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Inverse Spectral Problems for Schrödinger Operators with Energy Depending Potentials

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Abstract. We study an inverse problem for a class of Schrödinger operators with energy depending potentials. In particular, we show that introduction of the discrete spectrum generically does not lead to singularities of the corresponding soliton solutions. In our last chapter we derive some new trace formulas which could be considered as generalization of a standard trace formulas for Schrödinger operators.

1. Introduction

In this paper we consider the inverse problem for the operator

$$-\psi''_{xx}(x,k) + \big(2ku(x) + v(x)\big)\psi(x,k) = k^2\psi(x,k), \quad x \in \mathbb{R}. \tag{1.1}$$

In what follows we assume that the potential functions u and v are real-valued, smooth and exponentially decay at infinity together with all their derivatives

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}x^j} u(x)\right|, \quad \left|\frac{\mathrm{d}^j}{\mathrm{d}x^j} u(x)\right| \le C_j \exp(-\varepsilon|x|) \tag{1.2}$$

for $j \in \mathbb{N}$ and some $\varepsilon > 0$. In (1.1) k is a spectral parameter and if for some $k \in \mathbb{C}$ there is a L^2 solution of this equation, then we say that k is an eigenvalue.

Such operators are sometimes called Schrödinger operators with energy depending potentials and the scattering problem for them has been considered in the papers by Jaulent [3], Jaulent and Jean [4], Kaup [5] and also by Sattinger and Szmigielski [6, 7]. In [3, 4, 6] the authors used the Gel′fand – Levitan – Marchenko approach for the inverse problem on the line given in [2].

In fact, the inverse problem has been completely solved in [6] for the regular case (when the eigenvalues are absent) by using a so-called "vanishing lemma." If the data of the inverse problem have bound state, then the situation with solvability

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This is the final form of the paper.

becomes significantly more complicated. The main problem is that method for constructing the solution at the first step automatically gives a number of singularities. As it has been found in [6], for one-soliton solution these singularities generically cancel in the final formulas and for a two-soliton or more solutions problem, the cancellation of singularities has been stated as a conjecture.

The main aim of this paper is to clarify the situation where there is a finite number of eigenvalues. As it has been done in [7] we will use the Riemann – Hilbert as a method of solving inverse problem. The matrix solution of this problem has singularities in x if the data of the inverse problem contain the discrete spectrum. However we prove (see Theorem 4.1) that these singularities generically cancel at the second step of our construction.

In Section 4 we consider a class of reflectionless potentials and prove that the solutions u and v are generically smooth functions. We also consider a so-called “dressing up” transform (addition of one eigenvalue to the data of the inverse problem) and also prove that such a transform generically does not lead to singularities of the potential functions u and v.

In the end of the paper we derive some new trace formulas involving the potential functions u and v. If $u \equiv 0$ then these formulas coincide with the corresponding trace formulas for Schrödinger operators.

2. Properties of scattering data

In this section we recall some standard fact, cf. [7]. We begin with defining the Jost solution of the equation (1.1). Let us consider its two solutions satisfying the properties

$$f(x,k) = \mathrm{e}^{\mathrm{i}kx}\big(1 + o(1)\big), \qquad x \to +\infty, \tag{2.1}$$

$$g(x,k) = \mathrm{e}^{-\mathrm{i}kx}\big(1 + o(1)\big), \quad x \to -\infty. \tag{2.2}$$

We also introduce

$$\alpha(x) = \exp\left\{\mathrm{i}\int_x^\infty u(s)\,\mathrm{d}s\right\}, \quad \alpha_0 = \lim_{x\to-\infty}\alpha(x). \tag{2.3}$$

Since u is a real function $\alpha_0^{-1} = \overline{\alpha_0}$.

For a real-valued k we have two pairs of linear independent solutions:

$$\{f(x,k), \overline{f(x,k)}\}, \quad \{g(x,k), \overline{g(x,k)}\}.$$

In particular, the functions f and $\bar{f}$ could be written as linear combinations of g and $\bar{g}$

$$f(x,k) = a(k)\overline{g(x,k)} + b(k)g(x,k), \tag{2.4}$$

$$\overline{f(x,k)} = \overline{a(k)}g(x,k) + \overline{b(k)g(x,k)}. \tag{2.5}$$

Let us define the Wronskian

$$W[\varphi,\psi](x,k) = \varphi'_x(x,k)\psi(x,k) - \varphi(x,k)\psi'_x(x,k). \tag{2.6}$$

Proposition 2.1. *The following fundamental identity holds true*

$$|a(k)|^2 - |b(k)|^2 = 1. \tag{2.7}$$

Proof. The Wronskian of the functions f and $\bar{f}$ is constant with respect to x. Comparing the asymptotic behavior of $W[f,\bar{f}]$ at $+\infty$ and $-\infty$ we complete the proof. □

