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Lieb–Thirring Inequalities for Schrödinger Operators with Complex-valued Potentials

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Abstract. Inequalities are derived for power sums of the real part and the modulus of the eigenvalues of a Schrödinger operator with a complex-valued potential.

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1. Introduction

A motivation for the present paper was a challenge posed in a lecture by E.B. Davies about *non*-self-adjoint Schrödinger operators in $L^2(\mathbb{R}^d)$:

$$H = -\Delta + V(x). \tag{1}$$

where V is a complex-valued potential (see also the papers [1,2]). Theorem 4 in [1] states that when $d=1$ every eigenvalue λ of H that does not lie on the positive real axis satisfies

$$|\lambda| \leqslant \frac{1}{4}\left(\int_{\mathbb{R}} |V(x)|\ \mathrm{d}x\right)^2. \tag{2}$$

We note that the constant 1/4 in this inequality is optimal. The question was raised whether an estimate similar to (2) holds in dimension $d \geqslant 2$.

While we do not answer the question directly, we have succeeded in finding a version of the Lieb–Thirring inequality for the eigenvalue power sums (Riesz means) that holds for this non-self-adjoint operator. Since little is known about non-self-adjoint operators relative to self-adjoint operators, our results may be worth recording. The proofs are easy, but not entirely obvious.

We denote by λ_j, $j = 1, 2, 3, \ldots$, a listing of the (countably many) eigenvalues of H in the cut plane $\mathbb{C}\backslash[0,\infty)$, repeated according to their *algebraic* multiplicities. An eigenvalue is a solution to the equation $H\psi = \lambda\psi$ for some $\psi \in L^2$. A given number $\lambda \in \mathbb{C}$ may occur several times in this list of eigenvalues according to the dimension of the generalized eigenspace $\{\psi : (H-\lambda)^k\psi = 0 \text{ for some } k \in \mathbb{N}\}$, which is called the algebraic multiplicity. In principle a generalized eigenspace could have infinite dimension, but, as we shall see, this will not occur in the situations considered here.

Note that the dimension of a generalized eigenspace may be strictly larger than the number of linearly independent solutions of $H\psi = \lambda\psi$, i.e., the geometric multiplicity of λ. The algebraic multiplicity is known to be finite for sufficiently decaying potentials as a consequence of Weyl's theorem, but we do not need this fact in our proof; a simple corollary of our theorems is that the multiplicity is automatically finite when the appropriate power of the potential is integrable.

It is a pleasure to acknowledge some very fruitful discussions with Prof. E.B. Davies about this paper, especially with regard to the question of multiplicities. Our original version was formulated in terms of geometric multiplicities instead of algebraic multiplicities because we needed to use the actual eigenfunctions of H, and these exist only with geometric multiplicity. He pointed out that it is only necessary in our proof to have basis functions in the generalized eigenspace with eigenvalue λ such that $(\phi, H\phi) = \lambda(\phi,\phi) = \lambda\|\phi\|^2$.

Before stating our main results let us recall the standard Lieb–Thirring inequalities (see [5] and also the survey [4]). For real-valued potentials V one has the bound

$$\sum_j (\lambda_j)_-^\gamma \leqslant L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\, \mathrm{d}x \tag{3}$$

provided $\gamma \geqslant 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geqslant 0$ if $d \geqslant 3$. (Here and in the sequel $t_- := \max\{0, -t\}$ denotes the negative part of t.) By $L_{\gamma,d}$ we will always mean the *sharp* constant in (3) (which at present is only known for $\gamma = 1/2$ if $d = 1$ and for $\gamma \geqslant 3/2$ if $d \geqslant 1$, see [4]).

For general, complex-valued potentials we shall prove

THEOREM 1. Eigenvalue sums. *Let* $d \geqslant 1$ *and* $\gamma \geqslant 1$.

