FOLLYTONS
AND THE REMOVAL OF EIGENVALUES
FOR FOURTH ORDER DIFFERENTIAL OPERATORS

J. HOPPE, A. LAPTEV AND J. ÖSTENSSON

ABSTRACT. A non-linear functional $Q[u, v]$ is given that governs the loss, respectively gain, of (doubly degenerate) eigenvalues of fourth order differential operators $L = \partial^4 + \partial\, u\, \partial + v$ on the line. Apart from factorizing L as $A^* A + E_0$, providing several explicit examples, and deriving various relations between u, v and eigenfunctions of L, we find u and v such that L is isospectral to the free operator $L_0 = \partial^4$ up to one (multiplicity 2) eigenvalue $E_0 < 0$. Not unexpectedly, this choice of u, v leads to exact solutions of the corresponding time-dependent PDE's. Removal of eigenvalues allows us to obtain a sharp Lieb-Thirring inequality for a class of operators L whose negative eigenvalues are of multiplicity two.

1. FACTORIZATION OF THE OPERATOR $L = \partial^4 + \partial\, u\, \partial + v$.

Let us assume that u and v are real-valued functions and $u, v \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class of rapidly decaying functions. Let L be a linear fourth order selfadjoint operator in $L^2(\mathbb{R})$

$$L := \partial^4 + \partial\, u\, \partial + v \tag{1.1}$$

defined on functions from the Sobolev class $H^4(\mathbb{R})$.

A general inverse theory for higher order operators on the line was considered in [1], [2] and [5]. In [3] Lieb-Thirring inequalities for (matrix) Schrödinger operators were proven by using factorization of second order operators into products of first order operators.

Let us assume that the lowest eigenvalue $E_0 < 0$ of the operator (1.1) is of double multiplicity and therefore there exist two orthogonal in $L^2(\mathbb{R})$ eigenfunctions ψ_+ and ψ_- satisfying the equation

$$L\, \psi = E_0\, \psi. \tag{1.2}$$

As shown in the appendix, the Wronskian

$$W(x) := \psi_+(x)\, \psi_-'(x) - \psi_-(x)\, \psi_+'(x) \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary 34L40; Secondary 34L30.

is necessarily non-vanishing, $W(x) \neq 0$, $x \in \mathbb{R}$. Let us try to factorize $L - E_0$ as

$$A^* A = \left(-\partial^2 - f\partial + g - f'\right)\left(-\partial^2 + f\partial + g\right), \tag{1.4}$$

with f and g real-valued. Clearly,

$$\begin{cases} f' + f^2 + 2g &= -u \\ g^2 - (fg + g')' &= v - E_0. \end{cases} \tag{1.5}$$

Instead of discussing these non-linear differential equations directly, let us express f, g, u and v in terms of the functions ψ_+, ψ_-. Straightforwardly, one finds that since ψ_+ and ψ_- are eigenfunctions of A^*A with eigenvalue 0, we have $A\psi_+ = A\psi_- = 0$, which implies

$$\begin{cases} f\,W &= W' \\ -g\,W &= \psi'_+\,\psi''_- - \psi''_+\,\psi'_- =: W_{12}, \end{cases} \tag{1.6}$$

while $(L - E_0)\,\psi_+ = (L - E_0)\,\psi_- = 0$ implies

$$\begin{cases} u\,W &= 2\,W_{12} - W'' + \epsilon \\ (v - E_0)\,W &= u\,W_{12} + W''_{12} - W_{23}, \end{cases} \tag{1.7}$$

where ϵ is an integration constant and

$$W_{23} := \psi''_+\,\psi'''_- - \psi'''_+\,\psi''_- \tag{1.8}$$

is expressible in terms of W and W_{12} via

$$W\,W_{23} = W'_{12}\,W' - W_{12}\,W'' + W_{12}^2. \tag{1.9}$$

Equations (1.6) say that

$$f = \frac{W'}{W}, \quad g = -\frac{W_{12}}{W}. \tag{1.10}$$

Since $u\,W + W'' - 2\,W_{12}$ vanishes at infinity, ϵ has to be 0, and one finds, using equations (1.7)-(1.9), that