In follows from (1.1) that the function f satisfies the following Volterra equation

$$f(x,k) = \mathrm{e}^{\mathrm{i}kx} - \int_x^\infty \frac{\sin k(x-y)}{k}\big(ku(x)+v(x)\big)f(y,k)\,\mathrm{d}y. \tag{2.8}$$

Therefore f can be written as a convergent series obtained by standard iterations and this immediately implies that f is an analytic function with respect to $k \in \mathbb{C}_+$ in the upper half plane and can be continuously extended on the real line. In particular, this also implies the analyticity of the scattering coefficients $a(k)$ in $\mathbb{C}_+$ and therefore $\overline{a(\bar{k})}$ in $\mathbb{C}_-$. Similarly we find that the solution g is an analytic function in $\mathbb{C}_+$.

It is well known that if $u \equiv 0$ then $f(x,k) \to \exp(\mathrm{i}kx)$ as $k \to \infty$ ($\operatorname{Im} k > 0$). If $u \not\equiv 0$ then we obtain the following.

Proposition 2.2. *For* $\operatorname{Im} k > 0$ *functions* $f(x,k)$ *and* $g(x,k)$ *satisfy following asymptotic relations*

$$f(x,k) = \mathrm{e}^{\mathrm{i}kx} \exp\Big\{\mathrm{i}\int_x^\infty u(s)\,\mathrm{d}s\Big\}\big(1+o(1)\big), \qquad k \to \infty; \tag{2.9}$$

$$g(x,k) = \mathrm{e}^{-\mathrm{i}kx} \exp\Big\{\mathrm{i}\int_{-\infty}^x u(s)\,\mathrm{d}s\Big\}\big(1+o(1)\big), \qquad k \to \infty. \tag{2.10}$$

Similarly we find asymptotic properties of the functions $a(k)$ and $b(k)$.

Proposition 2.3. *The scattering coefficients* a *and* b *satisfy the asymptotic behavior as* $k \to \infty$

$$\lim_{k\to\infty} a(k) = \alpha_0^{-1}, \quad \operatorname{Im} k \geq 0,$$
$$\lim_{k\to\infty} b(k) = 0, \quad \operatorname{Im} k = 0.$$

From the asymptotical behavior of f at $\pm\infty$ we immediately find

Proposition 2.4. *Zeros of the scattering coefficient* a *in* $\mathbb{C}_+$ *coincide with the eigenvalues of the operator* (1.1).

Remark 2.5. Due to the assumption (1.2) the scattering coefficient $a(k)$ is also an analytical function for $\operatorname{Im} k > -\varepsilon$, $k \neq 0$, where $\varepsilon > 0$ is introduced in (1.2). Point $k = 0$ can be a pole for function $a(k)$ or it is regular point. Therefore it can only have a finite number of zeros in $\mathbb{C}_+$.

3. Riemann – Hilbert problem

Here we describe the inverse problem by using Riemann – Hilbert approach, see also [7]. Let us assume that the scattering coefficient $a(k)$ has N simple zeros $\{\varkappa_j\}_{j=1}^N$, $\varkappa_j \in \mathbb{C}_+$. Denote by $\Psi(x,k) = \binom{\psi_1(x,k)}{\psi_2(x,k)}$ the vector defined by

$$\Psi(x,k) = \begin{pmatrix} f(x,k) \\ g(x,k)/a(k) \end{pmatrix}, \quad \text{for } k \in \mathbb{C}_+,$$
$$\Psi(x,k) = \begin{pmatrix} \overline{g(x,\bar{k})}/\overline{a(\bar{k})} \\ \overline{f(x,\bar{k})} \end{pmatrix}, \quad \text{for } k \in \mathbb{C}_-.$$

By using the analytical properties of the functions f and g and also the properties of the scattering coefficients a and b described in the previous section, we find that the vector Ψ is meromorphic in $\mathbb{C}_\pm$ and can be continuously extended to the boundary.

Let $\Psi^+(x,k)$ and $\Psi^-(x,k)$ be the limits of the function $\Psi(x,k)$ as k approaches the real axis from upper and lower complex half plane respectively. One finds that the this vector satisfies following relation

$$\Psi^+(x,k) = G(k)\Psi^-(x,k), \tag{3.1}$$

$$G(k) = \begin{pmatrix} 1 & -\overline{r(k)} \\ r(k) & 1-|r(k)|^2 \end{pmatrix}, \tag{3.2}$$

where

$$r(k) = -\frac{\overline{b(k)}}{a(k)}, \quad k \in \mathbb{R}. \tag{3.3}$$

It follows from Proposition 2.1 that if $k \in \mathbb{R}\setminus\{0\}$, then $|r(k)| < 1$. Moreover, if we denote by $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then using (2.9) and (2.10) we find

$$\Psi(x,k) = \exp\{\mathrm{i}\sigma_3 kx\} \begin{pmatrix} \alpha(x) \\ \big(\alpha(x)\big)^{-1} \end{pmatrix} \big(1+o(1)\big), \quad \text{as } k \to \infty. \tag{3.4}$$

