A GEOMETRICAL VERSION OF HARDY'S INEQUALITY

M. HOFFMANN-OSTENHOF[1], T. HOFFMANN-OSTENHOF[2,3] AND A.LAPTEV[4]

ABSTRACT. We prove a version of Hardy's type inequality in a domain $\Omega \subset \mathbb{R}^n$ which involves the distance to the boundary and the volume of Ω. In partucular, we obtain a result which gives a positive answer to a question asked by H.Brezis and M.Marcus.

0. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary. It is known that the following extension of Hardy's inequality is valid

$$\int_\Omega |\nabla u(x)|^2\, dx \geq \mu \int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx, \qquad \forall u \in H_0^1(\Omega), \tag{0.1}$$

where μ is a positive constant and $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. The best constant $\mu = \mu(\Omega)$ in (0.1) depends on the domain Ω. It is also known that for convex domains $\mu(\Omega) = 1/4$, but there are smooth domains such that $\mu(\Omega) < 1/4$ (see[6], [7]).

H.Brezis and M.Marcus [3], Theorem I, have shown that for every domain Ω of class C^2 there exists a constant $\lambda = \lambda(\Omega) \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u(x)|^2\, dx \geq \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx + \lambda \int_\Omega |u(x)|^2\, dx, \qquad \forall u \in H_0^1(\Omega). \tag{0.2}$$

Note that there are examples ([6], [7]) which confirm that there are smooth domains with $\lambda \leq 0$. However, if Ω is convex then it is proved in [3] (see Theorem II) that

$$\lambda(\Omega) \geq \frac{1}{4\,\mathrm{diam}^2(\Omega)}. \tag{0.3}$$

In this paper Brezis and Marcus have asked whether the diameter of Ω, in (0.3) can be replaced by an expression depending on $|\Omega| := \mathrm{vol}\,\Omega$, namely, whether $\lambda \geq c\,|\Omega|^{-2/n}$ with some $c = c(n) > 0$.

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The aim of this short article is to prove that this is the case indeed and that for convex domains

$$\lambda \geq \frac{c(n)}{|\Omega|^{2/n}}, \qquad c(n) = \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4}, \tag{0.4}$$

where $s_{n-1} := |\mathbb{S}^{n-1}|$. In particular, if $n = 2$ then $c(2) = \pi/2$.

The proof of this result is based on a one-dimensional version of Hardy's inequality which is obtained in Section 1. In Section 2 we extend the one-dimensional result to the many-dimensional case using arguments of E.B.Davies [4], Ch.5.3. In Section 3 we prove (0.4) and consider some other generalizations of this result.

Note that various types of Hardy's inequalities can be found in books [8] and [9] and in the recent review article [5]. Some results from [3] were recently extended to cases with weights in [2] and to L^p spaces in [1].

1. One dimensional results

We start with a simple statement which is just a corollary of the Cauchy-Schwarz inequality and partial integration.

Let f be a function defined on $(0, b)$, $b > 0$, and whose derivative is finite on $(0, b)$. We say that f belongs to the class $\Phi(0, b)$ if f is real valued and there is a constant $C = C(f)$ such that

$$\sup_{0<t\leq b} \left(t|f(t)| + t^2|f'(t)|\right) \leq C. \tag{1.1}$$

Lemma 1.1. *Let $u \in C^1(0, b)$, $b > 0$, $u(0) = 0$ and let $f \in \Phi(0, b)$. Then*

$$\int_0^b \left|\frac{du}{dt}\right|^2 dt \geq \frac{1}{4} \, \frac{(\int_0^b f'(t)|u|^2\, dt)^2}{\int_0^b (f(t) - f(b))^2 |u|^2\, dt}. \tag{1.2}$$

Proof. For any constant c taking into account (1.1) we have

$$\begin{aligned}
\Big((f(b) - c)|u(b)|^2 - \int_0^b f'(t)\,|u|^2\, dt\Big)^2 &= \Big(\int_0^b (f(t) - c)\,(|u|^2)'\, dt\Big)^2 \\
&= \Big(\int_0^b (f(t) - c)\,(u'\bar{u} + u\bar{u}')\, dt\Big)^2 \\
&\leq 4\Big(\int_0^b |u'|^2\, dt\Big)\Big(\int_0^b (f(t) - c)^2\,|u|^2\, dt\Big).
\end{aligned}$$

We complete the proof by substituting $c = f(b)$.

The next result shows that (1.2) is often sharp unlikely many other Hardy's type inequalities.