$$u = \frac{2\,W_{12} - W''}{W} \tag{1.11}$$

$$v - E_0 = \frac{W_{12}^2}{W^2} + \left(\frac{W'_{12}}{W}\right)'. \tag{1.12}$$

Note that

$$\tilde{L} := AA^* + E_0 = L + 4\,\partial\,f'\,\partial + 2\,f\,g' - f\,f'' + f''' \tag{1.13}$$

will be isospectral to L, apart from E_0, which has been removed. To see why E_0 is not an eigenvalue of $\tilde{L}$, let us for simplicity assume that $u, v \in C_0^\infty(\mathbb{R})$, say that $\operatorname{supp} u$, $\operatorname{supp} v \subset (-c, c)$. Then,

$$\begin{array}{l} \psi_+(x) = \alpha_1 \, e^{-\kappa x} \, \cos(\kappa x) + \beta_1 \, e^{-\kappa x} \, \sin(\kappa x) \\ \psi_-(x) = \alpha_2 \, e^{-\kappa x} \, \cos(\kappa x) + \beta_2 \, e^{-\kappa x} \, \sin(\kappa x) \end{array}, \quad x > c,$$

where $E_0 = -4\,\kappa^4$, $k > 0$. This implies

$$\begin{array}{ll} W(x) & = \kappa \, e^{-2\kappa x} \, (\alpha_1 \, \beta_2 - \beta_1 \, \alpha_2) \\ W_{12}(x) & = 2\,\kappa^3 \, e^{-2\kappa x} \, (\alpha_1 \, \beta_2 - \beta_1 \, \alpha_2) \end{array}, \quad x > c.$$

(note that the bracket does not vanish, since ψ_+ and ψ_- are linearly independent.) This (and a similar investigation at the other end) implies that

$$f(x) = \mp 2\,\kappa, \quad g(x) = -2\,\kappa^2, \quad \text{for } \pm\, x > c.$$

Since $\tilde{L}\psi = E_0\,\psi$ implies $A^*\psi = 0$, we obtain

$$\psi'' - 2\kappa\,\psi' + 2\kappa^2\psi = 0, \quad x > c.$$

It clearly follows that ψ cannot be in $L^2(\mathbb{R})$ unless it vanishes identically.

Before giving some explicit examples, let us make some comments concerning the problem of actually finding f and g, or ψ_+ and ψ_-, when u and v are given. Instead of solving the non-linear system (1.5), or the spectral problem (1.2), one may also try to solve the Hirota-type equation which follows from (1.11), (1.12)

$$4\,(v - E_0) = \left(\frac{W''}{W} + u\right)^2 + 2\,\left(\frac{W'''}{W} + u' + u\,\frac{W'}{W}\right)', \tag{1.14}$$

and which for $u \equiv 0$ reads

$$4\,(v - E_0)\,W^2 = 2\,(W''''\,W - W'''\,W') + W''^2.$$

Once $W\,(\neq 0)$ is obtained, f and g can be given by the equations (1.10). With f and g defined in this way, equation (1.5) is satisfied and the factorization (1.4) is valid.

Note also the following: the functions ψ_+ and ψ_- are solutions of $A\,\psi = 0$, i.e.

$$-\psi'' + f\,\psi' + g\,\psi = 0.$$

By writing

$$\psi_\pm = \sqrt{W}\,\phi_\pm,$$

one finds that $\phi_+\,\phi_-' - \phi_+'\,\phi_- = 1$ and that $\phi_\pm$ are (oscillating) solutions of the equation in Liouville form

$$-\phi'' + \left(g + \frac{3}{4}\left(\frac{W'}{W}\right)^2 - \frac{1}{2}\,\frac{W''}{W}\right)\,\phi = 0,$$

i.e. associated to a second order self-adjoint differential operator.

