P\psi(PBP)P\Big)\ | \;\leqslant\; \frac{1}{2}\,\|\psi''\|_{L_\infty(K)}\,\|PB(I-P)\|^2_{\mathfrak{S}_2}\,. \tag{1.4}$$

Proof. See Appendix.

3. Let A be a selfadjoint, semibounded from below operator in H with a domain $\mathcal{D}(A)$, and let E be its spectral measure. We denote

$$P_\lambda \;=\; E\left((-\infty,\lambda)\right)\,, \qquad P_{\mu,\lambda} \;=\; P_\lambda - P_\mu \;=\; E\left([\mu,\lambda)\right)\,, \quad \mu\leqslant\lambda\,.$$

We shall now obtain an estimate of the Hilbert–Schmidt norm appearing in the right hand side of (1.4) with $P = P_\lambda$ and $I - P = I - P_\lambda = P_{\lambda,\infty}$. Without loss of generality we shall assume that $A > 0$; this can be always achieved by adding a sufficiently large positive constant to A.

Theorem 1.3. *Let $A > 0$ and $P_\lambda BA \in \mathfrak{S}_2$. Then for all $r > 0$ we have*

$$\|P_\lambda BP_{\lambda,\infty}\|^2_{\mathfrak{S}_2} \;\leqslant\; \|P_{\lambda-r,\lambda}BP_{\lambda,\infty}\|^2_{\mathfrak{S}_2} \;+\; \|(A-\lambda)^{-1}P_{\lambda-r}[A,B]P_{\lambda,\infty}\|^2_{\mathfrak{S}_2}\,. \tag{1.5}$$

Remark 1.4. Generally speaking, the commutator $[A,B]=AB-BA$ is not always well-defined. However, since A^{-1} is bounded, $P_\lambda BA\in\mathfrak{S}_2$ implies $P_\lambda B\in\mathfrak{S}_2$. Therefore

$$P_{\lambda-r}[A,B] \;=\; (A-\lambda)P_{\lambda-r}B \;-\; P_{\lambda-r}B(A-\lambda)\;\in\;\mathfrak{S}_2\,,\qquad \forall\, r>0.$$

Proof of Theorem 1.3. Obviously,

$$\|P_\lambda BP_{\lambda,\infty}\|^2_{\mathfrak{S}_2} \;=\; \|P_{\lambda-r,\lambda}BP_{\lambda,\infty}\|^2_{\mathfrak{S}_2} \;+\; \|P_{\lambda-r}BP_{\lambda,\infty}\|^2_{\mathfrak{S}_2}\,,$$

and we only need to estimate the second term in the right hand side.

Let χ and χ_r be the characteristic functions of the intervals $(-\infty,\lambda)$ and $(-\infty,\lambda-r)$ respectively. By Proposition 1.1 (with $E_1=E_2=E$) we have

$$\begin{aligned}P_{\lambda-r}BP_{\lambda,\infty} \;&=\; \iint \chi_r(t)\,(1-\chi(s))\,dE(t)\,P_{\lambda-r}BP_{\lambda,\infty}\,dE(s)\\ =\; &\iint \chi_r(t)\,(1-\chi(s))\,(t-\lambda)(t-s)^{-1}\,dE(t)\,(A-\lambda)^{-1}P_{\lambda-r}[A,B]P_{\lambda,\infty}\,dE(s).\end{aligned}$$

Since

$$0\;\leqslant\;\chi_r(t)\,(1-\chi(s))\,(t-\lambda)(t-s)^{-1}\;\leqslant\;1\,,\qquad \forall\, t,s\in\mathbb{R}^1\,,$$

(1.2) implies that the Hilbert–Schmidt norm of this integral is estimated by the second term in the right hand side of (1.5). $\square$

Combining Theorems 1.2 and 1.3 we immediately obtain

Theorem 1.5. *Let conditions of Theorem 1.3 be fulfilled. Then for all* $\psi\in W^2_\infty(K)$ *we have*

$$\begin{aligned}&|\operatorname{Tr}\Big(P_\lambda\psi(B)P_\lambda - P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda\Big)|\\ &\qquad\leqslant\;\frac12\,\|\psi''\|_{L_\infty(K)}\Big(\|P_{\lambda-r,\lambda}B\|^2_{\mathfrak{S}_2}\;+\;\|(A-\lambda)^{-1}P_{\lambda-r}[A,B]\|^2_{\mathfrak{S}_2}\Big).\end{aligned}\tag{1.6}$$

4. Assume now that $\operatorname{rank}P_\lambda<\infty$. This obviously implies that the operator A is semibounded from below. Let

$$N(\lambda)\;=\;\operatorname{rank}P_\lambda\,,\qquad N_r(\lambda)=\sup_{\mu\leqslant\lambda}\,\left(N(\mu)-N(\mu-r)\right)\,,\quad r>0\,.$$