Lemma 1.2. *Let us assume that* $f \in \Phi(0,b)$ *and*

- $\int_t^b f(s)ds \to +\infty$, *as* $t \to 0$.
- *There is a constant* a, $a > 0$ *such that*

$$f(t)e^{-a\int_t^b f(s)\,ds} \in L^2(0,b).$$

Then Lemma 1.1 is sharp.

Proof. The Cauchy-Schwarz inequality which has been used in the proof of Lemma 1.1 is sharp if there exists a real valued function u such that $u'(t)$ and $u(t)(f(t)-f(b))$ are linearly dependent

$$u'(t) = a\,u(t)(f(t)-f(b)). \tag{1.3}$$

Solving (1.3) we obtain

$$u(t) = Ce^{-a\int_t^b f(s)\,ds - af(b)\,t}.$$

If now $a > 0$, then the first assumption implies $u(0) = 0$. The second one provides the inclusion $du/dt \in L^2(0,b)$.

Example.
If $f(t) = 1/t$ and $a > 1/2$, then substituting in (1.2)

$$u(t) = t^a e^{-at/b}$$

we find that the left and the right hand sides of this inequality are the same. Moreover, $u(0) = 0$, $u \in C^1(0,b)$ and therefore the inequality (1.1) has an extremizer.

Although in many cases Lemma 1.1 gives sharp results, the right hand side in the inequality (1.2) is not linear with respect to u. We would now like to give the following linearized version of this inequality.

By using (1.2) we obviously have

$$\int_0^b |u'|^2\,dt \geq \frac{1}{4}\int_0^b \Big(2f'(t) - (f(t)-f(b))^2\Big)|u|^2\,dt. \tag{1.4}$$

If we rewrite this inequality for the interval $[b,2b]$ with $u \in C^1(b,2b)$, $u(2b) = 0$, then

$$\int_b^{2b} |u'|^2\,dt \geq \frac{1}{4}\int_b^{2b} \Big(2f'(2b-t) - (f(2b-t)-f(b)^2\Big)|u|^2\,dt. \tag{1.5}$$

Adding up (1.4) and (1.5) and by using standard density arguments we can finally state our main one-dimensional result.

Lemma 1.3. *Let* $u \in H_0^1(0,2b)$, $b > 0$ *and let* $f \in \Phi(0,b)$. *Then*

$$\int_0^{2b} |u'(t)|^2\,dt \geq \frac{1}{4}\int_0^{2b} \Big(2f'(\rho(t)) - (f(\rho(t))-f(b))^2\Big)|u|^2\,dt, \tag{1.6}$$

where

$$\rho(t) = \min(t, 2b - t).$$

2. A RESULT FOR HIGHER DIMENSIONS

Let Ω be a domain in $\mathbb{R}^n$. In order to formulate the main result of this section we need some notations. Denote by $\tau_\nu(x)$ the distance between $x \in \Omega$ and its nearest point belonging to the boundary $\partial\Omega$ in the direction $\nu \in \mathbb{S}^{n-1}$,

$$\tau_\nu(x) = \min\{s > 0 : x + s\nu \notin \Omega\}. \tag{2.1}$$

Let us also introduce the "distance" to the boundary ρ_ν and the "diameter" D_ν along the line defined by ν via:

$$\rho_\nu(x) = \min(\tau_\nu(x), \tau_{-\nu}(x)) \tag{2.2}$$

$$D_\nu(x) = \tau_\nu(x) + \tau_{-\nu}(x). \tag{2.3}$$

By $d\omega(\nu)$ we denote the normalized measure on the unit sphere $\mathbb{S}^{n-1}$, $\int_{\mathbb{S}^{n-1}} d\omega(\nu) = 1$.

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^n$, $D \in (0, \infty]$ be its diameter and $f \in \Phi(0, D/2)$. Then for any $u \in H_0^1(\Omega)$ we have*

$$\int_\Omega |\nabla u|^2\, dx \geq \frac{n}{4} \int_\Omega \Big(\int_{\mathbb{S}^{n-1}} (2f'(\rho_\nu(x)) - f^2(\rho_\nu(x)) + 2f(\rho_\nu(x)) f(D_\nu(x)/2) - f^2(D_\nu(x)/2))\, d\omega(\nu) \Big) |u(x)|^2\, dx, \tag{2.4}$$