2. ADDITION AND REMOVAL OF EIGENVALUES.

Although adding and removing eigenvalues may be thought to be a procedure that can be read both ways (symmetrically), the steps involved are actually quite different in both cases (in particular, it is not yet clear, which conditions on u and v allow for the addition of a doubly degenerate eigenvalue below the spectrum of $\partial^4 + \partial\, u\partial + v$). Let us therefore 'summarize' them separately, in both cases starting from a given operator

$$L_n := \partial^4 + \partial\, u_n \partial + v_n, \quad n \in \mathbb{N},$$

and the equation (1.14) with u, v replaced by u_n, v_n. This equation shall be referred to as $(1.14)_n$.

Removal of eigenvalues:
1. Solve $(1.14)_n$ $\left(\text{with } E_0 \to E_0^{(n)} = -4\kappa_n^4\right)$ for $W_n := W\ (\to 0)$ at infinity and define $W_{12}^{(n)}$ as $\frac{1}{2}\,(W_n\, u_n + W_n'')$, as is natural in accordance with equation $(1.11)_n$. Alternatively, if $\psi_\pm^{(n)}$ are known, calculate W_n and $W_{12}^{(n)}$ via their definitions, i.e. as

$$\begin{aligned} W_n &= \psi_+^{(n)}\, \psi_-^{(n)\prime} - \psi_-^{(n)}\, \psi_+^{(n)\prime} \\ W_{12}^{(n)} &= \psi_+^{(n)\prime}\, \psi_-^{(n)\prime\prime} - \psi_+^{(n)\prime\prime}\, \psi_-^{(n)\prime}. \end{aligned}$$

2. Define f_n and g_n according to $(1.10)_n$, thus solving the system (1.5), and obtaining the factorization

$$L_n = A_n^* A_n - 4\kappa_n^4.$$

3. The operator

$$\tilde{L}_n = A_n A_n^* - 4\kappa_n^4 =: L_{n-1}$$

will then be isospectral to L_n apart form the lowest eigenvalue $E_0^{(n)} = -4\kappa_n^4$ (of multiplicity 2), which has been removed.

Addition of eigenvalues:
1. Solve $(1.14)_n$ $\left(\text{with } E_0 \to E_0^{(n+1)} = -4\kappa_{n+1}^4\right)$ for $\hat{W}_{n+1} := W \sim e^{\pm 2\kappa_{n+1} x}$, as $x \to \pm\infty$, i.e. $\hat{W}_{n+1}$ diverging at infinity and non-vanishing for finite x. (As mentioned above, conditions on u_n, v_n ensuring the existence of $\hat{W}_{n+1}$ are still unclear.)
2. Define $W_{n+1} := \frac{1}{\hat{W}_{n+1}}$, which will then solve the (more complicated

looking) equation

$$40\,\frac{W'^4}{W^4} - 2\,\frac{W''''}{W} + 14\,\frac{W'''\,W'}{W^2} + 13\,\frac{W''^2}{W^2} - 64\,\frac{W''\,W'^2}{W^3} + \tag{2.1}$$
$$2u'' + u^2 - 2u'\,\frac{W'}{W} + 2u\left(\frac{W'^2}{W^2} - 2\left(\frac{W'}{W}\right)'\right) = 16\kappa^4 + 4v$$

(with $u, v \to u_n, v_n$ and $\kappa \to \kappa_{n+1}$). In fact, (2.1) is equivalent to

$$-2f''' + 6ff'' + 7f'^2 - 8f'f^2 + f^4 + 2u\left(f^2 - 2f'\right) - 2u'f + u^2 + 2u''$$
$$= 4v + 16\kappa^4$$

$\left(\text{via } f = \frac{W'_{n+1}}{W_{n+1}} =: f_{n+1},\ u, v \to u_n, v_n \text{ and } \kappa \to \kappa_{n+1}\right)$ that arises in the factorization of L_{n+1}.

3. Write

$$L_n = A_{n+1}A^*_{n+1} - 4\kappa^4_{n+1}$$

(implying $2\,g_{n+1} := 3\,f'_{n+1} - f^2_{n+1} - u_n$).

