Singular factorizations, self-adjoint extensions and applications to quantum many-body physics

Edwin Langmann[1], Ari Laptev[2] and Cornelius Paufler[3]

[1] Mathematical Physics, KTH Physics, AlbaNova, SE-106 91 Stockholm, Sweden
[2] Department of Mathematics, KTH, SE-100 44 Stockholm, Sweden
[3] Theoretische Elementarteilchen-Physik, Institut für Physik, Johannes-Gutenberg-Universität, D-55099 Mainz, Germany

Abstract

We study self-adjoint operators defined by factorizing second-order differential operators in first-order ones. We discuss examples where such factorizations introduce singular interactions into simple quantum-mechanical models such as the harmonic oscillator or the free particle on the circle. The generalization of these examples to the many-body case yields quantum models of *distinguishable* and interacting particles in one dimensions which can be solved explicitly and by simple means. Our considerations lead us to a simple method to construct exactly solvable quantum many-body systems of Calogero–Sutherland type.

PACS numbers: 02.30.Ik, 03.65.Db

1. Introduction

In this paper, we define and study peculiar self-adjoint extensions of simple differential operators like that defining the quantum-mechanical harmonic oscillator. We also find the explicit solution of some of the resulting quantum-mechanical models and discuss their physical interpretations. These examples provide amusing illustrations that the mathematical theory of Schrödinger operators [K, RS1, RS2, BS], and in particular the distinction between formally Hermitian and self-adjoint operators, can have important physical consequences even in elementary quantum mechanics. Moreover, these examples allow a straightforward generalization to the case of an arbitrary number of particles, and the resulting many-body models describe strongly interacting systems which can be solved by elementary means. These models include, as special cases, those studied by Girardeau [G], which have received considerable interest recently since it has become possible to realize and study them experimentally [P]. We stress that these models can be solved even in the general case of *distinguishable* particles and that they can be studied by much simpler methods than other well-known integrable quantum many-body systems [LL, Y, C2, S1, S2]. Moreover,

our considerations lead us to a remarkably simple and general approach to exactly solvable quantum many-body systems of Calogero–Sutherland type [C2, S1, S2, OP] which seems new.

The starting point for this work was a paradox for the differential operator $-\partial_x^2 + x^2$ defining quantum harmonic oscillator, $-\infty < x < \infty$ and $\partial_x := \partial/\partial x$. The paradox arises if one confronts the formal factorization

$$-\partial_x^2 + x^2 = \left(\partial_x - x + \frac{1}{x}\right)\left(-\partial_x - x + \frac{1}{x}\right) + 3 \equiv Q^*Q + 3 \tag{1}$$

and $Q^*Q \geqslant 0$, with the well-known fact that the groundstate eigenvalue of the harmonic oscillator Hamiltonian $H = -\partial_x^2 + x^2$ is 1. It is interesting to note that a variant of this paradox was already discussed by Calogero in 1969; see [C1], second paragraph after equation (3.24).

The paradox above is only one of many, and they are resolved by observing that each such factorization gives rise to a different quantum-mechanical Hamiltonian defined by the Friedrichs extension. These examples have in common that a simple differential operator with an obvious factorization also allows for alternative factorizations where singularities are introduced, and this implies a restriction on the domain of functions on which the resulting self-adjoint operator is defined. This restriction has a natural physical interpretation as strong repulsive interaction.

We will present several examples leading to an interesting quantum-mechanical model with additional interactions and which are exactly solvable in the sense that all eigenfunctions and eigenvalues can be found explicitly and by simple means[4]. In particular, for each non-negative integer n we will find a quantum-mechanical Hamiltonian H_n which is defined by the differential operator of the harmonic oscillator differential operator $-\partial_x^2 + x^2$ but is bounded from below by $2n + 1$. Moreover, in addition to the standard harmonic oscillator $H = H_0$ also the Hamiltonian H_1 defined by the factorization in equation (1) is exactly solvable, while the other $H_{n \geqslant 2}$ are only partially solvable, i.e., only a finite number of eigenstates and eigenvalues can be found exactly. Two other examples are provided by the differential operator $-\partial_x^2 - \gamma(\gamma + 1)\cosh^{-2}(x)$ often referred to as Pöschl–Teller potential and the differential operator $-\partial_x^2$ for $-\pi < x < \pi$ describing a free particle on the circle. In the latter example we will show that, for every $\alpha > 0$, the factorization

$$-\partial_x^2 = (\partial_x + \alpha\cot(\alpha x))(-\partial_x + \alpha\cot(\alpha x)) + \alpha^2$$

defines a unique self-adjoint operator H_α bounded from below by α^2. Moreover, this Hamiltonian is exactly solvable if and only if α is an integer or half integer.

While we believe that these examples are of pedagogical interest, our first main result is that much of what we say can be naturally extended to the many-body case. This yields many-body systems describing distinguishable particles moving in one dimension and interacting through a strong local two-body interactions, and also these models can be solved exactly and by simple means.

Our considerations described above lead us, somewhat unexpectedly, to our second main result which is on Calogero–Sutherland type systems: as we will see, all our one-particle examples mentioned above are related to Schrödinger operators $H = -\partial_x^2 + V(x)$ which have exact eigenfunctions of the form

$$\psi_n(x) = P_n(\eta(x))\,\mathrm{e}^{-w(x)}, \qquad n = 0, 1, 2, \ldots, N_{\max}, \tag{2}$$

where P_n is a polynomial of order n, η and w are real-valued C^2 functions and $N_{\max} \leqslant \infty$. These quantum-mechanical models are exactly solvable in the sense that these eigenfunctions

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[4] To illustrate what we mean by this we recall that the spectrum of the Hamiltonian defined by the differential operator $-\partial_x^2$ on all C^∞-functions on an interval and general boundary conditions is given by a transcendental equation in general which can be solved only numerically. These examples are not exactly solvable in this sense.

Table 1. Solutions of the Schrödinger equation $(-\partial_x^2 + V(x) - E_n)\psi_n(x) = 0$ associated with classical orthogonal polynomials; adapted from table 22.6 in [AS].