Proof. We proceed by using E.B.Davies' arguments (see [4]). Let ∂_ν denote partial differentiation in the direction $\nu \in \mathbb{S}^{n-1}$. Then Lemma 1.3 implies

$$\int_\Omega |\partial_\nu u|^2\, dx \geq \frac{1}{4} \int_\Omega \Big(2f'(\rho_\nu(x)) - f^2(\rho_\nu(x)) + 2f(\rho_\nu(x)) f(D_\nu(x)/2) - f^2(D_\nu(x)/2) \Big) |u|^2\, dx,$$

where the function ρ_ν and D_ν are defined in (2.2) and (2.3). Let us introduce an orthonormal basis $\{\bar{e}_j\}_{j=1}^n$ in $\mathbb{R}^n$. Then

$$\int_\Omega |\nabla u|^2\, dx \geq \frac{1}{4} \sum_{j=1}^n \int_\Omega (2f'(\rho_{\bar{e}_j}(x)) - f^2(\rho_{\bar{e}_j}(x)) + 2f(\rho_{\bar{e}_j}(x)) f(D_{\bar{e}_j}/2) - f^2(D_{\bar{e}_j}/2)) |u|^2\, dx.$$

Averaging both sides of the last inequality over orthonormal bases using the group $O(n)$ we complete the proof.

3. APLICATIONS OF THEOREM 2.1

3.1. **On a question of Brezis and Marcus.** Let

$$f(t) = -1/t, \qquad t > 0. \tag{3.1}$$

Then the integral over $\mathbb{S}^{n-1}$ in the right hand side of (2.4) becomes equal to

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \Big(2f'(\rho_\nu) - f^2(\rho_\nu) + 2f(\rho_\nu)f(D_\nu/2) - f^2(D_\nu/2)\Big)\, d\omega(\nu) \\ &\qquad = \int_{\mathbb{S}^{n-1}} \Big(\frac{1}{\rho_\nu^2} + \frac{4}{\rho_\nu D_\nu} - \frac{4}{D_\nu^2}\Big)\, d\omega(\nu). \end{aligned} \tag{3.2}$$

Let us consider the last two terms. It is clear that $\rho_\nu(x) \le \tau_\nu(x)$, $x \in \Omega$, where the functions ρ and τ are defined in (2.2) and (2.1). Since $D_\nu = \tau_\nu + \tau_{-\nu}$ (see (2.3)) we obtain

$$\begin{aligned} \frac{1}{\rho_\nu D_\nu} - \frac{1}{D_\nu^2} &\ge \frac{1}{\tau_\nu(\tau_\nu + \tau_{-\nu})} - \frac{1}{(\tau_\nu + \tau_{-\nu})^2} \\ &= \frac{\tau_{-\nu}}{\tau_\nu(\tau_\nu + \tau_{-\nu})^2}. \end{aligned}$$

This implies

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \Big(\frac{1}{\rho_\nu(x)D_\nu(x)} - \frac{1}{D_\nu^2(x)}\Big)\, d\omega(\nu) &\ge \int_{\mathbb{S}^{n-1}} \frac{\tau_{-\nu}}{\tau_\nu(\tau_\nu + \tau_{-\nu})^2}\, d\omega(\nu) \\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \Big(\frac{\tau_{-\nu}}{\tau_\nu(\tau_\nu + \tau_{-\nu})^2} + \frac{\tau_\nu}{\tau_{-\nu}(\tau_\nu + \tau_{-\nu})^2}\Big)\, d\omega(\nu) \\ &\ge \frac{1}{4}\int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu\tau_{-\nu}}\, d\omega(\nu). \end{aligned}$$

In order to estimate the latter integral we apply the Cauchy-Schwarz inequality twice and obtain

$$\begin{aligned} 1 &\le \int_{\mathbb{S}^{n-1}} \tau_\nu\tau_{-\nu}\, d\omega(\nu) \int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu\tau_{-\nu}}\, d\omega(\nu) \\ &\le \int_{\mathbb{S}^{n-1}} \tau_\nu^2\, d\omega(\nu) \int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu\tau_\nu}\, d\omega(\nu), \end{aligned}$$

where we have used that $\int_{\mathbb{S}^{n-1}} \tau_\nu^2\, d\omega(\nu) = \int_{\mathbb{S}^{n-1}} \tau_{-\nu}^2\, d\omega(\nu)$.