For the discrete spectrum we obtain

$$\operatorname{res}_{k=\varkappa_j} \psi_2(x,k) = C_j\psi_1(x,\varkappa_j), \quad \operatorname{res}_{k=\overline{\varkappa_j}} \psi_1(x,k) = \overline{C_j}\psi_2(x,\overline{\varkappa_j}), \\ C_j \neq 0, j = 1,\dots,N. \tag{3.5}$$

Therefore meromorthic vector function $\Psi(x,k)$ is the solution of the Riemann–Hilbert problem: (3.1), (3.4), (3.5). In order to exclude the direct appearance of the function $\alpha(x)$ from it we consider its matrix version which is slightly different. Namely, let us introduce a meromorthic in $\mathbb{C}_+$ and $\mathbb{C}_-$ matrix

$$\Xi = \begin{pmatrix} \xi_{11}(x,k) & \xi_{12}(x,k) \\ \xi_{21}(x,k) & \xi_{22}(x,k) \end{pmatrix}, \tag{3.6}$$

such that

$$\Xi^+ = G(k)\Xi^-, \quad k \in \mathbb{R}, \tag{3.7}$$

$$\Xi = \exp\{\mathrm{i}\sigma_3 kx\} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (1+o(1)), \quad k \to \infty.$$

$$\begin{aligned} &\operatorname{res}_{k=\varkappa_j} \xi_{21}(x,k) = C_j\xi_{11}(x,\varkappa_j), && \operatorname{res}_{k=\varkappa_j} \xi_{22}(x,k) = C_j\xi_{12}(x,\varkappa_j), \\ &\operatorname{res}_{k=\overline{\varkappa_j}} \xi_{11}(x,k) = \overline{C_j}\xi_{21}(x,\overline{\varkappa_j}), && \operatorname{res}_{k=\overline{\varkappa_j}} \xi_{12}(x,k) = \overline{C_j}\xi_{22}(x,\overline{\varkappa_j}), \\ & && C_j \neq 0, j=1,\dots,N. \end{aligned} \tag{3.8}$$

As before we denote by $\Xi^\pm(x,k)$, $k \in \mathbb{R}$, the limits in k of $\Xi(x,k)$ from the upper and the lower complex half plane.

The asymptotic formulas for the functions $\xi_{js}(x,k)$, $j,s = 1,2$, as $k \to \infty$, are given by

$$\xi_{js}(x,k) = \exp\{(-1)^{j-1}\mathrm{i}kx\}\left(\delta_{js} + \frac{\varphi_{js}(x)}{k} + O(k^{-2})\right).$$

The vector Ψ and the matrix Ξ are related as follows

$$\Psi(x,k) = \begin{pmatrix} \psi_1(x,k) \\ \psi_2(x,k) \end{pmatrix} = \alpha(x)\begin{pmatrix} \xi_{11}(x,k) \\ \xi_{21}(x,k) \end{pmatrix} + \big(\alpha(x)\big)^{-1}\begin{pmatrix} \xi_{12}(x,k) \\ \xi_{22}(x,k) \end{pmatrix}. \tag{3.9}$$

Let us now substitute the functions $\psi_1(x,k)$, $\psi_2(x,k)$ into the equation (1.1). Since they are solutions of this equation, then comparing the terms of order k^0, we obtain

$$\frac{\big(\alpha(x)\big)''}{\alpha(x)} + 2\mathrm{i}\big(\varphi_{11}(x)\big)' + 2\mathrm{i}\big((\alpha(x))^{-2}\big)'\varphi_{12}(x) + 2\mathrm{i}(\alpha(x))^{-2}(\varphi_{12}(x))' = v(x), \tag{3.10}$$

$$\frac{\Big(\big(\alpha(x)\big)^{-1}\Big)''}{\big(\alpha(x)\big)^{-1}} - 2\mathrm{i}\big(\varphi_{22}(x)\big)' - 2i\big((\alpha(x))^2\big)'\varphi_{21}(x) - 2\mathrm{i}\big(\alpha(x)\big)^2\big(\varphi_{21}(x)\big)' = v(x). \tag{3.11}$$