	Name	$\psi_n(x)$	$V(x)$	E_n
I	Jacobi	$\left(\sin\frac{x}{2}\right)^\alpha \left(\cos\frac{x}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos x)$	$\frac{4\alpha^2-1}{16\sin^2\frac{x}{2}} + \frac{4\beta^2-1}{16\cos^2\frac{x}{2}}$	$\left(n + \frac{\alpha+\beta+1}{2}\right)^2$
II	Gegenbauer	$(\sin x)^\alpha C_n^{(\alpha)}(\cos x)$	$\frac{\alpha(\alpha-1)}{\sin^2 x}$	$(n+\alpha)^2$
III	Chebyshev	$T_n(\cos x)$	0	n^2
IV	Laguerre	$e^{-x^2/2} x^{(\alpha+1)/2} L_n^{(\alpha)}(x^2)$	$x^2 + \frac{4\alpha^2-1}{4x^2}$	$4n + 2\alpha + 2$

ψ_n and the corresponding eigenvalues E_n can be found explicitly. While we will restrict our discussions to our three examples, harmonic oscillator, Pöschl–Teller potential and free particle on the circle, it is important to note that there are many more: in table 1, we list the well-known examples associated with the classical orthogonal polynomials, but even this list is not complete. Our second main result is a surprisingly simple argument showing that, to *any* such model, the corresponding quantum many body given by the Schrödinger operator

$$H_N = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + V(x_j)\right) + \lambda(\lambda-1) \sum_{1 \leqslant j < k \leqslant N} W(x_j, x_k) \tag{3}$$

and the following particular two-body interaction

$$W(x, y) = \frac{\eta'(x)^2 + \eta'(y)^2}{(\eta(x) - \eta(y))^2} + \frac{2\eta'(x)w'(x) - \eta''(x) - 2\eta'(y)w'(y) + \eta''(y)}{\eta(x) - \eta(y)} - W_0 \tag{4}$$

defines an exactly solvable quantum many-body system; $\eta'(x) := \partial_x \eta(x)$, and W_0 is an arbitrary constant which can be fixed to some convenient value. More specifically, this argument leads to a simple proof of the following:

Proposition. *Let $H = -\partial_x^2 + V(x)$ be a self-adjoint Schrödinger operator on $L^2([a, b])$, $-\infty \leqslant a < b \leqslant \infty$, with exact eigenfunctions as specified in equation (2) ff. Then the corresponding many-body Schrödinger operator H_N in equations (3) and (4) defines a quantum many-body systems with the following exact groundstate:*

$$\Psi_0(x_1, \ldots, x_N) = e^{-\sum_{j=1}^{N} w(x_j)} \prod_{1 \leqslant j < k \leqslant N} (\eta(x_j) - \eta(x_k))^\lambda, \tag{5}$$

provided $N < N_{max} + 1$ and $\lambda > 0$ is such that Ψ_0 is square integrable. The corresponding eigenvalue is determined by the one-particle eigenvalues E_n as follows:

$$\mathcal{E}_0 = \lambda^2 \sum_{n=0}^{N-1} (E_n - E_0) + N E_0 - \frac{1}{2}\lambda(\lambda-1)N(N-1)W_0. \tag{6}$$

(A concise proof can be found in the last paragraph of section 6.)

In fact, we found that it is possible to compute all other bound states and corresponding eigenvalues for all these systems exactly and in a unified manner [HL]. As we will discuss in section 7, special cases of this result yield various well-known exactly solvable systems of the Calogero–Sutherland type [C2, S1, S2, OP, BF], but there are other interesting examples which we have not seen discussed in the physics literature.

We mention that the factorization of Schrödinger operators which we use also play a central role in supersymmetric quantum mechanics; see e.g. [CKS] and references therein.

The plan of this paper is as follows. In the following section, we resolve in detail the paradox described above using the theory of Friedrichs extensions. This provides a general recipe for finding other such examples. In section 3, we present our one-body examples, and in section 4, we discuss their physical interpretation. In section 5, we discuss the extension of our results to the many-body case. Our method to find quantum many-body systems with exact groundstates is explained in section 6. We end with a few remarks in section 7.

2. Factorizations and self-adjoint extensions

We first recall some standard facts from the operator theory in Hilbert spaces. Let A_0 be a symmetric operator in a separable Hilbert space $\mathcal{H}$ with scalar product $(\cdot, \cdot)$ (linear in the second argument) and norm $\|\cdot\|$, and let $\mathcal{D}(A_0)$ be the domain of the operator A_0 which is a dense subset in $\mathcal{H}$. We assume that A_0 is bounded from below, i.e., there exists a constant $E_0 \in \mathbb{R}$ such that

$$(A_0 u, u) \geqslant E_0 \|u\|^2, \qquad u \in \mathcal{D}(A_0).$$

The sesquilinear form $a_0[u, v] = (A_0 u, v)$ defined for $u, v \in d[a_0] = \mathcal{D}(A_0)$ can be completed with respect to the norm defined by

$$a_0[u, u] + (E_0 + 1)\|u\|^2. \tag{7}$$

Such a completion is denoted by a and its domain by $d[a]$. By standard arguments (see for example [BS]) there is a unique self-adjoint operator A corresponding to the closed form a whose domain $\mathcal{D}(A) \subset d[a]$. This operator is called the Friedrichs extension of the operator A_0.