1. *For eigenvalues with non-positive real parts*

$$\sum_{\Re\lambda_j<0} (-\Re\lambda_j)^\gamma \leqslant L_{\gamma,d} \int_{\mathbb{R}^d} (\Re V(x))_-^{\gamma+d/2}\, \mathrm{d}x. \tag{4}$$

2. *If $\varkappa > 0$, then for eigenvalues outside the cone* $\{|\Im z| < \varkappa \Re z\}$,

$$\sum_{|\Im\lambda_j| \geqslant \varkappa \Re\lambda_j} |\lambda_j|^\gamma \leqslant C_{\gamma,d}(\varkappa) \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, \mathrm{d}x. \tag{5}$$

Here $L_{\gamma,d}$ is the same as the constant in (3) *and*
$C_{\gamma,d}(\varkappa) = 2^{1+\gamma/2+d/4}\left(1+\frac{2}{\varkappa}\right)^{\gamma+d/2} L_{\gamma,d}$.

As a consequence we obtain

COROLLARY 1. *Let $d \geqslant 1$ and $\gamma \geqslant 1$.*

1. *For eigenvalues with non-positive real parts*

$$\sum_{\Re\lambda_j < 0} |\lambda_j|^\gamma \leqslant C_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\, \mathrm{d}x. \tag{6}$$

2. *If $\varkappa > 0$, then for eigenvalues inside the cone* $\{|\Im z| \leqslant -\varkappa \Re z\}$

$$\sum_{|\Im\lambda_j| \leqslant -\varkappa \Re\lambda_j} |\lambda_j|^\gamma \leqslant L_{\gamma,d}(\varkappa) \int_{\mathbb{R}^d} (\Re V(x))_-^{\gamma+d/2}\, \mathrm{d}x. \tag{7}$$

Here $C_{\gamma,d} = 2^{1+\gamma/2+d/4} L_{\gamma,d}$ *and* $L_{\gamma,d}(\varkappa) = (1+\varkappa) L_{\gamma,d}$.

It is natural to conjecture that the estimates in Theorem 1 and Corollary 1 hold for all values of γ for which (3) holds, and not only for $\gamma \geqslant 1$.

The proof below shows that $|V(x)|$ in the bounds (5) and (6) can actually be replaced by $(1/\sqrt{2})\,((\Re V(x))_- + |\Im V(x)|)$.

Remark 1. We can replace $-\Delta$ in H by $(i\nabla + A(x))^2$, where A is an arbitrary, real vector-field. This replacement is valid for the usual (self-adjoint) Lieb–Thirring inequality (3), and so it is valid here because we use only the self-adjoint Lieb–Thirring inequality in our proof of the theorem. If $d=1$ or if $\gamma \geqslant 3/2$ the constant in (3) (and hence in Theorem 1 and Corollary 1) remains the same as in the case $A=0$. In general it is not known whether the constant $L_{\gamma,d}$ in (3) has to be increased when the A is added. It is a fact, however, that all known proofs of the Lieb–Thirring inequality (without the, as yet unknown, sharp constant) do not require an increase in the constant.

Remark 2. We can also replace $-\Delta$ in H by any operator for which Lieb–Thirring bounds for real–valued potentials hold (but making the appropriate change in the exponent of V on the right side of the inequalities). For example, we can replace $-\Delta$ in H by the “relativistic” operator $|i\nabla + A(x)|$, in which case $\gamma + d/2$ has to be replaced by $\gamma + d$.

We now state bounds on single eigenvalues. Let us denote by $L^1_{\gamma,d}$ the sharp constant in the inequality

$$(\inf \operatorname{spec}(-\Delta+V))^{\gamma}_{-} \leqslant L^1_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma+d/2}_{-}\,\mathrm{d}x \tag{8}$$

for real-valued potentials V. This estimate holds under the same condition on γ as (3) and one has, of course, $L^1_{\gamma,d} \leqslant L_{\gamma,d}$. The sharp value of $L^1_{\gamma,d}$ is known for $\gamma \geqslant 1/2$ if $d=1$ and for $\gamma=0$ if $d \geqslant 3$. Note that $L^1_{\gamma,d}$ plays a role in the Lieb–Thirring conjecture, see [5].