4. Then,

$$L_{n+1} := A^*_{n+1}A_{n+1} - 4\kappa^4_{n+1},$$

will be isospectral to L_n apart from having one additional (doubly degenerate) eigenvalue $E_0^{(n+1)}$ below the spectrum of L_n.

3. A non-linear functional Q and a system of PDE's associated with the operator L.

As observed 100 years ago [7], the operator $L = \partial^4 + \partial\,u\,\partial + v$ has a unique 4'th root in the form $L^{1/4} := \partial + \sum_{k=1}^{\infty} l_k(x)\partial^{-k}$. Define M to be the positive (differential operator) part of any integer power of $L^{1/4}$. Then it is well known, that

$$L_t = [L, M]\,,$$

where L_t is the operator defined by $L_t\,\varphi = \partial\,u_t\,\partial\,\varphi + v_t\,\varphi$, consistently defines evolution equations (for $u = u(x,t)$, $v = v(x,t)$) that have infinitely many conserved quantities (i.e. functionals of u and v, and their spatial derivatives, that do not depend on t). We shall make use of this by letting

$$M := 8\left(L^{3/4}\right)_+ = 8\,\partial^3 + 6\,u\,\partial + 3\,u',$$

and focusing on the quantity

$$Q[u,v] := \frac{1}{48}\int_{\mathbb{R}}\left(48\,v^2 + \frac{5}{4}\,u^4 - 12\,u^2\,v - 40\,u''\,v + \frac{13}{2}\,u^2\,u'' + 9\,u''^2\right)\,dx. \tag{3.1}$$

This quantity does not change when u and v evolve according to

$$\begin{cases} u_t &= 10\,u''' + 6\,u\,u' - 24\,v' \\ v_t &= 3\,(u''''' + u\,u''' + u'\,u'') - 8\,v''' - 6\,u\,v'. \end{cases} \tag{3.2}$$

The functional Q appears in the power series expansion for the Fredholm determinant $\det(L-z)(\partial^4-z)^{-1}$ and is one of the infinite number of integrals of motion for the dynamical system (3.2).

4. A Lieb-Thirring inequalitiy

It is clear that formula (1.13) for $\tilde{L} = \partial^4 + \partial\,\tilde{u}\,\partial + \tilde{v}$ implies that

$$\begin{cases} \tilde{u} - u &= 4\,f' \\ \tilde{v} - v &= 2\,f\,g' - f\,f'' + f'''. \end{cases} \tag{4.1}$$

By using the asymptotic properties of f and g ($f \to \mp 2\,\kappa$, $g \to -2\kappa^2$, as $x \to \pm\infty$), one can show that

$$\delta Q := \Big(Q[\tilde{u},\tilde{v}] - Q[u,v]\Big) = 2(4\kappa^4)^{7/4}\frac{64}{21\sqrt{2}}. \tag{4.2}$$

This result is similar to that for Schrödinger operators [3] and reflects the loss of a doubly degenerate eigenvalue $E_0 = -4\,\kappa^4$, when going from L to $\tilde{L}$.

The proof of (4.2), just as the derivation of (3.1), involves very lengthy calculations. When deriving (4.2) we have used (1.5) and (4.1) to write the expression for δQ as an integral of terms involving only the functions f and g, and their spatial derivatives. The crucial step then is to note that the integrand is a pure derivative of x, i.e. $\delta Q = \int \mathbb{Q}'\,dx$ for some function $\mathbb{Q}$, which makes it possible to evaluate the integral solely from the limits of f and g at infinity. Thus, to compute δQ, we have selected the terms in $\mathbb{Q}$ which are free of derivatives, as those are the only ones that contibute. The terms in $\mathbb{Q}$ still containing derivatives, for instance the ones quadratic in g and linear in f,

$$\frac{1}{48}\int\Big((96-48)\,g^2\,f''' - 2\cdot 96\,f\,g'\,g'' - 8\cdot 12\,g''\,f'\cdot 2g - 4\cdot 40\,f'''\,g^2 \\ + 160\,g''\,g'\,f - 16\cdot 26\,f''\,g'\,g - 16\cdot 13\,g'^2\,f'\Big)\,dx,$$

give zero.