Let V be a real function defined on open subset $\Omega \subseteq \mathbb{R}^N$, and let H be a self-adjoint Schrödinger operator in $L^2(\Omega)$ with some boundary conditions:

$$Hu(x) = -\Delta u(x) + V(x)u(x). \tag{8}$$

This operator generates a quadratic form $h[u, u]$ defined on functions from d. We assume that H has a discrete spectrum $\{E_n\}_{n=0}^K$, with the corresponding orthonormal in the $L^2(\Omega)$ system of eigenfunctions $\{u_n\}_{n=0}^K$ where $K \leqslant \infty$ (i.e. H may also have a continuous spectrum). It can easily be checked directly that the differential expression $-\Delta + V(x)$ can formally be factorized as

$$-\Delta + V = Q_n^* Q_n + E_n, \tag{9}$$

where

$$Q_n = -\nabla_x + \frac{\nabla u_n}{u_n}. \tag{10}$$

It is well known that the groundstate u_0 can be chosen strictly positive and factorization (9) does not give any problem. However, considering higher eigenfunctions we obtain singularities on the right-hand side of (10) due to the zero set S of u_n, $S = \{x : u_n(x) = 0\}$. In order to define the self-adjoint operator corresponding to the operator Q^*Q we begin with the quadratic form

$$h_n[u, u] = \int_\Omega \left| -\nabla_x u + \frac{\nabla u_n}{u_n} u \right|^2 \mathrm{d}^N x,$$

whose domain $d[h_n]$ consists of functions $u \in d$ such that $u(x) = 0$ for $x \in S$.

The corresponding Friedrichs extension gives us a self-adjoint operator which we denote by H_n. Although

$$H_n = Q_n^* Q_n + E_n = -\Delta + V \tag{11}$$

coincides with $H = -\Delta + V$ defined by (8) regarding its action on functions as a differential expression, its domain is very different and consists of functions not only just satisfying the boundary conditions of the original operator H but being equal to zero on S. This automatically makes the domain $d[h_n]$ smaller compared with the domain of the original operator H and 'lifts up' the spectrum.

In particular, this answers the question concerning the formal inequality which follows from (11), that is

$$H = Q_n^* Q_n + E_n \geqslant E_n.$$

The equality on the right-hand side of this latter equation is wrong because the domains of the operator H and $Q_n^* Q_n$ are different and therefore $H \neq Q_n^* Q_n + E_n$ as Hilbert space operators.

3. Some examples

Here we shall give some simple examples where the above construction can be worked out more explicitly.

3.1. Harmonic oscillator

Let H be the operator of the harmonic oscillator on $\mathbb{R}$,

$$H = -\partial_x^2 + x^2. \tag{12}$$

The eigenvalues and eigenfunctions of this operator are well known and equal to $E_n = 2n + 1$, $n = 0, 1, \ldots$, and

$$\psi_n(x) = \mathcal{H}_n(x)\, e^{-x^2/2}, \tag{13}$$

where $\mathcal{H}_n$ are Hermite polynomials,

$$\mathcal{H}_n(x) = (2^n n!)^{-1/2} \pi^{-1/4} (-1)^n\, e^{x^2} \partial_x^n\, e^{-x^2}.$$

As in (11) the operators H_n can be defined via

$$H_n = Q_n^* Q_n + E_n = \left(\partial_x + \frac{\psi_n'}{\psi_n} \right) \left(-\partial_x + \frac{\psi_n'}{\psi_n} \right) + E_n.$$

According to section 2, the latter expression coincides with $-\partial_x^2 + x^2$ as a differential expression but not as a self-adjoint operator. In particular, when $n = 0$,

$$\frac{\psi_0'}{\psi_0} = -x,$$

and we obtain the standard factorization of the harmonic oscillator

$$H_0 = -\partial_x^2 + x^2 = (\partial_x - x)(-\partial_x - x) + 1.$$

If $n = 1$ then

$$\frac{\psi_1'}{\psi_1} = -x + \frac{1}{x},$$

and we obtain the factorization

$$H_1 = -\partial_x^2 + x^2 = \left(\partial_x - x + \frac{1}{x} \right) \left(-\partial_x - x + \frac{1}{x} \right) + 3.$$

It is interesting to note that not only H_0 but also H_1 defines an exactly solvable system in the sense that all its eigenfunctions can be computed explicitly. Using that the eigenfunctions

$\psi_n(x)$ of H_0 in equation (13) are even and odd for even and odd integers n, respectively, we conclude that the eigenfunctions $u_n(x)$ of H_1 are given by

$$u_{2n+1}(x) = \psi_{2n+1}(x), \qquad u_{2n}(x) = \psi_{2n+1}(x)\,\text{sign}(x), \quad n = 0, 1, 2 \dots \tag{14}$$

with the usual sign function $\text{sign}(x)$: the odd eigenfunctions of H_1 are the same as for the standard harmonic oscillator, but the even ones are very different. The spectrum of H_1 thus has multiplicity two, $E_{2n} = E_{2n+1} = 4n + 3$.

The other exotic harmonic oscillator Hamiltonians $H_{n\geqslant 2}$ are only partially solvable: as discussed in section 4, we can determine the $(n + 1)$-fold degenerate groundstate of $H_{n\geqslant 2}$ explicitly, but it is not possible to compute its excited states analytically.

3.2. Pöschl–Teller potential

Let

$$H = -\partial_x^2 - \frac{\gamma(\gamma + 1)}{\cosh^2 x} \tag{15}$$

for $\gamma > 0$. It is well known that this operator has $[\gamma]$ (= largest integer smaller or equal to γ) square integrable eigenfunctions ψ_n, $n = 0, 1, \ldots, [\gamma] - 1$, with corresponding eigenvalues

$$E_n = -(\gamma - n)^2. \tag{16}$$

These eigenfunctions are given by

$$\psi_n(x) = \cosh^{-\gamma}(x) \sum_{0 \leqslant s \leqslant n/2} (-1)^s \frac{n!(\gamma - n)!}{4^s(n - 2s)!(\gamma + s - n)!s!} \sinh^{n-2s}(x) \tag{17}$$

with $r! := \Gamma(r + 1)$ for non-integers r, as can be checked by straightforward computations.