When now applying Hölder's inequality we recall that $\int_{\mathbb{S}^{n-1}} d\omega(\nu) = 1$. Therefore

$$\int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu(x)\tau_{-\nu}(x)}\, d\omega(\nu) \ge \Big(\int_{\mathbb{S}^{n-1}} \tau_\nu^2(x)\, d\omega(\nu)\Big)^{-1}$$
$$\ge \Big(\int_{\mathbb{S}^{n-1}} \tau_\nu^n(x)\, d\omega(\nu)\Big)^{-2/n}.$$

Let us introduce the domain $\Omega_x \subseteq \Omega$ defined as a part of Ω which can be "seen" from point x

$$\Omega_x := \{y \in \Omega : x + t(y-x) \in \Omega,\ \forall t \in [0,1]\}. \tag{3.3}$$

Then

$$\int_{\mathbb{S}^{n-1}} \tau_\nu^n(x)\, d\omega(\nu) = \frac{n}{s_{n-1}}|\Omega_x|,$$

which finally gives us

$$\int_{\mathbb{S}^{n-1}} \Big(\frac{4}{\rho_\nu(x)D_\nu(x)} - \frac{4}{D_\nu^2(x)}\Big)\, d\omega(\nu) \ge \Big(\frac{s_{n-1}}{n}\Big)^{2/n} \frac{1}{|\Omega_x|^{2/n}}. \tag{3.4}$$

Now (2.4), (3.2) and (3.4) imply the following reformulation of Theorem 2.1 in the case when the function f is defined by (3.1).

Theorem 3.1. *For any $\Omega \subset \mathbb{R}^n$ and any $u \in H_0^1(\Omega)$ we have*

$$\int_\Omega |\nabla u|^2\, dx \ge \frac{n}{4}\int_\Omega \int_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu^2(x)}\, d\omega(\nu)\, |u(x)|^2\, dx$$
$$+ \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4} \int_\Omega \frac{|u(x)|^2}{|\Omega_x|^{2/n}}\, dx. \tag{3.5}$$

Clearly in (3.5) the value $|\Omega_x|$ can always be replaced by $|\Omega|$. If Ω is convex then it is known (see for example [4] Exercise 5.7 and [5]) that

$$\frac{n}{4}\int_{\mathbb{S}^{n-1}} \rho_\nu^{-2}(x)\, d\omega(\nu) \ge \frac{1}{4}\frac{1}{\delta^2(x)}, \tag{3.6}$$

Moreover, in this case $\Omega_x = \Omega$, $x \in \Omega$, and we obtain

Theorem 3.2. *For any convex domain $\Omega \subset \mathbb{R}^n$ and any $u \in H_0^1(\Omega)$*

$$\int_\Omega |\nabla u|^2\, dx \ge \frac{1}{4}\int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx \ + \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4\,|\Omega|^{2/n}} \int_\Omega |u(x)|^2\, dx.$$

Note that the domain Ω in Theorem 3.1 can be unbounded and it is valid for a variety of domains with fractal boundaries, for example, such as the Koch snowflake in $\mathbb{R}^2$. So often the inequality (3.6) might hold true with a constant $\mu < 1/4$ instead of $1/4$ in the right hand side, whereas the second integral of the inequality (3.5) is very stable.

In particular, following E.B.Davies, Lemma 3 from [5] we can obtain:

Corollary 3.1. *Suppose that there is a constant κ such that for each $y \in \partial\Omega$ and each $a > 0$ there exists a disjoint from Ω ball B with centre z and radius $\beta \geq a\kappa$, where $|z - y| = a$. Then there exists a constant $\mu \leq 1/4$ such that*

$$\frac{n}{4}\int_{\mathbb{S}^{n-1}} \rho_\nu^{-2}(x)\, d\omega(\nu) \geq \mu\, \frac{1}{\delta^2(x)}$$

and hence

$$\int_\Omega |\nabla u|^2\, dx \geq \mu \int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx + \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4} \int_\Omega \frac{|u(x)|^2}{|\Omega_x|^{2/n}}\, dx.$$

3.2. **Some refined inequalities.** The next application of Theorem 2.1 concerns the function

$$f(t) = -\frac{1}{t} + \frac{1}{t(1-\ln(\alpha t/D))}, \qquad 0 < t < D/2, \tag{3.7}$$

where $D = \operatorname{diam}\Omega$ and $0 < \alpha \leq 2$. In this case the expression appearing in the right hand side of (2.4) is equal to

$$\begin{aligned}
&2f'(\rho_\nu) - f^2(\rho_\nu) + 2f(\rho_\nu)f(D_\nu/2) - f^2(D_\nu/2)\\
&\quad = \frac{1}{\rho_\nu^2} + \frac{1}{\rho_\nu^2\,(1-\ln(\alpha\rho_\nu/D))^2}\\
&\quad + 4\Big(\frac{\ln(\alpha\rho_\nu/D)\,\ln(\alpha D_\nu/2D)}{\rho_\nu D_\nu(1-\ln(\alpha\rho_\nu/D))(1-\ln(\alpha D_\nu/2D))} - \frac{\ln^2(\alpha D_\nu/2D)}{D_\nu^2(1-\ln(\alpha D_\nu/2D))^2}\Big)\\
&\quad \geq \frac{1}{\rho_\nu^2} + \frac{1}{\rho_\nu^2\,(1-\ln(\alpha\rho_\nu/D))^2} + 4\Big(\frac{1}{\rho_\nu D_\nu} - \frac{1}{D_\nu^2}\Big)\frac{\ln^2(\alpha/2)}{(1-\ln(\alpha/2))^2}.
\end{aligned}$$