Theorem 1.6. *Let* $A>0$ *and* $\operatorname{rank}P_\lambda<\infty$. *Then for all* $\psi\in W^2_\infty(K)$ *and for all* $\lambda>0$, $r>0$, $\varkappa\geqslant0$ *we have*

$$\begin{aligned}&|\operatorname{Tr}\Big(P_\lambda\psi(B)P_\lambda - P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda\Big)|\\ &\quad\leqslant\;\frac12\,\|\psi''\|_{L_\infty(K)}\,N_r(\lambda)\,\Big(\,\|P_{\lambda-r,\lambda}B\|^2+\frac{\pi^2}{6}\,\frac{\lambda^{2\varkappa}}{r^2}\,\|A^{-\varkappa}P_{\lambda-r}[A,B]\|^2\Big).\end{aligned}\tag{1.7}$$

Proof. The operator $P_\lambda BA\in\mathfrak{S}_2$ is of finite rank, so we can apply Theorem 1.5. Since

$$\begin{aligned}\|P_{\lambda-r,\lambda}B\|^2_{\mathfrak{S}_2}\;&\leqslant\;\|P_{\lambda-r,\lambda}\|^2_{\mathfrak{S}_2}\,\|P_{\lambda-r,\lambda}B\|^2\\ &=\;(N(\lambda)-N(\lambda-r))\,\|P_{\lambda-r,\lambda}B\|^2\;\leqslant\;N_r(\lambda)\,\|P_{\lambda-r,\lambda}B\|^2\,,\end{aligned}$$

it is sufficient to prove that

$$\|(A-\lambda)^{-1}P_{\lambda-r}[A,B]\|_{\mathfrak{S}_2}^2 \;\leqslant\; \frac{\pi^2}{6}\,\frac{\lambda^{2\varkappa}}{r^2}\,N_r(\lambda)\,\|A^{-\varkappa}P_{\lambda-r}[A,B]\|^2\,. \tag{1.8}$$

Obviously,

$$\begin{aligned}
&\|(A-\lambda)^{-1}P_{\lambda-r}[A,B]\|_{\mathfrak{S}_2}^2\\
&\qquad = \|(A-\lambda)^{-1}P_{\lambda-r}A^{\varkappa}A^{-\varkappa}P_{\lambda-r}[A,B]\|_{\mathfrak{S}_2}^2\\
&\qquad\qquad \leqslant \|(A-\lambda)^{-1}P_{\lambda-r}A^{\varkappa}\|_{\mathfrak{S}_2}^2\,\|A^{-\varkappa}P_{\lambda-r}[A,B]\|^2\,,
\end{aligned}$$

and

$$\|(A-\lambda)^{-1}P_{\lambda-r}A^{\varkappa}\|_{\mathfrak{S}_2}^2 \;=\; \sum_{\lambda_j<\lambda-r}\frac{\lambda_j^{2\varkappa}}{(\lambda_j-\lambda)^2} \;=\; \int_0^{\lambda-r}\frac{\mu^{2\varkappa}}{(\mu-\lambda)^2}\,dN(\mu)$$

where $\lambda_j>0$, $j=1,2,\dots$, are the eigenvalues of the operator A. Let $J=[\lambda r^{-1}]$ be the integer part of λr^{-1}. Then the right hand side is estimated as follows

$$\begin{aligned}
\int_0^{\lambda-r}\frac{\mu^{2\varkappa}}{(\mu-\lambda)^2}\,dN(\mu) \;&\leqslant\; \lambda^{2\varkappa}\int_0^{\lambda-r}\frac{dN(\mu)}{(\mu-\lambda)^2} \;=\; \lambda^{2\varkappa}\sum_{j=1}^{J}\int_{\lambda-r(j+1)}^{\lambda-rj}\frac{dN(\mu)}{(\mu-\lambda)^2}\\
&\leqslant\; \lambda^{2\varkappa}N_r(\lambda)\sum_{j=1}^{J}r^{-2}j^{-2} \;\leqslant\; \frac{\lambda^{2\varkappa}}{r^2}N_r(\lambda)\sum_{j=1}^{\infty}j^{-2} \;=\; \frac{\pi^2}{6}\,\frac{\lambda^{2\varkappa}}{r^2}\,N_r(\lambda).
\end{aligned} \tag{1.9}$$

This implies (1.8). $\square$

2. Applications

In this section we apply Theorems 1.5 and 1.6 to some particular classes of operators A and B and obtain the corresponding Szegö type theorems.

In the following A will be a selfadjoint (pseudo)differential operator. We shall denote by $e(x,y,\lambda)$ the spectral function of A, i.e., the Schwartz kernel of the spectral projection P_λ. If the spectrum of A is discrete then

$$e(x,y;\lambda) \;=\; \sum_{\lambda_j<\lambda} w_j(x)\,\overline{w_j(y)}\,,$$

where $0<\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_j\ldots$ are the eigenvalues of A and $w_j\in L_2$ are the corresponding orthonormed eigenfunctions. In this case $e(x,x,\lambda)$ belongs to the corresponding space L_1 for each fixed λ.