In particular, the lowest eigenvalue of this operator is $E_0 = -\gamma^2$ and the corresponding eigenfunction is $\psi_0(x) = \cosh^{-\gamma}(x)$. Then

$$\frac{\psi_0'}{\psi_0} = -\gamma \tanh(x)$$

and therefore one can obtain the following factorization,

$$H_0 = -\partial_x^2 - \frac{\gamma(\gamma + 1)}{\cosh^2 x} = (\partial_x - \gamma \tanh(x))(-\partial_x - \gamma \tanh(x)) - \gamma^2,$$

defining the standard self-adjoint operator $H_0 = H$ given by the differential operator $-\partial_x^2 + \gamma(\gamma + 1)\cosh^{-2}(x)$. The second eigenvalue is $E_1 = -(\gamma - 1)^2$, and the corresponding eigenfunction is given by $\psi_1(x) = \cosh^{-\gamma}(x)\sinh(x)$, for $\gamma > 1$. Then

$$\frac{\psi_1'}{\psi_1} = \coth(x) - \gamma \tanh(x)$$

which implies that

$$H_1 = -\partial_x^2 - \frac{\gamma(\gamma + 1)}{\cosh^2 x} = (\partial_x + \coth(x) - \gamma \tanh(x))(-\partial_x + \coth(x) - \gamma \tanh(x)) - (\gamma - 1)^2$$

which defines another self-adjoint operator H_1 and similarly for $H_{n\geqslant 2}$. Again it is possible to compute all eigenfunctions and eigenvalues of H_1 in terms of those of H_0, similarly as in the harmonic oscillator case.

3.3. Free particle on the circle

This example is slightly more subtle. For arbitrary $\alpha > 0$, we define a self-adjoint operator on $L^2([-\pi, \pi])$ by the factorization

$$H_\alpha = -\partial_x^2 = (\partial_x + \alpha \cot(\alpha x))(-\partial_x + \alpha \cot(\alpha x)) + \alpha^2$$

which is obtained with the odd eigenfunctions $\sin(\alpha x)$ of $-\partial_x^2$. The subtle point now concerns the correct boundary conditions at $x = \pm\pi$. According to our general theory in section 2, we need to start with the self-adjoint operator $H = -\partial_x^2$ with boundary conditions such that $\sin(\alpha x)$ is indeed its eigenfunction, and it is not difficult to see that these conditions are as follows,

$$u'(-\pi) = u'(\pi) \qquad \text{and} \qquad u(\pi) - u(-\pi) = -2\tan(\alpha\pi)u'(\pi), \qquad (18)$$

for functions u in the domain of H. Thus only for integers α this corresponds to periodic boundary conditions, and only in this case we can interpret H as Hamiltonian of a particle on a circle. We conclude that the eigenfunctions of H_α are defined by the same boundary conditions at $x = \pm\pi$ together with the requirements that they vanish at all points x where $\sin(\alpha x) = 0$.

If we interpret H_α as Hamiltonian of a particle moving on a circle, then we not only have infinitely strong repulsive interactions at the points x where $\sin(\alpha x)$ vanishes but, in addition, a singular interaction at the point $x = \pm\pi$ whose strength depends on α so that it vanishes at integer values of α and becomes infinitely strong if α is a half integer.

In particular, if $\alpha = 1/2$, then the operator

$$H_{1/2} = -\partial_x^2 = \left(\partial_x + \tfrac{1}{2}\cot(x/2)\right)\left(-\partial_x + \tfrac{1}{2}\cot(x/2)\right) + \tfrac{1}{4}$$

is defined on functions u satisfying a Dirichlet boundary condition at zero (from the singularity of cot at zero) and Neumann boundary conditions at $-\pi$ and π,

$$u(0) = u'(-\pi) = u'(\pi) = 0.$$

Our discussion above implies that, for integers and half integers α, one can determine all eigenfunctions and eigenvalues of H_α explicitly and by elementary computations, but otherwise part of the spectrum is determined by a transcendental equation: the eigenfunctions are determined by the conditions $u(k\pi/\alpha) = 0$ for an integer k and the boundary conditions in equation (18), and the latter are non-trivial unless 2α is an integer.

For example, for $\alpha = 1$ these eigenfunctions are given by

$$u_{\pm,n}(x) = \Theta(\pm x)\sin(nx)$$

with $n = 1, 2, \ldots$. The corresponding eigenvalues are n^2 and two-fold degenerate. For $\alpha = 1/2$, the explicit eigenfunctions are

$$u_{\pm,n} = \Theta(\pm x)\sin((n + 1/2)x) \qquad n = 0, 1, 2, \ldots$$

with the corresponding two-fold degenerate eigenvalues $(n + 1/2)^2$.

We finally mention that it is amusing to visualize the α-dependence of the singular interactions and the spectrum of the Hamiltonian H_α.

4. Physical interpretation

We first give a physics argument providing an intuitive resolution of the paradox mentioned after equation (1).

The paradox arises due to the singularity in the factorization in equation (1) at $x = 0$. One can regularize this singularity and consider the following family of Hamiltonians,

$$H^{(\varepsilon)} = \left(\partial_x - x + \frac{x}{x^2 + \varepsilon^2}\right)\left(-\partial_x - x + \frac{x}{x^2 + \varepsilon^2}\right) + 3,$$

with a regularization parameter $\varepsilon > 0$. By a straightforward computation one finds

$$H^{(\varepsilon)} = -\partial_x^2 + x^2 + \frac{\varepsilon^2(1 + 2\varepsilon^2 + 2x^2)}{(x^2 + \varepsilon^2)^2}.$$

Even though the second term vanishes for $\varepsilon = 0$, it is obviously relevant in the limit $\varepsilon \downarrow 0$. Recalling that $\delta_\varepsilon(x) = \varepsilon/(x^2 + \varepsilon^2)\pi$ is a sequence of C^∞ functions converging to the delta distribution as $\varepsilon \downarrow 0$, we see that this term gives an additional singular interaction of infinite strength at $x = 0$: formally, the second term converges to $\lim_{\varepsilon \downarrow 0}(\pi \delta_\varepsilon(x))^2 = \pi^2\delta(0)\delta(x) = \lim_{g \to +\infty} g\delta(x)$. This interaction forces the eigenfunctions of the Hamiltonian $\lim_{\varepsilon \downarrow 0} H^{(\varepsilon)}$ to vanish at $x = 0$. It is plausible that this latter Hamiltonian is identical with that defined by the factorization in (1), and this is indeed the case, as discussed below.