Note that due to the symmetry of the Riemann – Hilbert problem, the equation (3.11) is complex conjugated to the equation (3.10). Note also that the function

$$m(x,k) = \det \Xi(x,k) = \xi_{11}(x,k)\xi_{22}(x,k) - \xi_{12}(x,k)\xi_{21}(x,k) \tag{3.12}$$

is analytic in $\mathbb{C}_\pm$ and solves the trivial Riemann – Hilbert problem

$$m^+(x,k) = m^-(x,k), \quad k \in \mathbb{R}, \qquad m(x,k) \to 1, \quad k \to \infty.$$

Thus $m(x,k) \equiv 1$ and therefore comparing the terms of order k^{-1} and in view of (3.12) we derive

$$\varphi_{11}(x,k) + \varphi_{22}(x,k) = 0. \tag{3.13}$$

Using (3.10), (3.11) and (3.13) we obtain that the function $t(x) := \big(\alpha(x)\big)^2$ satisfies the Riccati equation

$$(t(x))' + 2\mathrm{i}\varphi_{21}(x)(t(x))^2 + 2\mathrm{i}\varphi_{12}(x) = 0, \qquad t(x) \to 1, \quad \text{as } x \to +\infty. \tag{3.14}$$

Solving this equation leads to finding the potential u

$$u(x) = \frac{\mathrm{i}}{2}\frac{t'(x)}{t(x)}.$$

The latter formula is not very explicit and relies on the solution of the equation (3.14). Therefore we use a different approach (see [6]). Obviously $f(x,0) = \overline{f(x,0)}$ and thus

$$\alpha(x)\psi_{11}^+(x,0) + \big(\alpha(x)\big)^{-1}\psi_{12}^+(x,0) = \alpha(x)\psi_{21}^-(x,0) + \big(\alpha(x)\big)^{-1}\psi_{22}^-(x,0).$$

Thus

$$t(x) = \big(\alpha(x)\big)^2 = \exp\bigg\{-2\mathrm{i}\int_\infty^x u(s)\,\mathrm{d}s\bigg\} = \frac{\xi_{22}^-(x,0) - \xi_{12}^+(x,0)}{\xi_{11}^+(x,0) - \xi_{21}^-(x,0)}. \tag{3.15}$$

Remark 3.1. We emphasize that $|t(x)| = 1$ and thus *potential u is regular.*

Remark 3.2. Assume now that the inverse problem does not contain eigenvalues. If reflection coefficient $r(k)$ satisfies the conditions which are usually imposed for the Schrödinger operator then the similar arguments as in the Schrödinger operator case show that the Riemann – Hilbert problem can be solved and thus potentials u and v can be uniquely reconstructed. We do not go into details. The complete proof via Gel′fand – Levitan equations can be found in [6]. The close result in terms of Riemann – Hilbert problem has been obtained in [7].

4. Class of reflectionless potentials

In [6] a particular example of one-soliton solution was constructed. It was pointed out that the determinant of the system which appears when solving Riemann – Hilbert problem (3.7) – (3.8) (or the corresponding Gel′fand – Levitan problem) has zeros if the eigenvalues are included.

It has been proved in [6] that if there is only one soliton solution then this singularity cancels after substituting it into (3.9).

In the present section we show that this cancellation is generic. Note also that, in general, one cannot expect the global solvability. The difficulties appearing here are similar to the difficulties when considering the Schrödinger operator with complex potential. We prove that generically we obtain smooth reflectionless potentials but if we choose some "wrong" coefficients, then potential functions u and v can be singular. Already the case of one-soliton potential shows that, in general, the set of "wrong" coefficients although "small" but its structure is sufficiently complicated.

Assume that $r(k) = 0$ and the set of simple eigenvalues $\{\varkappa_j\}$, $\mathrm{Im}(\varkappa_j) > 0$, $j = 1, \dots, N$, is given. Since the potentials u and v are real functions we obtain

$$\xi_{11} = \mathrm{e}^{\mathrm{i}kx}\left(1 + \sum_{j=1}^{N} \frac{h_j(x)}{k - \overline{\varkappa_j}}\right), \qquad \xi_{12} = \mathrm{e}^{\mathrm{i}kx} \sum_{j=1}^{N} \frac{w_j(x)}{k - \overline{\varkappa_j}}, \tag{4.1}$$

$$\xi_{21} = \mathrm{e}^{-\mathrm{i}kx} \sum_{j=1}^{N} \frac{\overline{w_j}(x)}{k - \varkappa_j}, \qquad \xi_{22} = \mathrm{e}^{-\mathrm{i}kx}\left(1 + \sum_{j=1}^{N} \frac{\overline{h_j}(x)}{k - \varkappa_j}\right). \tag{4.2}$$

We define $h := (h_1, \dots, h_N)^t$, $w := (w_1, \dots, w_N)^t$. Let $B = B(x)$ be the block $(2N \times 2N)$-matrix such that

$$\begin{aligned} B_{11} &= \overline{B_{22}} = A, & B_{12} &= \overline{B_{21}} = J; \\ A_{js} &= -\frac{1}{\varkappa_j - \overline{\varkappa_s}}, & J &= \mathrm{diag}\{C_j^{-1} e^{-2i\varkappa_j x}\}. \end{aligned} \tag{4.3}$$

It follows from (3.8) that

$$B \begin{pmatrix} h \\ \overline{w} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{4.4}$$

By using the properties (4.3), we find that the determinant $d(x) := \det B(x)$ is real-valued. Zeros of d define points of singularity of vector-valued functions h and w. We are going to show that these singularities generically cancel.

