Lieb-Thirring inequalities with improved constants

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Abstract

Following Eden and Foias we obtain a matrix version of a generalised Sobolev inequality in one-dimension. This allows us to improve on the known estimates of best constants in Lieb-Thirring inequalities for the sum of the negative eigenvalues for multi-dimensional Schrödinger operators.

Key-words: Sobolev inequalities; Schrödinger operator; Lieb-Thirring inequalities.
MSC (2000): Primary: 35P15; Secondary: 81Q10

1 Introduction

Let H be a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - V \tag{1}$$

For a real-valued potential V we consider Lieb-Thirring inequalities for the negative eigenvalues $\{\lambda_n\}$ of the operator H

$$\sum |\lambda_n|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V_+^{d/2+\gamma}(x)\, dx\,, \tag{2}$$

where $V_+ = (|V| + V)/2$ is the positive part of V.

Eden and Foias have obtained in [3] a version of a one-dimensional generalised Sobolev inequality which gives best known estimates for the constants in the inequality (2) for $1 \leq \gamma < 3/2$. The aim of this short article is to extend the method from [3] to a class of matrix-valued potentials. By using ideas from [6] this automatically improves on the known estimates of best constants in (2) for multidimensional Schrödinger operators.

Lieb-Thirring inequalities for matrix-valued potentials for the value $\gamma = 3/2$ were obtained in [6] and also in [2]. Here we state a result corresponding to $\gamma = 1$.

Theorem 1. *Let $V \geq 0$ be a Hermitian $m \times m$ matrix-function defined on $\mathbb{R}$ and let λ_n be all negative eigenvalues of the operator* (1)*. Then*

$$\sum |\lambda_n| \leq \frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \operatorname{Tr}\left[V^{3/2}(x)\right] dx\,. \tag{3}$$

Remark 1. *The constant* $\frac{2}{3\sqrt{3}}$ *should be compared with the Lieb-Thirring constant found in [7] for a class of single eigenvalue potentials and with the constant obtained in [5] which is twice as large as the semi-classical one*

$$\frac{4}{3\sqrt{3}\,\pi} < \frac{2}{3\sqrt{3}} < 2 \times \frac{2}{3\pi} = 2 \times \frac{1}{2\pi} \int_{\mathbb{R}} (1-\xi^2)_+ \, d\xi \,.$$

This is about $0,2450\dots < 0,3849\dots < 0,4244\dots$.

Remark 2. *Note that the values of the best constants for the range* $1/2 < \gamma < 3/2$ *remain unknown.*

Let $\mathcal{A}(x) = (a_1(x), \dots, a_d(x))$ be a magnetic vector potential with real valued entries $a_k \in L^2_{\text{loc}}(\mathbb{R}^d)$ and let

$$H(\mathcal{A}) = (i\,\nabla + \mathcal{A})^2 - V\,,$$

where $V \geq 0$ is a real-valued function.

Denote the ratio of $2/3\sqrt{3}$ and the semi-classical constant by

$$R := \frac{2}{3\sqrt{3}} \times \left(\frac{2}{3\pi}\right)^{-1} = 1.8138\dots.$$

By using the Aizenmann-Lieb argument [1], a "lifting" with respect to dimension [6], [5], and Theorem 1 we obtain the following result:

Theorem 2. *For any* $\gamma \geq 1$ *and any dimension* $d \geq 1$*, the negative eigenvalues of the operator* $H(\mathcal{A})$ *satisfy inequalities*

$$\sum |\lambda_n|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V^{d/2+\gamma}(x)\, dx\,,$$

where

$$L_{d,\gamma} \leq R \times L^{cl}_{d,\gamma} = R \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1-|\xi|)^\gamma_+ \, d\xi\,.$$

Remark 3. *Theorem 2 allows us to improve on the estimates of best constants in Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials recently obtained in [4].*