As shown in the previous section, the eigenfunctions of Hamiltonians $H = -\partial_x^2 + V(x)$ on $L^2([a, b])$, $-\infty \leqslant a < b \leqslant \infty$, defined by the factorization

$$H_n = -\partial_x^2 + V(x) = (\partial_x + u_n'(x)/u_n(x))(-\partial_x + u_n'(x)/u_n(x)) + E_n$$

can be fully characterized by the (simple) zeros $x_{n,j}$ of the eigenfunction u_n of $-\partial_x + V(x)$: $u_n(x_{n,j}) = 0$ for $j = 1, 2, \ldots, n$ implies that $u(x_{n,j}) = 0$ for all eigenfunctions u of H_n. One can thus interpret H_n as the strong coupling limit $g \to \infty$ of the following Hamiltonian:

$$H_n^{(g)} = -\partial_x^2 + V(x) + \sum_{j=1}^n g\delta(x - x_{n,j}).$$

Indeed, the eigenfunctions u of this latter Hamiltonian are defined by $-u'' + Vu = Eu$ for $x \neq x_{n,j}$, together with the boundary conditions

$$u(x_{n,j} + 0) = u(x_{n,j} - 0), \qquad u'(x_{n,j} + 0) - u'(x_{n,j} - 0) = gu(x_{n,j} + 0),$$

and for $g \to \infty$ the latter reduce to $u(x_{n,j}) = 0$. Thus, the factorization corresponds to adding infinitely strong repelling delta-potentials at the locations of the zeros of the eigenfunction u_n. This implies that the groundstate of H_n is $(n + 1)$-fold degenerate: all states $u_{0,j}(x)$ which are equal to $u_n(x)$ in an interval between two adjacent zeros $x_{n,j}$ and zero otherwise are eigenfunctions of H_n with the eigenvalue E_n. However, the excited states cannot be found in such a simple manner in general.

5. Exactly solvable many-body systems in 1D

We now show that the examples above allow a straightforward extension to the N-particle case, and this gives rise to exactly solvable interacting many-body systems of *distinguishable* particles.

5.1. Interacting particles in a harmonic oscillator potential

Our first example is for the operator

$$H_N = \sum_{j=1}^N \left(-\partial_{x_j}^2 + x_j^2\right) \tag{19}$$

on $\mathbb{R}^N$. This self-adjoint operator is defined by the standard factorization

$$H_N = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + x_j^2\right) = \sum_{j=1}^{N} \left(\partial_{x_j} - x_j\right)\left(-\partial_{x_j} - x_j\right) + N,$$

and it describes an arbitrary number, N, of non-interacting, identical particles moving in an external harmonic oscillator potential. The eigenfunctions of H_N obviously are given by products of the one-particle eigenfunctions defined in section 3.1:

$$\psi_{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{j=1}^{N} \psi_{n_j}(x_j), \qquad n_j = 0, 1, 2, \ldots,$$

where we use the shorthand notation $\boldsymbol{x} = (x_1, \ldots, x_N)$ and similarly for $\boldsymbol{n}$, and the eigenvalue is

$$\mathcal{E}_{\boldsymbol{n}} = \sum_{j=1}^{N}(2n_j + 1). \tag{20}$$

Using the huge degeneracy of these eigenfunctions we can construct eigenfunctions of H_N which vanish at all coinciding points $x_j = x_k$ by antisymmetrization:

$$f_{\boldsymbol{n}}(\boldsymbol{x}) = \sum_{P \in S_N} (-1)^{|P|} \prod_{j=1}^{N} \psi_{n_j}(x_{Pj}), \qquad 0 \leqslant n_1 < n_2 < \cdots < n_N < \infty. \tag{21}$$

These are exactly the fermion eigenfunctions of H_N, of course, exchanging any two particles gives a minus sign. The fermion wavefunction with the lowest eigenvalue is

$$f_0(\boldsymbol{x}) = \text{const} \det_{1 \leqslant j,k \leqslant N} (\psi_{j-1}(x_k)), \tag{22}$$

where const is some irrelevant nonzero constant which we will later fix to some convenient value. To compute f_0 we use the explicit form of the harmonic oscillator eigenfunctions in equation (13):

$$f_0(\boldsymbol{x}) = e^{-\sum_j x_j^2/2} \det_{1 \leqslant j,k \leqslant N} (\mathcal{H}_{j-1}(x_k)).$$

Since the Hermite polynomials are of the form $\mathcal{H}_n(x) = c_n x^n + \cdots$ with dots some linear combinations of $\mathcal{H}_{m<n}$ and c_n nonzero constants, we can replace in this determinant $\mathcal{H}_{j-1}(x_k)$ by $c_{j-1} x_k^{j-1}$, and a well-known identity of the Vandermonde determinant implies that

$$f_0(\boldsymbol{x}) = e^{-\sum_{j=1}^{N} x_j^2/2} \prod_{1 \leqslant j < k \leqslant N} (x_j - x_k)$$

is the fermion groundstate of the operator H_N with the eigenvalue $\sum_{j=0}^{N-1}(2j+1) = N^2$ (we fixed const so as to absorb factors of c_j and a possible sign). Then obviously

$$\frac{1}{f_0(\boldsymbol{x})} \frac{\partial}{\partial x_j} f_0(\boldsymbol{x}) = -x_j + \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{x_j - x_k}.$$

We thus conclude that the differential operator defining H_N can also be factorized as follows:

$$H_{N,1} = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + x_j^2\right) = \sum_{j=1}^{N} Q_j^* Q_j + N^2, \qquad Q_j = -\partial_{x_j} - x_j + \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{x_j - x_k}, \tag{23}$$

and this defines another self-adjoint operator.