Let us define

$$p = d\left(\sum_j \frac{\overline{w_j}}{\varkappa_j} - \sum_j \frac{h_j}{\overline{\varkappa_j}}\right). \tag{4.5}$$

Note that functions $d\, h_j$ and $d\, w_j$, $j = 1, \dots, N$, do not have singularities. It follows from (3.15), (4.1) – (4.2) that

$$\alpha^2 = \frac{d + \bar{p}}{d + p}. \tag{4.6}$$

The function α is regular and thus the potential u is regular. We are going to prove the regularity of $\psi_1(x, k)$, $\psi_2(x, k)$. It follows from (3.9) that it is sufficient to prove regularity of the vector-valued function $s := \alpha^2 h + w$.

Theorem 4.1. *Let $d(x_0) = 0$. Assume that*

(1) $d'(x_0) \neq 0$.
(2) $\dim \ker B(x_0) = 1$.
(3) $p(x_0) \neq 0$.

Then $(sd)(x_0) = 0$ *and thus* s *and, correspondingly,* ψ_1 *and* ψ_2 *are regular functions at* x_0.

PROOF. From (4.4) and the equality $d(x_0) = 0$ we obtain that vector $\left(\begin{smallmatrix} dh \\ d\overline{w} \end{smallmatrix}\right)(x_0)$ belongs to the kernel of $B(x_0)$. It follows from the structure of B (see (4.3)) that the vector $\left(\begin{smallmatrix} dw \\ d\bar{h} \end{smallmatrix}\right)(x_0)$ also belongs to the kernel of $B(x_0)$ (recall that d is real-valued). Due to the condition (2) there exists a constant σ such that

$$\begin{pmatrix} dh \\ d\overline{w} \end{pmatrix}(x_0) = \sigma \begin{pmatrix} dw \\ d\bar{h} \end{pmatrix}(x_0). \tag{4.7}$$

Note that if $\left(\begin{smallmatrix} dh \\ d\overline{w} \end{smallmatrix}\right)(x_0) = 0$ then theorem is proved. Otherwise, it follows from (4.7) that $(dh)(x_0) = \sigma(dw)(x_0) = |\sigma|^2(dh)(x_0)$ and thus,

$$|\sigma| = 1. \tag{4.8}$$

Now, the direct calculations show that (4.5) – (4.8), the equality $d(x_0) = 0$ and the condition (3) imply

$$(ds)(x_0) = 0.$$

Indeed,

$$\begin{aligned} (ds)(x_0) &= \alpha^2(x_0)(dh)(x_0) + (dw)(x_0) = \left(\frac{\overline{p(x_0)}}{p(x_0)}\sigma + 1\right)(dw)(x_0) \\ &= \left(\left(\frac{\sum_j (dw_j)(x_0)/\overline{\varkappa_j} - \overline{\sigma}\sum_j (d\overline{w_j})(x_0)/\varkappa_j}{\sum_j (d\overline{w_j})(x_0)/\varkappa_j - \sigma\sum_j (dw_j)(x_0)/\overline{\varkappa_j}}\right)\sigma + 1\right)(dw)(x_0) = 0. \end{aligned}$$

Now it follows from (1) that s is a regular function at x_0. □

5. "Dressing up" process

Now we proceed to the general case. Let u and v be regular potentials corresponding to some scattering data with eigenvalues $\varkappa_j$, $j = 1, \dots, N$ and coefficient of reflection $r(k)$. Let matrix $\{\xi_{js}\}$, $j, s = 1, 2$, solve the Riemann – Hilbert problem (3.7) – (3.8). We would like to consider the possibility to add one more eigenvalue. So, let $\varkappa_0 \in C_+$, $\varkappa_0 \neq \varkappa_j$, $j = 1, \dots, N$. We are going to construct new potentials $\widetilde{u}$, $\widetilde{v}$ by extended initial data. Let $\{\widetilde{\xi}_{js}\}$, $j, s = 1, 2$, solve the Riemann – Hilbert problem (3.7) – (3.8) with additional condition