1. Differential operators with constant coefficients.

Let A be a differential operator in $\mathbb{R}^n$ with constant coefficients,

$$A(D) \;=\; \sum_{|\alpha|\leqslant 2m} a_\alpha\,D^\alpha\,, \qquad D^\alpha \;=\; i^{-|\alpha|}\,\frac{\partial^\alpha}{\partial x^\alpha}\,,$$

and let

$$a(\xi) \;=\; \sum_{|\alpha|\leqslant 2m} a_\alpha\,\xi^\alpha\,, \qquad a_{2m}(\xi) \;=\; \sum_{|\alpha|=2m} a_\alpha\,\xi^\alpha$$

be its full and principal symbols respectively. We assume that a is real, and that $a_{2m}(\xi) > 0$ for all $\xi \neq 0$. Then A is elliptic and defines a selfadjoint operator in $L_2(\mathbb{R}^n)$. The spectral projections and spectral function of A are given by

$$P_\lambda\, u \;=\; (2\pi)^{-n}\int_{\widetilde{\mathcal{O}}_\lambda} e^{ix\xi}\,\hat{u}(\xi)\,d\xi\,, \qquad e(x,y,\lambda) \;=\; (2\pi)^{-n}\int_{\widetilde{\mathcal{O}}_\lambda} e^{i(x-y)\xi}\,d\xi\,,$$

where $\widetilde{\mathcal{O}}_\lambda = \{\xi : a(\xi) < \lambda\}$.

Theorem 2.1. *Let B be the multiplication by a real function $b \in C_0^{2m}(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$, and let $\psi \in W^2_\infty(\mathbb{R}^1)$. If $\psi(0) = 0$ then*

$$\begin{aligned} \operatorname{Tr} P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda \;&=\; \int \psi\,(b(x))\; e(x,x,\lambda)\,dx \;+\; O(\lambda^{(n-1)/2m}) \\ &=\; (2\pi)^{-n}\,\lambda^{n/2m}\int_{a_{2m}<1} d\xi \int \psi\,(b(x))\;dx \;+\; O(\lambda^{(n-1)/2m}) \end{aligned} \tag{2.1}$$

as $\lambda \to +\infty$.

Proof. We assume that $a > 0$ and, consequently, $A > 0$ (otherwise we add to A a sufficiently large positive constant).

Let $\mathcal{O}_\lambda = \{\xi : a_{2m}(\xi) < \lambda\}$. It is obvious that

$$a_{2m} \;\leqslant\; a \;+\; \text{const}\;(a^{(2m-1)/2m} + 1)\,, \qquad a \;\leqslant\; a_{2m} \;+\; \text{const}\;(a_{2m}^{(2m-1)/2m} + 1)$$

with a sufficiently large positive constant. Thus it follows that

$$\mathcal{O}_{\mu_-} \;\subset\; \widetilde{\mathcal{O}}_\lambda \;\subset\; \mathcal{O}_{\mu_+}\,, \qquad \mu_\pm = \mu_\pm(\lambda) = \lambda \pm C(\lambda^{(2m-1)/2m} + 1) \tag{2.2}$$

with some constant $C > 0$.

Denote

$$\widetilde{N}(\lambda) \;=\; \text{vol}\;\widetilde{\mathcal{O}}_\lambda\,, \qquad \widetilde{N}_r(\lambda) = \sup_{\mu\leqslant\lambda}\,[\widetilde{N}(\mu) - \widetilde{N}(\mu - r)]\,.$$

Since $\text{vol}\,\mathcal{O}_\lambda = \int_{a_{2m}<\lambda} d\xi = \lambda^{n/2m}\int_{a_{2m}<1} d\xi$, (2.2) implies

$$\widetilde{N}(\lambda) \;=\; \lambda^{n/2m}\int_{a_{2m}<1} d\xi \;+\; O(\lambda^{(n-1)/2m})\,, \qquad \lambda \to +\infty\,. \tag{2.3}$$

The commutator $[A,B]$ is a differential operator of order $2m-1$ which can be written in the form

$$[A,B]\,u(x) \;=\; \sum_{|\alpha|\leqslant 2m-1} D^\alpha\,(c_\alpha(x)u(x))$$

with $c_\alpha \in C_0^{|\alpha|}(\mathbb{R}^n)$. Then

$$P_\lambda BA\,u(x) \;=\; P_\lambda(AB - [A,B])\,u(x) \;=\; \int \mathcal{K}(x,y)\,u(y)\,dy$$

where

$$\mathcal{K}(x,y) \;=\; (2\pi)^{-n} \int_{\widetilde{\mathcal{O}}_\lambda} e^{i(x-y)\xi} \Big(a(\xi)\,b(y) \;-\; \sum_{|\alpha| \leqslant 2m-1} \xi^\alpha\, c_\alpha(y) \Big)\, d\xi\,.$$

Since $\mathcal{K} \in L_2(\mathbb{R}^n \times \mathbb{R}^n)$, we obtain $P_\lambda BA \in \mathfrak{S}_2$ and can therefore apply Theorem 1.5. We shall take $r = \lambda^{(2m-1)/2m}$, then by (2.3)

$$\widetilde{N}_r(\lambda) \;=\; O(\lambda^{(n-1)/2m})\,, \qquad \lambda \to +\infty\,.$$