We now discuss how to determine all eigenfunctions $f(x)$ of $H_{N,1}$; see [G] for a similar discussion. As in section 2, we conclude that the eigenfunctions have to vanish at coinciding points,

$$f(x_1, x_2, \ldots, x_N) = 0 \qquad \text{if} \quad x_j = x_k \qquad \text{for some} \quad j \neq k, \quad j, k = 1, 2, \ldots, N,$$

(24)

and thus all fermion eigenfunctions f_n of H_N in equation (21) are also eigenfunctions of $H_{N,1}$ with the same eigenvalues. It is not difficult to see that all these are C^∞ eigenfunctions of $H_{N,1}$, but they obviously do not span the full Hilbert space $L^2(\mathbb{R}^N)$ but only its fermion subspace. To find the missing eigenfunctions we divide the space $\mathbb{R}^N$ in fundamental domains which can be parameterized by permutations $P \in S_N$:

$$\Delta_P := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \, | -\infty < x_{P1} < x_{P2} < \cdots < x_{PN} < \infty\}.$$

(25)

There are $N!$ distinct such domains, and the closure of the union of all these domains equals full space, $\overline{\cup_{P \in S_N} \Delta_P} = \mathbb{R}^N$. Moreover, the boundaries of these domains correspond to the hyperplanes $x_j = x_k$. We thus need to find all functions which are C^∞ and obey the eigenvalue equation for H_N in the interior of all these domains Δ_P and vanish at all their boundaries. The functions in equation (21) obey both these conditions, but since the derivatives of eigenfunctions need not be continuous at coinciding point also the functions

$$f_n^P(x) = \begin{cases} f_n(x) & \text{for } x \in \Delta_P \\ 0 & \text{otherwise,} \end{cases} \qquad 0 \leqslant n_1 < n_2 < \cdots < n_N < \infty, \quad P \in S_N \qquad (26)$$

are legitimate eigenfunctions: they obey the correct eigenvalue equation with the eigenvalue in equation (20) at non-coinciding points, and they also obey all boundary conditions in equation (24). It is not difficult to verify that these eigenfunctions $f_n(x)$ span the full Hilbert space $L^2(\mathbb{R}^N)$.

Similarly as discussed in section 4, we can interpret $H_{N,1}$ as the strong coupling limit, $g \to \infty$, of the following Hamiltonian:

$$H_N^{(g)} = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + x_j^2 \right) + g \sum_{1 \leqslant j < k \leqslant N} \delta(x_j - x_k)$$

describing N particles moving on the real line in the harmonic oscillator potentials and interacting with two-body delta function potentials. This corresponds to the fact that the quadratic form of the operator $H_{N,1}$ is split into $N!$ independent problems in $L^2(\Delta_P)$ with Dirichlet boundary conditions.

5.2. Interacting particles in a Pöschl–Teller potential

We now consider the operator

$$H_N = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 - \gamma(\gamma + 1) \frac{1}{\cosh^2(x_j)} \right)$$

(27)

describing N non-interacting particles in a weakly confining $\cosh^{-2}$-potential. For $\gamma > N$ the fermion groundstate of this Hamiltonian is f_0 as in (22) but now with the eigenfunctions ψ_n in (17), and by an argument as in section 5.1 we conclude that

$$f_0(x) = \prod_{j=1}^{N} \cosh^{-\gamma}(x_j) \prod_{1 \leqslant j < k \leqslant N} (\sinh(x_j) - \sinh(x_k))$$

with the corresponding eigenvalue

$$\mathcal{E}_0 = -\sum_{n=0}^{N-1}(\gamma - n)^2 = -\frac{N}{6}(2N-1)(N-1) + \gamma N(N-1) - N\gamma^2.$$

We thus can define the following Hamiltonian different from H_N:

$$H_{N,1} = \sum_{j=1}^{N} Q_j^* Q_j + \mathcal{E}_0 \tag{28}$$

with

$$Q_j = -\partial_{x_j} + \frac{1}{f_0}\partial_{x_j} f_0 = -\partial_{x_j} - \gamma \tanh(x_j) + \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{\cosh(x_j)}{\sinh(x_j) - \sin(x_k)}. \tag{29}$$

As in section 5.1, we can interpret $H_{N,1}$ as the strong coupling limit of the Hamiltonian H_N in (27) with two-body delta interactions enforcing that the eigenfunctions vanish at coinciding points $x_j = x_k$, and all eigenvalues and eigenfunctions of this interacting many-body Hamiltonian can be determined explicitly from the fermion wavefunctions and corresponding eigenvalues of the non-interacting Hamiltonian H_N. The example here is, however, quite interesting from a physical point of view since, in addition to many-body bound states, there are also scattering states and, in general, one has mixed states where only some of the particles are trapped and the others are free.

5.3. Interacting particles on the circle

Let

$$H_N = -\Delta = -\sum_{j=1}^{N} \partial_{x_j}^2 \tag{30}$$

on N-dimensional torus $\mathbb{T}^N$. The right-hand side of (30) can be considered as a trivial factorization of the periodic Laplace operator. We note that

$$\psi_n(x) = (e^{ix})^n e^{-i(N-1)x/2}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

all are eigenfunctions of the corresponding one-particle Hamiltonian $-\partial_x^2$ with the eigenvalue $\left(n - \frac{1}{2}(N-1)\right)^2$, and thus f_0 as defined in (22) is an fermion eigenstate of operator (30) on $\mathbb{T}^N$ with the eigenvalue $\sum_{n=0}^{N-1}\left(n - \frac{1}{2}(N-1)\right)^2 = \frac{1}{12}N(N^2-1)$. By straightforward computations we find that

$$f_0(\boldsymbol{x}) = \prod_{1 \leqslant j < k \leqslant N} \sin\frac{1}{2}(x_j - x_k),$$

and therefore, by analogy with (23), we can consider the factorization

$$H_{N,1} = -\sum_{j=1}^{N} \partial_{x_j}^2 = \sum_{j=1}^{N} Q_j^* Q_j + \frac{1}{12}N(N^2 - 1), \tag{31}$$

where

$$Q_j = -\partial_{x_j} - \frac{1}{f_0}\partial_{x_j} f_0 = -\partial_{x_j} + \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{2}\cot\frac{1}{2}(x_j - x_k). \tag{32}$$

Note that the singularities enforce that eigenfunctions of $H_{N,1}$ vanish at points $x_j = x_k$ modulo 2π, and thus we can interpret $H_{N,1}$ as describing particles moving on a circle of length 2π, $-\pi \leqslant x_j \leqslant \pi$, interacting with a strong two-body delta interaction.