THEOREM 2. Bounds on single eigenvalues. *Let $\gamma \geqslant 1/2$ if $d=1$, $\gamma>0$ if $d=2$ and $\gamma \geqslant 0$ if $d \geqslant 3$.*

1. *For any eigenvalue with non-positive real part*

$$(-\Re\lambda_j)^{\gamma} \leqslant L^1_{\gamma,d} \int_{\mathbb{R}^d} (\Re V(x))^{\gamma+d/2}_{-}\,\mathrm{d}x \tag{9}$$

and

$$|\lambda_j|^{\gamma} \leqslant C^1_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\,\mathrm{d}x. \tag{10}$$

2. *For any eigenvalue with non-negative real part*

$$|\lambda_j|^{\gamma} \leqslant C^1_{\gamma,d} \left(1+\frac{2\Re\lambda_j}{|\Im\lambda_j|}\right)^{\gamma+d/2} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\,\mathrm{d}x. \tag{11}$$

Here $L^1_{\gamma,d}$ is the same as the constant in (8) *and $C^1_{\gamma,d} = 2^{\gamma/2+d/4} L^1_{\gamma,d}$.*

Remark 3. This theorem yields a region in $\mathbb{C}$ in which there are no eigenvalues. This region is far from optimal; in particular, it does not approach the positive real axis as λ gets large. The paper [2] of Davies–Nath has a much better result for $d=1$.

2. Proof of Theorem 1

The core of Theorem 1 is contained in

LEMMA 1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be an arbitrary finite family of eigenvalues of H in $\mathbb{C}\setminus[0,\infty)$.* (*In the case of algebraic multiplicity $k>1$ a given number $\lambda \in \mathbb{C}$ might occur several times in our family, but no more than k times.*) *Then, for any $\alpha \in \mathbb{R}$ and $\gamma \geqslant 1$,*

$$\sum_{j=1}^{N} (\Re\lambda_j + \alpha\Im\lambda_j)^{\gamma}_{-} \leqslant \operatorname{Tr}(-\Delta+\Re V+\alpha\Im V)^{\gamma}_{-}. \tag{12}$$

Proof. We begin with the case $\gamma = 1$. Let $\alpha \in \mathbb{R}$. By removing some of the λ_j we can assume without loss of generality that $-\Re\lambda_j - \alpha\Im\lambda_j > 0$ for all $1 \leqslant j \leqslant N$.

Special attention must be given to a number λ that occurs several times in our list owing to an algebraic multiplicity >1. Suppose that this λ occurs k times (while the algebraic multiplicity is $\geqslant k$). We can always find orthonormal functions $\varphi_1, \dots, \varphi_k$ in the invariant subspace belonging to λ such that $H\varphi_1 = \lambda\varphi_1$ and

$$H\varphi_j = \lambda_j \varphi_j + \sum_{k<j} \alpha_{kj}\varphi_k. \tag{13}$$

This is the upper triangular representation familiar from elementary linear algebra.

The collection of all the φ_j for the different eigenvalues in our family yields N linearly independent functions, which we denote by ψ_j, $j = 1, \dots, N$. We introduce the function of N variables in $\mathbb{R}^d$

$$\Psi(x_1, \dots, x_N) := \det\left(\psi_j(x_k)\right), \qquad (x_1, \dots, x_N) \in \mathbb{R}^{dN}.$$