Now we have

$$\begin{aligned} \|P_{\lambda-r,\lambda} B\|^2_{\mathfrak{S}_2} \;&=\; \int_{\widetilde{\mathcal{O}}_\lambda \setminus \widetilde{\mathcal{O}}_{\lambda-r}} d\xi \int b^2(x)\,dx \\ &\leqslant\; \widetilde{N}_r(\lambda) \int b^2(x)\,dx \;=\; O(\lambda^{(n-1)/2m})\,, \end{aligned}$$

$$\begin{aligned} &\|(A-\lambda)^{-1} P_{\lambda-r}[A,B]\|^2_{\mathfrak{S}_2} \\ &\quad=\; \int\int_{\widetilde{\mathcal{O}}_{\lambda-r}} \Big(\sum_{|\alpha| \leqslant 2m-1} c_\alpha(x)\,\xi^\alpha \Big)^2 \,(a(\xi)-\lambda)^{-2}\; d\xi\,dx \\ &\leqslant\; \text{const} \int_{\widetilde{\mathcal{O}}_{\lambda-r}} a^{(2m-1)/m}\,(a-\lambda)^{-2}\; d\xi \;=\; \text{const} \int_{\widetilde{\mathcal{O}}_{\lambda-r}} \frac{\mu^{(2m-1)/m}}{(\mu-\lambda)^2}\, d\widetilde{N}(\mu)\,. \end{aligned}$$

Applying (1.9) with $\varkappa = (2m-1)/2m$ we obtain

$$\|(A-\lambda)^{-1} P_{\lambda-r}[A,B]\|^2_{\mathfrak{S}_2} \;=\; O(\lambda^{(n-1)/2m})\,.$$

Now from (1.6) it follows that

$$\operatorname{Tr}\Big(P_\lambda \psi(B) P_\lambda - P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda \Big) \;=\; O(\lambda^{(n-1)/2m})\,.$$

If $\psi(0) = 0$ then $P_\lambda \psi(B) P_\lambda \in \mathfrak{S}_1$, and by (2.3)

$$\begin{aligned} \operatorname{Tr} P_\lambda \psi(B) P_\lambda \;&=\; \int \psi\,(b(x))\; e(x,x,\lambda)\,dx \;=\; (2\pi)^{-n} \int \psi\,(b(x))\;dx \int_{a<\lambda} d\xi \\ &=\; (2\pi)^{-n}\, \lambda^{n/2m} \int_{a_{2m}<1} d\xi \int \psi\,(b(x))\;dx \;+\; O(\lambda^{(n-1)/2m})\,. \end{aligned}$$

This implies (2.1). $\square$

2. Operators on closed manifolds.

Let M be a smooth compact manifold without boundary, $\dim M = n$. Let A and B be selfadjoint pseudodifferential operators (PDOs) acting in the space $L_2(M)$ of half-densities on M. We assume that A is a positive elliptic PDO of order $m \geqslant 1$ and that B is a PDO of order 0.

Theorem 2.2. *For any function $\psi \in C^\infty(\mathbb{R}^1)$ we have*

$$\operatorname{Tr} P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda = (2\pi)^{-n}\, \lambda^{n/m} \int_{a(x,\xi)<1} \psi(b(x,\xi))\, dx\, d\xi \; + \; O(\lambda^{(n-1)/m}), \tag{2.4}$$

where $a(x,\xi)$ and $b(x,\xi)$ are the principal symbols of the PDOs A and B respectively, and $dx\,d\xi$ is the invariant measure on the cotangent bundle T^*M.

This theorem has been announced in [LS1]. Its proof is based on the following well-known result (see, for example, [S]).

Lemma 2.3. *For an arbitrary PDO Q of order zero*

$$\begin{aligned}\operatorname{Tr} P_\lambda Q \; &= \; \int_M \left(Q(x,D_x)\, e(x,y,\lambda)\right)\big|_{y=x}\, dx \\ &= \; (2\pi)^{-n}\, \lambda^{n/m} \int_{a(x,\xi)<1} q(x,\xi)\, dx\, d\xi \; + \; O(\lambda^{(n-1)/m}),\end{aligned} \tag{2.5}$$

where q is the principal symbol of the PDO Q.

Proof of Theorem 2.2. The spectrum of the operator A consists of isolated positive eigenvalues tending to $+\infty$. Therefore for each fixed λ the rank of the projection P_λ is finite.

For all $s \in \mathbb{R}^1$ the operator $P_\lambda A^s$ is bounded and $\|P_\lambda A^s\| \leqslant \lambda^s$. The operator $A^{-s}BA$ is a PDO of order $(1-s)m$. If s is sufficiently large then this operator is from $\mathfrak{S}_2$. Therefore $P_\lambda BA = (P_\lambda A^s)(A^{-s}BA) \in \mathfrak{S}_2$. Thus, the operators A and B satisfy the conditions of Theorem 1.6.