Theorem 2.1 therefore gives

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\, dx\\
&\qquad \geq \frac{n}{4}\int_\Omega\int_{\mathbb{S}^{n-1}} \Big(\frac{1}{\rho_\nu^2(x)} + \frac{1}{\rho_\nu^2\,(1-\ln(\alpha\rho_\nu/D))^2}\Big)\, d\omega(\nu)\; |u(x)|^2\, dx\\
&\qquad + \frac{n\,\ln^2(\alpha/2)}{(1-\ln(\alpha/2))^2}\int_\Omega\int_{\mathbb{S}^{n-1}}\Big(\frac{1}{\rho_\nu D_\nu} - \frac{1}{D_\nu^2}\Big)\, d\omega(\nu)\, |u(x)|^2\, dx.
\end{aligned}$$

Application of (3.4) leads to a more refined version of Theorem 3.1.

Theorem 3.3. *Let $0 < \alpha \le 2$. Then for any $\Omega \subset \mathbb{R}^n$ and any $u \in H_0^1(\Omega)$ we have*

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx \\ \ge \frac{n}{4}\int_\Omega\int_{\mathbb{S}^{n-1}} \Big(\frac{1}{\rho_\nu^2(x)} + \frac{1}{\rho_\nu^2\,(1-\ln(\alpha\rho_\nu/D))^2}\Big)\,d\omega(\nu)\ |u(x)|^2\,dx \\ + \frac{n^{(n-2)/n} s_{n-1}^{2/n}\ln^2(\alpha/2)}{4\,(1-\ln(\alpha/2))^2}\int_\Omega \frac{|u(x)|^2}{|\Omega_x|^{2/n}}\,dx.\end{aligned}$$

Remark. Theorem 3.3 is a stronger result than Theorem 3.1. Indeed, we obtain Theorem 3.1 from Theorem 3.3 if we let $\alpha \to 0$.

For Ω convex we obtain via (3.6) that

$$\begin{aligned}\frac{n}{4}\int_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu^2(x)}\Big(1 + \frac{1}{(1-\ln(\alpha\rho_\nu(x)/D))^2}\Big)\,d\omega(\nu) \\ \ge \frac{1}{4}\frac{1}{\delta^2(x)}\Big(1+\frac{1}{(1-\ln(\alpha\delta(x)/D))^2}\Big).\end{aligned}$$

The latter inequality and Theorem 3.3 implies a version of Theorem 3.2:

Theorem 3.4. *Let $0 < \alpha \le 2$. For any convex domain $\Omega \subset \mathbb{R}^n$ and any $u \in H_0^1(\Omega)$*

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx \ge \frac{1}{4}\int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\Big(1+\frac{1}{(1-\ln(\alpha\delta(x)/D))^2}\Big)\,dx \\ + \frac{n^{(n-2)/n} s_{n-1}^{2/n}\ln^2(\alpha/2)}{4\,(1-\ln(\alpha/2))^2}\,\frac{1}{|\Omega|^{2/n}}\int_\Omega |u(x)|^2\,dx.\end{aligned} \tag{3.8}$$

Remark. The last statement is an improvment of Theorem 5.1 from [3], where for convex domains Ω and $u \in H_0^1(\Omega)$ the authors obtain the inequality

$$\int_\Omega |\nabla u(x)|^2\,dx \ge \frac{1}{4}\int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\Big(1+\frac{1}{(1-\ln(\delta(x)/D))^2}\Big)\,dx. \tag{3.9}$$

Indeed, if we choose $\alpha = 1$ in Theorem 3.4, then the first integral in the right hand side of (3.8) coincides with the right hand side of (3.9). However, $\alpha = 1$ still allows to have an additional non-zero term in (3.8).

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Institut für Mathematik, Universität Wien[1]
Institut für Theoretische Chemie, Universität Wien[2]

International Erwin Schrödinger Institute for Mathematical Physics[3]
Department of Mathmetics, KTH, 100 44 Stockholm, Sweden[4]