The linear independence of the ψ_j implies that $\Psi \not\equiv 0$. An easy calculation using (13) shows that

$$\sum_{j=1}^{N} \lambda_j \int_{\mathbb{R}^{dN}} |\Psi|^2 \, \mathrm{d}x_1 \dots \mathrm{d}x_N = \sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} \left(|\nabla_j \Psi|^2 + V(x_j)|\Psi|^2\right) \mathrm{d}x_1 \dots \mathrm{d}x_N,$$

where ∇_j denotes the gradient with respect to the variable x_j. Taking the real part in this relation we find that

$$\|\Psi\|^2 \sum_{j=1}^{N} \Re\lambda_j = (\Psi, H^{(N)}\Psi),$$

where $H^{(N)} := \sum_{j=1}^{N} (-\Delta_j + \Re V(x_j))$ acting on antisymmetric functions in $L^2(\mathbb{R}^{dN})$. A similar equality holds for $\Im\lambda_j$. Adding these two equations we have that

$$\|\Psi\|^2 \sum_{j=1}^{N} (\Re\lambda_j + \alpha\Im\lambda_j) = (\Psi, \tilde{H}^{(N)}\Psi), \tag{14}$$

where now $\tilde{H}^{(N)} := \sum_{j=1}^{N} (-\Delta_j + \Re V(x_j) + \alpha\Im V(x_j))$.

The variational principle together with (14) implies

$$\sum_{j=1}^{N} (\Re\lambda_j + \alpha\Im\lambda_j) \geqslant \inf \operatorname{spec}\left(\tilde{H}^{(N)}\right) \geqslant -\operatorname{Tr}(-\Delta + \Re V + \alpha\Im V)_-.$$

This proves (12) in the case $\gamma = 1$.

Now we reduce the case $\gamma > 1$ to the previous one following an idea of Aizenman–Lieb in [3]. There is a constant C_γ such that

$$C_\gamma s_-^\gamma = \int_0^\infty t^{\gamma-2}(s+t)_- \, \mathrm{d}t. \tag{15}$$

(Indeed, C_γ can be expressed in terms of the beta function, but we will not need this.) Hence

$$C_\gamma \sum_{j=1}^{N} (\Re\lambda_j + \alpha\Im\lambda_j)_-^\gamma = \int_0^\infty t^{\gamma-2} \sum_{j=1}^{N} (\Re\lambda_j(t) + \alpha\Im\lambda_j(t))_- \,\mathrm{d}t$$

where $\lambda_j(t) := \lambda_j + t$. The numbers $\lambda_j(t)$ are the eigenvalues of the operator $-\Delta + V_t$, $V_t(x) := V(x) + t$. Applying the result for $\gamma = 1$ that we have already proved we get

$$\sum_{j=1}^{N} (\Re\lambda_j(t) + \alpha\Im\lambda_j(t))_- \leqslant \mathrm{Tr}(-\Delta + \Re V_t + \alpha\Im V_t)_- = \mathrm{Tr}(-\Delta + \Re V + \alpha\Im V + t)_-.$$

Using (15) once more we conclude that

$$C_\gamma \sum_{j=1}^{N} (\Re\lambda_j + \alpha\Im\lambda_j)_-^\gamma \leqslant \int_0^\infty t^{\gamma-2} \mathrm{Tr}(-\Delta + \Re V + \alpha\Im V + t)_- \,\mathrm{d}t = \\ = C_\gamma \,\mathrm{Tr}(-\Delta + \Re V + \alpha\Im V)_-^\gamma,$$

as claimed. □

Now everything is in place for the

Proof of Theorem 1. The estimate (4) follows immediately from (12) with $\alpha = 0$ and (3).