We apply the estimate (1.7) with $\varkappa = (m-1)/m$ and $r = \lambda^\varkappa$. From (2.5) with $Q = I$ it follows that $N_{\lambda^\varkappa}(\lambda) = O(\lambda^{(n-1)/m})$. The operators B and $A^{-\varkappa}[A,B]$ are PDOs of order zero, so they are bounded in $L_2(M)$. Now (1.7) yields

$$\operatorname{Tr}\Big(P_\lambda \psi(B) P_\lambda - P_\lambda \psi(P_\lambda B P_\lambda) P_\lambda\Big) \; = \; O(\lambda^{(n-1)/m}). \tag{2.6}$$

The operator $\psi(B)$ is a PDO of order zero with the principal symbol $\psi(b)$ (see, for example, [T, Section 12.1]). Therefore (2.4) follows from (2.5) with $Q = \psi(B)$ and (2.6). $\square$

Remark 2.4. Theorem 2.2 remains valid if we only assume that $a(x,\xi) > 0$ as $\xi \neq 0$. Then the operator A is semibounded from below, and we can consider the positive operator $A + \lambda_0 I$ instead of A.

Remark 2.5. We have proved Theorem 2.2 assuming that $\psi \in C^\infty(\mathbb{R}^1)$, whereas Theorem 1.6 is valid for all $\psi \in W^2_\infty(K)$. This stronger condition is needed in order for $\psi(B)$ to be a PDO. If B is the operator of multiplication by a smooth function then Theorem 2.2 remains valid for all $\psi \in W^2_\infty(K)$.

3. Boundary value problems for second order differential operators.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $C^\infty(\overline{\Omega})$ be the space of functions from $C^\infty(\mathbb{R}^n)$ restricted to Ω, and $\rho \in L_1(\Omega)$ be a non-negative density. Then

$$C_0^\infty(\Omega) \;\subset\; C^\infty(\overline{\Omega}) \;\subset\; L_2(\Omega, \rho dx)$$

and $C_0^\infty(\Omega)$ is dense in $L_2(\Omega, \rho dx)$.

In this subsection we deal with the differential operator

$$A(x, \partial_x) \;=\; -\sum_{i,j} a_{ij}(x)\,\partial_{x_i}\partial_{x_j} \;-\; \sum_i a_i(x)\,\partial_{x_i} \;+\; a_0(x)$$

in Ω, where a_{ij} , a_j and a_0 are real functions from $L_2(\Omega, \rho dx)$ such that

(1) $\{a_{ij}\}$ is a real symmetric non-negative $n \times n$-matrix-function;
(2) $\sum_j \partial_{x_j}(a_{ij}\,\rho) \;=\; a_i\,\rho\,, \qquad i = 1, \ldots, n\,;$
(3) $a_0 \;\geqslant\; \text{const} \;>\; 0\,.$

Obviously, $A : C^\infty(\overline{\Omega}) \to L_2(\Omega, \rho dx)$ and

$$(Au, v)_{L_2(\Omega,\rho dx)} = \int_\Omega \sum_{i,j} a_{ij}\,u_{x_i}\,\bar v_{x_j}\,\rho\,dx \;+\; \int_\Omega a_0\,u\,\bar v\,\rho\,dx \tag{2.7}$$

for all $u, v \in C_0^\infty(\Omega)$.

We fix a linear subspace $\mathcal{D}_0$ such that $C_0^\infty(\Omega) \subset \mathcal{D}_0 \subset C^\infty(\overline{\Omega})$, and denote by $[u, v]_a$ the sesquilinear form (2.7) extended to $\mathcal{D}_0$. Let $\mathcal{D}_a$ be the completion of $\mathcal{D}_0$ with respect to $[\cdot,\cdot]_a$, and let A be the corresponding Friedrichs extension of the differential operator $A(x, \partial_x)$. Then A is a positive operator in $L_2(\Omega, \rho dx)$ and

$$(A^{1/2}u, A^{1/2}v)_{L_2(\Omega,\rho dx)} \;=\; [u, v]_a\,, \qquad \forall\, u, v \in \mathcal{D}_a = \mathcal{D}(A^{1/2})\,.$$

The domain $\mathcal{D}(A)$ is a dense subset of the Hilbert space $\mathcal{D}_a$ with the inner product $[\cdot,\cdot]_a$. By definition $\mathcal{D}(A)$ consists of $u \in L_2(\Omega, \rho dx)$ such that

$$[u, v]_a \;=\; (f, v)_{L_2(\Omega,\rho dx)}\,, \qquad \forall\, v \in \mathcal{D}_a\,, \tag{2.8}$$

for some $f \in L_2(\Omega, \rho dx)$, and then $Au = f$. In particular, integrating by parts we obtain $C_0^\infty(\Omega) \in \mathcal{D}(A)$.