2 One-dimensional generalised Sobolev inequality for matrices

Let $\{\phi_n\}_{n=0}^N$ be an ortho-normal system of vector-functions in $L^2(\mathbb{R}, \mathbb{C}^M)$, $M \in \mathbb{N}$,

$$(\phi_n, \phi_m) = (\phi_n, \phi_m)_{L^2(\mathbb{R},\mathbb{C}^M)} = \sum_{j=1}^M \int_{\mathbb{R}} \phi_n(x,j)\, \overline{\phi_m(x,j)}\, dx = \delta_{nm}\,,$$

where δ_{nm} is the Kronecker symbol. Let us introduce an $M \times M$ matrix U with entries

$$u_{j,k}(x,y) = \sum_{n=0}^{N} \phi_n(x,j)\, \overline{\phi_n(y,k)}\,.$$

Clearly

$$U^*(x,y) = U(y,x)\,. \tag{4}$$

The fact that the functions ϕ_n are orthonormal can be written in a compact form

$$\int_{\mathbb{R}} U(x,y)\, U(y,z)\, dy = U(x,z)\,. \tag{5}$$

The latter two properties (4) and (5) prove that $U(x,y)$ is the Schwartz kernel of an orthogonal projection P in $L^2(\mathbb{R},\mathbb{C}^M)$ whose image is the subspace of vector-functions spanned by $\{\phi_n\}_{n=1}^N$.

Theorem 3. *Let us assume that the vector-function ϕ_n, $n = 1,2,\dots N$, are from the Sobolev class $H^1(\mathbb{R},\mathbb{C}^M)$. Then*

$$\int_{\mathbb{R}} \mathrm{Tr}\left[U(x,x)^3\right] dx \leq \sum_{n=1}^{N}\sum_{j=1}^{M} \int_{\mathbb{R}} |\phi_n'(x,j)|^2\, dx\,.$$

Proof.

$$\begin{aligned} &\frac{d}{dy}\,\mathrm{Tr}\left[U(x,y)\,U(y,x)\,U(x,x)\right] \\ &= \mathrm{Tr}\left[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\right] + \mathrm{Tr}\left[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\right] \end{aligned} \tag{6}$$

By integrating (6) and taking absolute values one obtains

$$\begin{aligned} &\frac{1}{2}\,\mathrm{Tr}\left[U(x,z)\,U(z,x)\,U(x,x)\right] \\ &\qquad \leq \frac{1}{2}\int_{-\infty}^{z} \Big|\, \mathrm{Tr}\left[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\right] \\ &\qquad\qquad + \mathrm{Tr}\left[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\right]\Big|\, dy \end{aligned}$$

and

$$
\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}\,\Big[U(x,z)\,U(z,x)\,U(x,x)\Big]&\\
\leq \frac{1}{2}\int_z^\infty \Big|\,\mathrm{Tr}\,\Big[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\Big]&\\
+\,\mathrm{Tr}\,\Big[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\Big]\Big|\,dy\,.&
\end{aligned}
$$

Taking absolute values and adding the two inequalities yields for any $z\in\mathbb{R}$

$$
\begin{aligned}
\Big|\,\mathrm{Tr}\,\Big[U(x,z)\,U(z,x)\,U(x,x)\Big]\Big|&\\
\leq \frac{1}{2}\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[\Big(\frac{d}{dy}\,U(x,y)\Big)\,U(y,x)\,U(x,x)\Big]\Big|\,dy&\\
+\frac{1}{2}\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[U(x,y)\,\Big(\frac{d}{dy}\,U(y,x)\Big)\,U(x,x)\Big]\Big|\,dy\,.& \qquad (7)
\end{aligned}
$$

Note that we have reproved the inequality

$$
|f(x)|^2\leq\int_{\mathbb{R}}|f(y)\,f'(y)|\,dy
$$

for traces of matrices. By using properties of traces, the Cauchy-Schwarz inequality for matrix-functions and also properties (4) and (5), we find that for all $z\in\mathbb{R}$

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[\Big(\frac{d}{dy}\,U(x,y)\Big)\;U(y,x)\,U(x,x)\Big]\Big|\;dy\Big)^2\\
&\leq\int_{\mathbb{R}}\mathrm{Tr}\,\Big[\frac{d}{dy}\,U(x,y)^*\,\frac{d}{dy}\,U(x,y)\Big]\,dy\;\int_{\mathbb{R}}\mathrm{Tr}\,\big[U(x,y)^*\,U^2(x,x)\,U(x,y)\big]\,dy\\
&=\int_{\mathbb{R}}\mathrm{Tr}\,\Big[\frac{d}{dy}\,U(y,x)\,\frac{d}{dy}\,U(x,y)\Big]\,dy\;\int_{\mathbb{R}}\mathrm{Tr}\,\big[U^2(x,x)\,U(x,y)\,U(y,x)\big]\,dy\\
&=\int_{\mathbb{R}}\mathrm{Tr}\,\Big[\frac{d}{dy}\,U(x,y)\,\frac{d}{dy}\,U(y,x)\Big]\,dy\;\;\mathrm{Tr}\,\big[U(x,x)^3\big]\,,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}}\Big|\,\mathrm{Tr}\,\Big[U(x,y)\,\frac{d}{dy}\,U(y,x)\,U(x,x)\Big]\Big|\;dy\Big)^2\\
&\leq\int_{\mathbb{R}}\mathrm{Tr}\,\Big[\frac{d}{dy}\,U(x,y)\,\frac{d}{dy}\,U(y,x)\Big]\,dy\;\;\mathrm{Tr}\,\big[U(x,x)^3\big]\,.
\end{aligned}
$$