As in the previous examples this factorization splits the operator $H_{N,1}$ into an orthogonal sum of operators in fundamental domains parametrized by permutations $P \in S_N$, and the eigenfunctions of $H_{N,1}$ can all be determined explicitly as antisymmetrized products of the eigenfunctions of the corresponding non-interacting Hamiltonian H_N.

6. A method to find Calogero–Sutherland type systems

As already mentioned, our considerations in the previous section are closely related to arguments which are well known in the context of Calogero–Sutherland type systems [C1]. In this section, we explore this relation in more detail. Through that we are naturally lead to a surprisingly simple approach to a large class of Calogero–Sutherland type systems.

One well-known method to find quantum integrable systems is to write down a many-body state of the Jastrow form

$$\Psi_0(\boldsymbol{x}) = \prod_{j=1}^{N} e^{-w(x_j)} \prod_{1 \leqslant j < k \leqslant N} g(x_j, x_k)^\lambda, \qquad \lambda \geqslant 0,$$

for some C^2 functions w and g, and then construct a many-particle Hamiltonian

$$H_N = \sum_{j=1}^{N} \mathcal{Q}_j^* \mathcal{Q}_j - \mathcal{E}_0, \qquad \mathcal{Q}_j = -\partial_{x_j} + \frac{1}{\Psi_0} \partial_{x_j} \Psi_0$$

with $\mathcal{E}_0$ some real, conveniently chosen constant. Then H_N obviously defines self-adjoint operator $\geqslant \mathcal{E}_0$, and $\mathcal{Q}_j \Psi_0 = 0 \ \forall j$ implies that Ψ_0 is an exact groundstate of H_N with the eigenvalue $\mathcal{E}_0$. In general, such a Hamiltonian H_N describes a system of particles with two- and three-body interactions, and only for very particular functions w and g the three-body terms cancel. These cases are of particular interest since they lead to physically interesting Hamiltonians of the form as in equation (3) which, in many cases, are exactly solvable. Famous such examples are $w(x) = \frac{1}{2}x^2$, $g(x, y) = (x - y)$ and $w(x) = 0$, $g(x, y) = \sin \frac{1}{2}(x - y)$ corresponding to the models discovered by Calogero [C2]:

$$H_N = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + x_j^2 \right) + \lambda(\lambda - 1) \sum_{1 \leqslant j < k \leqslant N} \frac{2}{(x - y)^2}$$

and Sutherland [S1, S2]:

$$H_N = -\sum_{j=1}^{N} \partial_{x_j}^2 + \lambda(\lambda - 1) \sum_{1 \leqslant j < k \leqslant N} \frac{1}{2 \sin^2 \frac{1}{2}(x_j - x_k)},$$

respectively. Olshanetsky and Perelomov found interesting variants of the Calogero–Sutherland models related to root systems; see [OP] and references therein.

Obviously, the Calogero and Sutherland models are very closely related to our examples discussed in sections 5.1 and 5.3, respectively. It is interesting to note that our considerations there provide an alternative proof of the key identity leading to these models. Indeed, if we check the identity in (23) by a direct computation, we find that it is equivalent to the well-known functional identity which cancels the three-body terms in the Calogero model:

$$\frac{1}{(x_j - x_k)(x_j - x_\ell)} + \text{cycl} = 0$$

for j, k, ℓ all different, where '+cycl' is short for adding the two terms of the same form obtained by cyclic permutations of the indices, $(j, k, \ell) \rightarrow (k, \ell, j)$ and $(j, k, \ell) \rightarrow (\ell, j, k)$. However, since we have a direct proof of (23) based on a basic identity of the Vandermonde determinant, we found an alternative proof of this functional identity. Similarly, the identity in equations (31) and (32) is equivalent to the well-known functional identity

$$\cot\left(\tfrac{1}{2}(x_j - x_k)\right) \cot\left(\tfrac{1}{2}(x_j - x_\ell)\right) + \text{cycl} = -1,$$

used to cancel the three-body terms in the Sutherland model.

It is therefore natural to ask: Is there a Calogero–Sutherland type system corresponding to our example in section 5.2? We can now use a similar method as above to *find* such a system: we can use equations (28) and (29) to derive the functional identity

$$\frac{\cosh^2(x_j)}{(\sinh(x_j) - \sinh(x_k))(\sinh(x_j) - \sinh(x_\ell))} + \text{cycl} = 1,$$

which then can be used to cancel three-body terms and prove that

$$\Psi_0(\boldsymbol{x}) = \prod_{j=1}^{N} \cosh^{-\gamma}(x_j) \prod_{1 \leqslant j < k \leqslant N} (\sinh(x_j) - \sinh(x_k))^\lambda \tag{33}$$

is the exact groundstate of the Hamiltonian

$$H_N = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 - \frac{\gamma(\gamma + 1)}{\cosh^2(x_j)} \right) + \lambda(\lambda - 1) \sum_{1 \leqslant j < k \leqslant N} \frac{\cosh^2(x_j) + \cosh^2(x_k)}{(\sinh(x_j) - \sinh(x_k))^2}. \tag{34}$$

Moreover, it is straightforward to show that all other bound states and corresponding eigenvalues can be found using Sutherland's method [S2].[5]