Let $b \in L_\infty(\Omega)$ be a real function such that $\partial_x^\alpha b \in L_{1,\mathrm{loc}}(\Omega)$ for all $|\alpha| \leqslant 2$, and B be the operator of multiplication by b in $L_2(\Omega, \rho dx)$. Obviously, B is bounded and selfadjoint. We assume that

$$a_0^{-1}\,|\sum_{i,j} a_{ij}\,b_{x_i x_j} + \sum_i a_i\,b_{x_i}\,|^2 \;\in\; L_\infty(\Omega, \rho dx) \tag{2.9}$$

and

$$\sum_{i,j} a_{ij}\,b_{x_i}\,b_{x_j} \;\in\; L_\infty(\Omega, \rho dx)\,. \tag{2.10}$$

Then $A(x, \partial_x)(bu) \in L_2(\Omega, \rho dx)$ and, consequently, $bu \in \mathcal{D}(A)$ for all $u \in C_0^\infty(\Omega)$. Moreover,

$$[A, B]\,u \;=\; -2\sum_{i,j} a_{ij}\,b_{x_i}\,u_{x_j} \;-\; \sum_{i,j} a_{ij}\,b_{x_i x_j}\,u \;-\; \sum_i a_i\,b_{x_i}\,u \tag{2.11}$$

for all $u \in C_0^\infty(\Omega)$, and

$$([A, B]u, v)_{L_2(\Omega,\rho dx)} = -(u, [A, B]v)_{L_2(\Omega,\rho dx)}\,, \qquad \forall\, u, v \in C_0^\infty(\Omega)\,. \tag{2.12}$$

Clearly, the operator $[A, B]$ can be extended to $\mathcal{D}_0$, and then (2.11) and (2.12) remain valid for all $u \in \mathcal{D}_0$ and $v \in C_0^\infty(\Omega)$.

Proposition 2.6. *Under the conditions (2.9) and (2.10) the operator $A^{-1/2}[A,B]$ is bounded in $L_2(\Omega,\rho dx)$.*

Proof. The inclusions (2.9) and (2.10) imply

$$\|[A,B]u\|^2_{L_2(\Omega,\rho dx)} \leqslant \text{ const } [u,u]_a = \text{ const } \|A^{1/2}u\|^2_{L_2(\Omega,\rho dx)}, \qquad \forall u\in\mathcal{D}_0 .$$

Therefore for all $u\in A^{1/2}\mathcal{D}_0$ and $v\in C_0^\infty(\Omega)$ we have

$$\begin{aligned} |\,(u, A^{-1/2}[A,B]v)_{L_2(\Omega,\rho dx)}| &= |\,([A,B]A^{-1/2}u, v)_{L_2(\Omega,\rho dx)}| \\ &\leqslant \text{ const } \|u\|_{L_2(\Omega,\rho dx)}\|v\|_{L_2(\Omega,\rho dx)} . \end{aligned} \tag{2.13}$$

Since $\mathcal{D}_0$ is dense in $\mathcal{D}_a$ with respect to $[\cdot,\cdot]_a$, the set $A^{1/2}\mathcal{D}_0$ is dense in $L_2(\Omega,\rho dx)$. Thus, the estimate (2.13) holds uniformly on a dense set in $L_2(\Omega,\rho dx)\times L_2(\Omega,\rho dx)$, which implies that $A^{-1/2}[A,B]$ is bounded. □

Let us now assume that the spectrum of A is discrete and that

$$N(\lambda+\lambda^{1/2}) - N(\lambda) = O(\lambda^\theta), \qquad \lambda\to+\infty, \tag{2.14}$$

or

$$N(\lambda+\lambda^{1/2}) - N(\lambda) = o(\lambda^\theta), \qquad \lambda\to+\infty. \tag{2.15}$$

The estimates (2.14) and (2.15) follow from the asymptotic formulae

$$N(\lambda) = c_0\,\lambda^p + O(\lambda^\theta), \qquad \lambda\to+\infty,$$

and

$$N(\lambda) = c_0\,\lambda^p + o(\lambda^\theta), \qquad \lambda\to+\infty,$$

respectively. Here c_0 is some constant and $\theta\leqslant p\leqslant\theta+1/2$. The asymptotic formulae of these types have been obtained under various additional assumptions on the operator A (see, for example, [BS3], [RShS]). For non-degenerate problems c_0 is the standard Weyl coefficient and $p=n/2$, and in the "regular" case $\theta=(n-1)/2$ (see [Iv2], [V]).

Theorem 2.7. *For all $\psi\in W^2_\infty(K)$*

$$\operatorname{Tr} P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda = \int_\Omega \psi\,(b(x))\; e(x,x,\lambda)\,\rho(x)\,dx + \begin{cases} O(\lambda^\theta), & \text{if } (2.14), \\ o(\lambda^\theta), & \text{if } (2.15). \end{cases}$$

Proof. If $q\in L_\infty(\Omega)$ and Q is the corresponding multiplication operator then

$$\operatorname{Tr} P_\lambda QP_\lambda = \int_\Omega q(x)\,e(x,x,\lambda)\,\rho(x)\,dx . \tag{2.16}$$

Therefore the theorem immediately follows from (1.7) with $\varkappa=1/2$ and $r=\lambda^{1/2}$ and Proposition 2.6. □

Thus, the study of $\operatorname{Tr} P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda$ is reduced to the computation of asymptotics of (2.16) with $q=\psi(b)$. Clearly, the latter is a much simpler object. However, it has not been studied so actively as the counting function

$$N(\lambda) = \int_\Omega e(x,x,\lambda)\,\rho(x)\,dx ,$$

and we are unaware of any general results concerning the asymptotic behavior of (2.16) for non-regular problems.