Thus, using this, and setting $x = z$ in (7), we arrive at

$$\Big| \operatorname{Tr} \Big[U(x,x)^3 \Big] \Big| \le \int_{\mathbb{R}} \operatorname{Tr} \left[\frac{d}{dy} U(x,y) \, \frac{d}{dy} U(y,x) \right] dy \, .$$

Integrating with respect to x we finally obtain

$$\begin{aligned} &\int_{\mathbb{R}} \Big| \operatorname{Tr} \Big[U(x,x)^3 \Big] \Big| \, dx \\ &\qquad \le \sum_{n,k=1}^{N} \sum_{i,j=1}^{M} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_n(x,i) \, \overline{\phi_n'(y,j)} \, \phi_k'(y,j) \, \overline{\phi_k(x,i)} \, dx \, dy \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \sum_{n=1}^{N} \sum_{j=1}^{M} \int_{\mathbb{R}} |\phi_n'(x,j)|^2 \, dx \, , \end{aligned}$$

which completes the proof. □

3 Lieb-Thirring inequalities for Schrödinger operators with matrix-valued potentials

Let us assume that $V \in C_0^\infty(\mathbb{R})$, $V \ge 0$, be a $M \times M$ Hermitian matrix-valued potential with entries $\{v_{ij}\}_{i,j=1}^M$. Then the negative spectrum of the Schrödinger operator $H = -\frac{d^2}{dx^2} - V$ in $L^2(\mathbb{R})$ is finite. For general potentials the result is obtained by an approximation argument.

Denote by $\{\phi_n\}$ the ortho-normal system of eigen-vector functions corresponding to the eigenvalues $\{\lambda_n\}_{n=1}^N$

$$-\frac{d^2}{dx^2} \, \phi_n - V \phi_n = \lambda_n \, \phi_n \, .$$

Clearly,

$$\sum_n \lambda_n = \sum_{n,j} \int_{\mathbb{R}} |\phi_n'(x,j)|^2 \, dx - \operatorname{Tr} \left[\int_{\mathbb{R}} V(x) \, U(x,x) \, dx \right]$$

and by Hölder's inequality for traces,

$$\int_{\mathbb{R}} \operatorname{Tr} \left[V(x) \, U(x,x) \right] \, dx \le \left(\int_{\mathbb{R}} \operatorname{Tr} \left[V^{3/2}(x) \right] dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}} \operatorname{Tr} \left[U(x,x)^3 \right] \, dx \right)^{\frac{1}{3}} ,$$

so that using Theorem 3

$$\sum_n \lambda_n \ge X - \left(\int_{\mathbb{R}} \operatorname{Tr} \left[V^{3/2}(x) \right] \, dx \right)^{\frac{2}{3}} \, X^{\frac{1}{3}}$$

with $X := \int_{\mathbb{R}} \mathrm{Tr}\, \left[U(x,x)^3\right] dx$. Minimising the right hand side with respect to X we finally complete the proof of Theorem 1

$$\sum_n \lambda_n \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \mathrm{Tr}\, \left[V^{3/2}(x)\right] dx\,.$$

Acknowledgements. The authors are grateful to the organisers of the meeting "Functional Inequalities: Probability and PDE's", Université Paris-X, June 4-6, 2007, where this paper came to fruition. We would like to thank Robert Seiringer for pointing out an omission in the formulation of Theorem 2. A.L. thanks the Department of Mathematics of the University Paris Dauphine for its hospitality and also the ESF Programme SPECT. M. L. would like to acknowledge partial support through NSF grant DMS-0600037.

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September 4, 2007