To obtain the estimate (5) we apply (12) with $\alpha = 1 + \frac{2}{\varkappa}$, considering only those eigenvalues with $\Im\lambda_j \leqslant 0$ and $\varkappa\Re\lambda_j \leqslant -\Im\lambda_j$. (If there are infinitely many eigenvalues we consider a finite subset and pass to the limit). We get

$$\sum_{\substack{\Im\lambda_j \leqslant 0, \\ \varkappa\Re\lambda_j \leqslant -\Im\lambda_j}} \left(-\Re\lambda_j - \left(1 + \frac{2}{\varkappa}\right)\Im\lambda_j\right)^\gamma \leqslant \mathrm{Tr}\left(-\Delta + \Re V + \left(1 + \frac{2}{\varkappa}\right)\Im V\right)_-^\gamma.$$

Now replace V in this inequality by its complex conjugate $\overline{V}$ and note that the eigenvalues of the operator $-\Delta + \overline{V}$ are $\overline{\lambda_j}$. Hence

$$\sum_{\substack{\Im\lambda_j \geqslant 0, \\ \varkappa\Re\lambda_j \leqslant \Im\lambda_j}} \left(-\Re\lambda_j + \left(1 + \frac{2}{\varkappa}\right)\Im\lambda_j\right)^\gamma \leqslant \mathrm{Tr}\left(-\Delta + \Re V - \left(1 + \frac{2}{\varkappa}\right)\Im V\right)_-^\gamma.$$

Adding the two previous relations yields

$$\sum_{\varkappa\Re\lambda_j \leqslant |\Im\lambda_j|} \left(-\Re\lambda_j + \left(1 + \frac{2}{\varkappa}\right)|\Im\lambda_j|\right)^\gamma \leqslant \\ \leqslant \mathrm{Tr}\left(-\Delta + \Re V + \left(1 + \frac{2}{\varkappa}\right)\Im V\right)_-^\gamma + \\ + \mathrm{Tr}\left(-\Delta + \Re V - \left(1 + \frac{2}{\varkappa}\right)\Im V\right)_-^\gamma.$$

Now (5) follows from (3) by means of the elementary inequality $\sqrt{a^2+b^2} \leqslant a + b \leqslant \sqrt{2}\sqrt{a^2+b^2}$ for $a, b \geqslant 0$ and the bound

$$-\Re\lambda_j + \left(1+\frac{2}{\varkappa}\right)|\Im\lambda_j| \geqslant |\Re\lambda_j| + |\Im\lambda_j|$$

provided $\varkappa\Re\lambda_j \leqslant |\Im\lambda_j|$. □

Proof of Corollary 1. The estimate (6) follows from (5) by letting $\varkappa \to \infty$, and the estimate (7) follows by noting that $|\Re\lambda_j| + |\Im\lambda_j| \leqslant (1+\varkappa)|\Re\lambda_j|$ provided $-\varkappa\Re\lambda_j \geqslant |\Im\lambda_j|$. □

3. Proof of Theorem 2

We proceed similarly as in the proof of the previous theorem. Let ψ_j be an eigenfunction corresponding to an eigenvalue λ_j. Considering the real and imaginary parts of the equation

$$\int \left(|\nabla\psi_j|^2 + V|\psi_j|^2\right) \mathrm{d}x = \lambda_j \int |\psi_j|^2 \,\mathrm{d}x$$

we find that for any $\alpha \in \mathbb{R}$

$$\int \left(|\nabla\psi_j|^2 + \Re V|\psi_j|^2 + \alpha\Im V|\psi_j|^2\right) \mathrm{d}x = (\Re\lambda_j + \alpha\Im\lambda_j)\int |\psi_j|^2 \,\mathrm{d}x.$$

The variational principle implies

$$\inf \operatorname{spec}(-\Delta + \Re V + \alpha\Im V) \leqslant \Re\lambda_j + \alpha\Im\lambda_j.$$

The estimates (9), (10) for eigenvalues with non-positive real part follow now with the choices $\alpha = 0$ and $\alpha = -\operatorname{sign}\Im\lambda_j$, respectively, from (8). Similarly, (11) for eigenvalues with non-negative real part is obtained by the choice $\alpha = (-\operatorname{sign}\Im\lambda_j)\left(1+\frac{2\Re\lambda_j}{|\Im\lambda_j|}\right)$.

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