4. Boundary value problems — the regular case.

If $\rho \equiv 1$, A is uniformly elliptic, and $\partial\Omega$, b and the coefficients of A are infinitely smooth, then

$$\int_\Omega \psi\,(b(x))\; e(x,x,\lambda)\,dx \;=\; (2\pi)^{-n}\lambda^{n/2}\iint_{a(x,\xi)\leqslant 1}\psi\,(b(x))\;dx\,d\xi \;+\; O(\lambda^{(n-1)/2})\,,$$

where $a(x,\xi) = \sum_{i,j} a_{ij}(x)\,\xi_i\,\xi_j$ is the principal symbol of A (see, for example, [Iv1]). Therefore by Theorem 2.7

$$\begin{aligned}&\operatorname{Tr} P_\lambda\psi(P_\lambda BP_\lambda)P_\lambda\\ &\qquad\qquad = \;(2\pi)^{-n}\lambda^{n/2}\iint_{a(x,\xi)\leqslant 1}\psi\,(b(x))\;dx\,d\xi \;+\; O(\lambda^{(n-1)/2})\,.\end{aligned}\tag{2.17}$$

Remark 2.8. Using the results from [Iv2], [V] one can easily extend (2.17) to domains (or smooth manifolds) with a piecewise smooth boundary.

Appendix

In the early seventies F. Berezin [B] and E. Lieb [L] (see also [Si]) independently obtained a Jensen type inequality for convex functions of selfadjoint operators. It has been generalized in [LS2], where we have extended this inequality to wider classes of functions and operators. For the sake of completeness we give here the version of the Berezin–Lieb inequality which is needed in this paper, and then prove Theorem 1.2.

Let B be a selfadjoint operator, P be an orthogonal projection in the Hilbert space H and K be the set defined by (1.3).

Theorem A.1 (Berezin–Lieb inequality). *Let φ be a convex function on K. Assume that PB is a compact operator and that*

$$P\varphi(B)P \;-\; P\varphi(PBP)P \;\in\; \mathfrak{S}_1.$$

Then

$$\operatorname{Tr}\Big(P\varphi(B)P \;-\; P\varphi(PBP)P\Big) \;\geqslant\; 0. \tag{A.1}$$

Proof. Let $\{\xi_k\}$ be an orthonormal basis in PH formed by the eigenfunctions ξ_k of the compact selfadjoint operator PBP. We denote by E_B the spectral measure of the operator B. If ν_k are the positive measures with $d\nu_k(t) = (dE_B(t)\xi_k,\xi_k)$ then

$$\begin{aligned}(P\varphi(PBP)P\xi_k,\xi_k) \;&=\; (\varphi(PBP)\xi_k,\xi_k) \;=\; \varphi\big((PBP\xi_k,\xi_k)\big)\\ &=\; \varphi\big((B\xi_k,\xi_k)\big) \;=\; \varphi\Big(\int t(dE_B(t)\xi_k,\xi_k)\Big) \;=\; \varphi\Big(\int t\,d\nu_k(t)\Big).\end{aligned}\tag{A.2}$$

Clearly, $\nu_k(\mathbb{R}^1) = 1$. By applying the Jensen inequality for convex functions, we obtain from (A.2)

$$\begin{aligned}(P\varphi(B)P\xi_k,\xi_k) \;-\; (P\varphi(PBP)P\xi_k,\xi_k) \;&=\; (\varphi(B)\xi_k,\xi_k) \;-\; (\varphi(PBP)\xi_k,\xi_k)\\ &=\; \int\varphi(t)\,d\nu_k(t) \;-\; \varphi\Big(\int t\,d\nu_k(t)\Big) \;\geqslant\; 0.\end{aligned}$$

This implies (A.1). $\square$

Proof of Theorem 1.2. Since the operator $P\psi(B)P - P\psi(PBP)P$ does not change when we add a linear function to ψ, we can assume without loss of generality that

$$|\psi(t)| \leqslant \frac{t^2}{2}\,\|\psi''\|_{L_\infty(K)}. \tag{A.3}$$

Let $\varphi(t) = t^2/2$. In view of (A.3) we have

$$|\,(P\psi(B)P\zeta_k,\zeta_k)\,| \leqslant \|\psi''\|_{L_\infty(K)}\,(P\varphi(B)P\zeta_k,\zeta_k), \tag{A.4}$$

$$|\,(P\psi(PBP)P\zeta_k,\zeta_k)\,| \leqslant \|\psi''\|_{L_\infty(K)}\,(P\varphi(PBP)P\zeta_k,\zeta_k), \tag{A.5}$$

for any orthonormal basis $\{\zeta_k\}$ in H. Since $PB \in \mathfrak{S}_2$, we also have

$$P\varphi(B)P = \frac{1}{2}\,PB^2P \in \mathfrak{S}_1 \quad\text{and}\quad P\varphi(PBP)P = \frac{1}{2}\,PBPBP \in \mathfrak{S}_1. \tag{A.6}$$

From (A.4), (A.5) and (A.6) it follows that

$$P\psi(B)P \in \mathfrak{S}_1 \quad\text{and}\quad P\psi(PBP)P \in \mathfrak{S}_1$$