$$\begin{aligned} &\operatorname{res}_{k=\varkappa_0} \widetilde{\xi}_{21}(x,k) = C_0\widetilde{\xi}_{11}(x,\varkappa_0), \qquad \operatorname{res}_{k=\varkappa_0} \widetilde{\xi}_{22}(x,k) = C_0\widetilde{\xi}_{12}(x,\varkappa_0), \\ &\operatorname{res}_{k=\overline{\varkappa_0}} \widetilde{\xi}_{11}(x,k) = \overline{C_0}\widetilde{\xi}_{21}(x,\overline{\varkappa_0}), \qquad \operatorname{res}_{k=\overline{\varkappa_0}} \widetilde{\xi}_{12}(x,k) = \overline{C_0}\widetilde{\xi}_{22}(x,\overline{\varkappa_0}), \\ &\hspace{22em} C_0 \neq 0. \end{aligned} \tag{5.1}$$

The solution can be written in the following form.

$$\begin{pmatrix} \widetilde{\xi}_{11} \\ \widetilde{\xi}_{21} \end{pmatrix} = \begin{pmatrix} \xi_{11} \\ \xi_{21} \end{pmatrix}\left(1 + \frac{h_1}{k-\varkappa_0} + \frac{h_2}{k-\overline{\varkappa_0}}\right) + \begin{pmatrix} \xi_{12} \\ \xi_{22} \end{pmatrix}\left(\frac{w_1}{k-\varkappa_0} + \frac{w_2}{k-\overline{\varkappa_0}}\right), \tag{5.2}$$

$$\begin{pmatrix} \widetilde{\xi}_{12} \\ \widetilde{\xi}_{22} \end{pmatrix} = \begin{pmatrix} \xi_{11} \\ \xi_{21} \end{pmatrix}\left(\frac{\overline{w_2}}{k-\varkappa_0} + \frac{\overline{w_1}}{k-\overline{\varkappa_0}}\right) + \begin{pmatrix} \xi_{12} \\ \xi_{22} \end{pmatrix}\left(1 + \frac{\overline{h_2}}{k-\varkappa_0} + \frac{\overline{h_1}}{k-\overline{\varkappa_0}}\right). \tag{5.3}$$

Since $\widetilde{\xi}_{11}$, $\widetilde{\xi}_{12}$ are analytic in $\mathbb{C}_+$ and $\widetilde{\xi}_{22}$, $\widetilde{\xi}_{21}$ are analytic in $\mathbb{C}_-$ we get two additional equations

$$\xi_{11}(\varkappa_0)h_1 + \xi_{12}(\varkappa_0)w_1 = 0, \quad \xi_{21}(\overline{\varkappa_0})h_2 + \xi_{22}(\overline{\varkappa_0})w_2 = 0. \tag{5.4}$$

Similarly to (3.15) we have

$$\widetilde{\alpha}^2 := \exp\Big\{-2\mathrm{i}\int_\infty^x \widetilde{u}\,\mathrm{d}s\Big\} = \frac{\widetilde{\xi}_{22}^-(0) - \widetilde{\xi}_{12}^+(0)}{\widetilde{\xi}_{11}^+(0) - \widetilde{\xi}_{21}^-(0)}. \tag{5.5}$$

The potential $\widetilde{u}$ is always regular. Note that $\widetilde{\alpha}^2 = \overline{\widetilde{\alpha}^{-2}}$. Recall also that

$$\xi_{11}\xi_{22} - \xi_{12}\xi_{21} = 1, \tag{5.6}$$

and due to the fact that both potentials u and v are real

$$\xi_{11}(k) = \overline{\xi_{22}(\bar{k})}, \quad \xi_{12}(k) = \overline{\xi_{21}(\bar{k})}. \tag{5.7}$$

Using (5.4) we express h_1 and w_2 via w_1 and h_2 correspondingly. Substituting it into (5.2), (5.3) and taking into account (5.6), (5.7) we arrive after simple calculations at the equalities

$$\begin{pmatrix}\widetilde{\xi}_{11}\\ \widetilde{\xi}_{21}\end{pmatrix} = \begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix} + \frac{\chi_1}{k-\varkappa_0}\left(\xi_{11}(\varkappa_0)\begin{pmatrix}\xi_{12}\\ \xi_{22}\end{pmatrix} - \xi_{12}(\varkappa_0)\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix}\right) + \frac{\chi_2}{k-\overline{\varkappa_0}}\left(\overline{\xi_{11}(\varkappa_0)}\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix} - \overline{\xi_{12}(\varkappa_0)}\begin{pmatrix}\xi_{12}\\ \xi_{22}\end{pmatrix}\right), \tag{5.8}$$

$$\begin{pmatrix}\widetilde{\xi}_{12}\\ \widetilde{\xi}_{22}\end{pmatrix} = \begin{pmatrix}\xi_{12}\\ \xi_{22}\end{pmatrix} + \frac{\overline{\chi_2}}{k-\varkappa_0}\left(\xi_{11}(\varkappa_0)\begin{pmatrix}\xi_{12}\\ \xi_{22}\end{pmatrix} - \xi_{12}(\varkappa_0)\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix}\right) + \frac{\overline{\chi_1}}{k-\overline{\varkappa_0}}\left(\overline{\xi_{11}(\varkappa_0)}\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix} - \overline{\xi_{12}(\varkappa_0)}\begin{pmatrix}\xi_{12}\\ \xi_{22}\end{pmatrix}\right). \tag{5.9}$$