It is interesting to note that our argument can be easily extended to a rather large class of examples: assume a one-particle Hamiltonian $H = -\partial_x^2 + V(x)$ which has exact eigenstates of the form

$$\psi_n(x) = P_n(\eta(x)) \, \mathrm{e}^{-w(x)}, \quad n = 0, 1, 2, \ldots$$

with the eigenvalue E_n, where P_n is a polynomial of order n, and w and η are C^2 functions. Then f_0 in (22) is obviously an eigenstate of the many-body Hamiltonian

$$\sum_j \left(-\partial_{x_j}^2 + V(x_j) \right)$$

with the eigenvalue $\mathcal{E}_0 = \sum_{j=0}^{N-1} E_n$. As in section 5, we can conclude that

$$f_0(\boldsymbol{x}) = \prod_{j=1}^{N} \mathrm{e}^{-w(x_j)} \prod_{1 \leqslant j < k \leqslant N} (\eta(x_j) - \eta(x_k)),$$

and using this formula for f_0 the identity

$$\sum_{j=1}^{N} \left(-\partial_{x_j}^2 + V(x_j) \right) = \sum_{j=1}^{N} Q_j^* Q_j + \mathcal{E}_0, \qquad Q_j = -\partial_{x_j} + \frac{1}{f_0} \partial_{x_j} f_0$$

implies a functional identity expressing three-body terms through two-body terms. We now compute

$$\mathcal{H} = \sum_{j=1} \mathcal{Q}_j^* \mathcal{Q}_j, \qquad \mathcal{Q}_j = -\partial_{x_j} + \frac{1}{\Psi_0} \partial_{x_j} \Psi_0$$

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[5] Langmann E, unpublished.

with Ψ_0 in (5), and using this functional identity we can cancel all three-body terms and thus find that $\mathcal{H} = H_N - \mathcal{E}_0$ with H_N in (3) and (4) and $\mathcal{E}_0$ in (6). Since obviously $\mathcal{H}\Psi_0 = 0$ and $\mathcal{H} \geqslant 0$, we obtain the result stated in the proposition in section 1. Table 1 gives other interesting examples of this result. We plan to give a comprehensive discussion of all these models and their solution in a future publication [HL].

7. Final comments

We should mention that the model in (34) is closely related to the well-known C_N-variant of the Sutherland model defined by

$$H_{C_N} = \sum_{j=1}^{N} \left(-\partial_{x_j}^2 + \frac{\gamma(\gamma+1)}{\sinh^2(x_j)} \right) + \lambda(\lambda-1)$$
$$\times \sum_{1 \leqslant j < k \leqslant N} \left(\frac{1}{2\sin^2 \frac{1}{2}(x_j - x_j)} + \frac{1}{2\sin^2 \frac{1}{2}(x_j + x_j)} \right);$$

see [OP] and references therein. This Hamiltonian describes identical particles in an external, repulsive potential and with repulsive two-body interactions, and it only has scattering states (purely continuous spectrum). One can show that by shifting $x_j \to x_j + i\pi/2$, this Hamiltonian is converted to that in (34), up to an additive constant. However, these shifts change the physics of the model dramatically: the Hamiltonian in (34) describes interacting particles in a weakly confining potential, and it thus has, in addition to bound states and scattering states, also mixed states where some of the particles are confined and others are free. To our knowledge, this situation has not been studied in the context of Calogero–Sutherland type systems. We therefore think it would be interesting to study this model in detail which, however, is beyond the scope of the present paper.

There are other special cases which seem less known in the physics literature even though, to our opinion, they also of interest in physics. The examples listed in table 1 lead to systems which were previously known in the mathematics literature; see [BF, DX] and references therein. We should mention that these latter systems are closely related to the BC_N variants of the Calogero (IV) and Sutherland model (I–III) [OP], respectively. However, our result nevertheless seems useful since it is the starting point of an alternative, systematic and unified approach to these systems and their solutions, including examples which are hardly known [HL].

Acknowledgments

References

[AGHH] Albeverio S, Gesztesy F, Hœegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (New York: Springer)

[AS]   Abramowitz A and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)

[BF]   Baker T H and Forrester P J 1997 The Calogero–Sutherland model and generalized classical polynomials *Commun. Math. Phys.* **188** 175

[BS]   Birman M and Solomyak M 1987 *Spectral theory of selfadjoint operators in Hilbert space, Mathematics and its Applications (Soviet Series)* (Dordrecht: Reidel)

[C1]   Calogero F 1969 Solution of a three-body problem in one dimension *J. Math. Phys.* **10** 2191

[C2]   Calogero F 1971 Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419

[CKS]   Cooper F, Khare A and Sukhatme U 1995 Supersymmetry and quantum mechanics *Phys. Rep.* **251** 267

[DX]   Dunkl C F and Xu Y 2001 *Orthogonal Polynomials of Several Variables* (Cambridge: Cambridge University Press)

[G]   Girardeau M 1960 Relationship between systems of impenetrable bosons and fermions in one dimensions *J. Math. Phys.* **1** 516

[K]   Kato T 1966 *Perturbation theory for linear operator, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen* vol 132 (Berlin: Springer)

[HL]   Hallnäs M and Langmann E in preparation

[L]   Lindblad G Quantum Mechanics with MATLAB, available on internet http://mathphys.physics.kth.se/schrodinger.html

[LL]   Lieb E H and Liniger W 1963 Exact analysis of an interacting bose gas: 1. The general solution and the ground state *Phys. Rev.* **130** 1605

[OP]   Olshanetsky M A and Perelomov A M 1983 Quantum integrable systems related to lie algebras *Phys. Rep.* **94** 313

[P]   Peredes B *et al* 2004 Tonks–Girardeau gas of ultracold atoms in optical lattice *Nature* **429** 277

[RS1]   Reed M and Simon B 1972 *Methods of Modern Mathematical Physics: I. Functional Analysis* (New York: Academic)

[RS2]   Reed M and Simon B 1975 *Methods of Modern Mathematical Physics: II. Fourier Analysis, Self-Adjointness* (New York: Academic)

[S1]   Sutherland B 1971 Exact results for a quantum many body problem in one-dimension *Phys. Rev.* A **4** 2019

[S2]   Sutherland B 1972 Exact results for a quantum many body problem in one-dimension II *Phys. Rev.* A **5** 1372

[Y]   Yang C N 1967 Some exact results for the many body problems in one dimension with repulsive delta function interaction *Phys. Rev. Lett.* **19** 1312