(see, for example, [RS, Ch. VI, Problem 26]). Now applying Theorem A.1 to convex functions

$$\varphi_+(t) = \frac{t^2}{2}\,\|\psi''\|_{L_\infty(K)} + \psi(t) \quad\text{and}\quad \varphi_-(t) = \frac{t^2}{2}\,\|\psi''\|_{L_\infty(K)} - \psi(t)$$

and taking into account the equality

$$\operatorname{Tr}\Big(P\varphi(B)P - P\varphi(PBP)P\Big) = \frac{1}{2}\,\|PB(I-P)\|^2_{\mathfrak{S}_2},$$

we obtain (1.4). $\square$

References

[B] F. Berezin, *Convex functions of operators.*, Mat.sb. **88** (1972), 268–276. (Russian)

[BS1] M. Birman and M. Solomyak, *Stieltjes double operator integrals.* I, Problems in Math. Physics. **1** (1966), 33–67; II, Problems in Math. Physics. **2** (1967); III, Problems in Math. Physics. **6** (1973). (Russian)

[BS2] ______, *Spectral theory of self-adjoint operators in Hilbert space*, Leningrad Univ., Leningrad, 1980 (Russian); English translation D. Reidel Publ. Co., 1987.

[BS3] ______, *Asymptotic properties of the spectrum of differential equations*, Itogi Nauki i Tekhniki. Mat. analiz., vol. 14, VINITI, Moscow, 1977 (Russian); English translation in J. Sov. Math **12** (1979), 247–283.

[G] V. Guillemin, *Some classical theorems in spectral theory revisited*, seminar on sing. of sol. of diff. equations, Princeton University Press, Princeton, 1979, pp. 219–259.

[H] L. Hörmander, *The analysis of linear partial differential operators.* IV, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

[Iv1] V. Ivrii, *Precise spectral asymptotics for elliptic operators*, Lect. Notes Math., vol. 1100, Springer Verlag, 1984.

[Iv2] ______, *Weyl's asymptotic formula for the Laplace-Beltrami operator in Riemann polyhedra and in domains with conical singularities of the boundary*, Dokl. AN SSSR **288** (1986), 35-38; English transl. in Soviet Math. Dokl. **38** (1986).

[J] K. Johansson, *On Szegö asymptotic formula for Toeplitz determinants and generalizations*, Bull. Sc. math. **112** (1988), 257–304.

[L] E.H. Lieb, *The classical limit of quantum spin systems*, Comm.Math.Phys. **31** (1973), 327–340.

[LS1] A. Laptev and Yu. Safarov, *Error estimate in the generalized Szegö theorem*, Equations aux derivees parielles, Saint-Jean-De-Monts (1991), XV-1— XV-7.

[LS2] ———, *A generalization of the Berezin-Lieb inequality*, Report No.15, Institut Mittag-Leffler, 1993.

[Ok] K. Okikiolu, *The analogue of the strong Szegö limit theorem on the torus and the 3-sphere*, PhD Diss.,, Dept. Math. UCLA, Los Angeles, California, 1991.

[R] D. Robert, *Remarks on the paper of S. Zelditch: "Szegö limit theorems in quantum mechanics"*, J. Funct. Anal. **53** (1983), 304–308.

[RS] M. Reed and B. Simon, *Methods of modern mathematical physics*, Academic Press, New York London, 1972.

[RShS] G. Rozenblum, M. Shubin and M. Solomyak, *Spectral theory of differential operators*, Modern problems in Mathematics, Fundamental directions, Vol.64, VINITI, Moscow, 1989. (Russian)

[S] Yu. Safarov, *Asymptotics of the spectral function of a positive elliptic operator without non-trapping condition*, Funk. Analis i Ego Pril. **22:3** (1988), 53–65 (Russian); English transl. in Functional Anal. Appl. **22** (1988).

[Si] B. Simon, *Trace ideals and their applications*, Cambridge University Press, London New York Melbourne, 1979.

[Sz] G. Szegö, *Beiträge zur theorie der Toeplizschen formen*, Math. Z. **6** (1920), 167–202.

[T] M. Taylor, *Pseudodifferential operators*, Princeton University Press, Princeton, 1981.

[V] D. Vassiliev, *Two-term asymptotics of the spectrum of a boundary value problem in the case of a piecewise smooth boundary*, Dokl. AN SSSR **286** (1986), 1043–1046; English transl. in Soviet Math. Dokl. **33** (1986), 227–230.

[Wi1] H. Widom, *Asymptotic expansions for pseudodifferential operators in bounded domains*, Lect. Notes in Math., vol. 1152, Springer Verlag, 1985.

[Wi2] H. Widom, *On a class of integral operators on a half-space with discontinuous symbol*, J. Funct. Anal. **88** (1990), 166–193.

[Ya] D. Yafaev, *Mathematical scattering theory : general theory*, American Mathematical Society, Providence, R.I., 1992.

[Z] S. Zelditch, *Szegö limit theorems in quantum mechanics*, J. Funct. Anal. **50** (1983), 67–80.

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