Here, $\chi_1 = w_1/\xi_{11}(\varkappa_0)$, $\chi_2 = h_2/\overline{\xi_{11}(\varkappa_0)}$. Substituting it into (5.1) we obtain the system

$$\widetilde{B}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = \begin{pmatrix}C_0\xi_{11}(\varkappa_0)\\ C_0\xi_{12}(\varkappa_0)\end{pmatrix}, \tag{5.10}$$

where $\widetilde{B}$ is 2×2-matrix with the following entries

$$\begin{aligned}\widetilde{B}_{11} = \overline{\widetilde{B}_{22}} &= 1 - C_0\left(\frac{\xi_{12}(k)\xi_{11}(\varkappa_0) - \xi_{12}(\varkappa_0)\xi_{11}(k)}{k-\varkappa_0}\right)\Bigg|_{k=\varkappa_0},\\ \widetilde{B}_{12} = \overline{\widetilde{B}_{21}} &= C_0\frac{|\xi_{11}(\varkappa_0)|^2 - |\xi_{12}(\varkappa_0)|^2}{\overline{\varkappa_0}-\varkappa_0}.\end{aligned} \tag{5.11}$$

Denote $\tilde{d} := \det\widetilde{B}$. Obviously, $\tilde{d}$ is real-valued. In general, $\tilde{d}$ must have zeros and thus χ_1, χ_2 are singular. But the theorem below shows that generically the vector-valued function (the solutions of the equation)

$$\widetilde{\Psi}(x,k) = \begin{pmatrix}\widetilde{\psi}_1(x,k)\\ \widetilde{\psi}_2(x,k)\end{pmatrix} := \widetilde{\alpha}(x)\begin{pmatrix}\widetilde{\xi}_{11}(x,k)\\ \widetilde{\xi}_{21}(x,k)\end{pmatrix} + (\widetilde{\alpha}(x))^{-1}\begin{pmatrix}\widetilde{\xi}_{12}(x,k)\\ \widetilde{\xi}_{22}(x,k)\end{pmatrix} \tag{5.12}$$

will be regular and so will be the potential $\widetilde{v}$.

Define

$$\widetilde{s}(x) := \tilde{d}(x)(\widetilde{\alpha}^2(x)\chi_1(x) + \overline{\chi}_2(x)), \tag{5.13}$$

and

$$\widetilde{p}(x) := \tilde{d}(x)(\widetilde{\xi}^{+}_{11}(x,0) - \widetilde{\xi}^{-}_{21}(x,0)). \tag{5.14}$$

Theorem 5.1. *Let $\tilde{d}(x_0) = 0$. Assume that*

(1) $\tilde{d}'(x_0) \neq 0$.
(2) $\dim\ker \widetilde{B}(x_0) = 1$.
(3) $\widetilde{p}(x_0) \neq 0$.

Then $\widetilde{\xi}$ and thus $\widetilde{v}$ is regular at point x_0.

Proof. It follows from (5.8), (5.9), and (5.12) that to prove regularity of $\widetilde{\Psi}$ at point x_0 it is sufficient to show that the function $\widetilde{\alpha}^2(x)\chi_1(x) + \overline{\chi}_2(x)$ is regular at x_0. Due to the condition (1) of the theorem the result will follow if we prove the equality $\widetilde{s}(x_0) = 0$. The same arguments as in the proof of Theorem 4.1 show that conditions (1), (2) together with the equality $\tilde{d}(x_0) = 0$ imply the identity

$$(\tilde{d}\overline{\chi_2})(x_0) = \widetilde{\sigma}(\tilde{d}\chi_1)(x_0) \tag{5.15}$$

for some constant $\widetilde{\sigma}$ such that

$$|\widetilde{\sigma}| = 1. \tag{5.16}$$

Substituting (5.8), (5.9), (5.15) into (5.5) and taking into account (5.16) and condition (3) of the theorem we obtain that

$$\widetilde{\alpha}^2(x_0) = \frac{\Big(\tilde{d}(x)\big(\widetilde{\xi}^{-}_{22}(x,0) - \widetilde{\xi}^{+}_{12}(x,0)\big)\Big)(x_0)}{\Big(\tilde{d}(x)\big(\widetilde{\xi}^{+}_{11}(x,0) - \widetilde{\xi}^{-}_{21}(x,0)\big)\Big)(x_0)} = -\widetilde{\sigma}. \tag{5.17}$$

From here and (5.13), (5.15) we obtain the result. □

6. Trace formulas

Let us rewrite the equation (1.1)

$$-f''_{xx}(x,k) + (2ku(x) + v(x))f(x,k) = k^2 f(x,k)$$

in the terms of function $h(x)$

$$f = \exp\Big(\mathrm{i}kx - \mathrm{i}\int_{\infty}^{x} u(t)\,\mathrm{d}t + \int_{\infty}^{x} h(t)\,\mathrm{d}t\Big).$$

Then

$$u^2 - h^2 - 2\mathrm{i}kh + 2\mathrm{i}uh + v + \mathrm{i}u' - h' = 0. \tag{6.1}$$

Due to asymptotic properties of the function $f(x,k)$ function $h(x,k)$ has the following asymptotic expansion as $k \to \infty$

$$h(x) = \sum_{j=1}^{\infty} \frac{h_j}{k^j}.$$

Substituting it to the equation (6.1) we have

$$u^2 + v - 2\mathrm{i}h_1 + \mathrm{i}u' = 0,$$

$$-\sum_{p=1}^{l-1} h_p h_{l-p} - 2\mathrm{i}h_{l+1} + 2\mathrm{i}uh_l - h'_l = 0, \quad l > 0.$$

In particular, we obtain

$$\begin{aligned} h_1 &= \frac{1}{2\mathrm{i}}(u^2 + v + \mathrm{i}u'), \\ h_2 &= \frac{1}{2\mathrm{i}}(u^3 + vu) + uu' + \frac{1}{4}(v' + \mathrm{i}u''), \\ h_3 &= \frac{1}{2\mathrm{i}}(2\mathrm{i}h_2 - h_1^2 - h_2') \\ &= \frac{1}{2\mathrm{i}}\left(u^4 + vu^2 + \frac{1}{4}(u^2+v)^2 - h_2'\right) + \frac{1}{4}(vu)' + \frac{5}{12}(u^3)' - \frac{1}{8\mathrm{i}}(2uu'' + (u')^2). \end{aligned}$$

On the other hand, due to analytical properties of the function $a(k)$ it follows that

$$a(k) = \prod_1^n \frac{k - k_l}{k - \overline{k_l}} \exp\left(\mathrm{i}\int_{-\infty}^{\infty} u(x)\,\mathrm{d}x + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\ln|a(s)|^2}{s-k}\,\mathrm{d}s\right).$$

Let us consider asymptotic expansion of $a(k)$ as $k \to \infty$

$$\ln a(k) = \mathrm{i}\int_{-\infty}^{\infty} u(x)\,\mathrm{d}x + \sum_1^{\infty} \frac{a_j}{k^j}.$$

A simple calculation shows that

$$a_j = \frac{-2\mathrm{i}}{j}\sum_{l=1}^n \mathrm{Im}(k_l^j) - \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \ln|a(s)|^2 s^{j-1}\,\mathrm{d}s. \tag{6.2}$$

Note that for $\mathrm{Im}\, k > 0$

$$\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = \mathrm{i}\int_{-\infty}^{\infty} u(x)\,\mathrm{d}x - \ln a(k). \tag{6.3}$$

From (6.2) and (6.3) it follows that

$$\int_{-\infty}^{\infty} h_j(x)\,\mathrm{d}x = \frac{2\mathrm{i}}{j}\sum_{l=1}^n \mathrm{Im}(k_l^j) + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \ln|a(s)|^2 s^{j-1}\,\mathrm{d}s.$$

These relations are trace formulas for equation (1.1). In particular, first three trace formulas have the following form

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1}{2\mathrm{i}}(u^2 + v)\,\mathrm{d}x = 2\mathrm{i}\sum_{l=1}^n \mathrm{Im}(k_l) + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \ln|a(s)|^2\,\mathrm{d}s, \\ &\int_{-\infty}^{\infty} \frac{1}{2\mathrm{i}}(u^3 + vu)\,\mathrm{d}x = \mathrm{i}\sum_{l=1}^n \mathrm{Im}(k_l^2) + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \ln|a(s)|^2 s\,\mathrm{d}s, \\ &\int_{-\infty}^{\infty} \frac{1}{2\mathrm{i}}\left(u^4 + vu^2 + \frac{1}{4}\big((u^2+v)^2 + (u')^2\big)\right)\mathrm{d}x \\ &\qquad\qquad = \frac{2}{3}\mathrm{i}\sum_{l=1}^n \mathrm{Im}(k_l^3) + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \ln|a(s)|^2 s^2\,\mathrm{d}s. \end{aligned}$$

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