The constant in the right hand side of (4.2) is related to the semiclassical constant $L^{cl}_{4,7/4,1}$ appearing in the asymptotic formula

$$\lim_{\alpha\to\infty}\alpha^{-2}\,\mathrm{Tr}\,(\partial^4 + \alpha v)_-^{7/4} = L^{cl}_{4,7/4,1}\int v_-^2\,dx,$$

where

$$L^{cl}_{4,7/4,1} = (2\pi)^{-1} \int_{-1}^{1} (1-\xi^4)^{7/4}\, d\xi = \frac{21\sqrt{2}}{128}.$$

This constant also appears in the trace formula for a fourth order differential operator $\partial^4 + v$ considered in [6] and its generalization for the operator $\partial^4 + \partial u \partial + v$.

If we assume that the operator $L = \partial^4 + \partial\, u\, \partial + v$ has n negative eigenvalues of multiplicity two, we can annihilate them by using the procedure described in Section 2 and obtain new potentials u_n, v_n. Formula (4.2) allows us to state the following result:

Theorem 4.1. *Let L be an operator* (1.1) *that has $2n$ negative eigenvalues $\{\lambda_j\}_{j=1}^{2n}$, counted with their multiplicity and let all of them be of multiplicity two, $\lambda_{2k-1} = \lambda_{2k}$, $k = 1, 2, \dots, n$. Assume that $Q[u_n, v_n] \geq 0$, where u_n and v_n are obtained by using the removal of eigenvalues described in Section 2. Then*

$$\sum_{j=1}^{2n} |\lambda_j|^{7/4} \leq 2\, L^{cl}_{4,7/4,1}\, Q[u, v]. \tag{4.3}$$

If u and v are reflectionless potentials for which we end up with $u_n = v_n \equiv 0$ (see Section 6), then instead of inequality in (4.3) we obtain equality.

Corollary 4.1. *The constant $2\, L^{cl}_{4,7/4,1}$ in Theorem 4.1 cannot be improved.*

Note that Q is the integral of a quadratic form in v, u^2 and u'' which has two positive but one (very small) negative eigenvalue, so Q is not obviously positive for all u and v. The eigenvalues of this quadratic form approximately are $\frac{1}{48}(57.2566, 1.1592, -0.1657)$.

Rather involved functions u and v have recently been constructed [4] for which (3.1) is actually negative.

5. Some examples.

Example 1. The operator

$$L = \partial^4 - 5\,\partial^2 + \partial\, \frac{12}{\cosh^2 x}\, \partial - \frac{6}{\cosh^2 x} = A^* A - 4$$

with

$$A = -\partial^2 - 3\, \tanh x\, \partial - 2$$

has 2 linearly independent eigenfunctions with eigenvalue $E_0 = -4$,

$$\psi_+(x) = \frac{1}{\cosh^2 x}, \quad \psi_-(x) = \frac{\sinh x}{\cosh^2 x}.$$

One can easily check that $A\,\psi_\pm = 0$ and that u, v are reflectionless, as

$$\tilde{L} = AA^* - 4 = \partial^4 - 5\,\partial^2$$

(note that ψ_+ and ψ_- have different fall-off behaviour at ∞ and that $W(x) = \cosh^{-3} x$).

Example 2. The operator

$$L = \partial^4 + 16\,\partial\,\frac{1}{\cosh^2 x}\,\partial + \frac{40}{\cosh^4 x} - \frac{88}{\cosh^2 x} = A^*A - 64$$

with

$$A = -\partial^2 - 4\,\tanh x\,\partial - 8 + \frac{2}{\cosh^2 x}$$

has 2 linearly independent eigenfunctions with eigenvalue $E_0 = -64$,

$$\psi_+(x) = \frac{\cos 2x}{\cosh^2 x}, \quad \psi_-(x) = \frac{\sin 2x}{\cosh^2 x}.$$

One easily verifies that $A\,\psi_\pm = 0$, and that

$$\tilde{L} = AA^* - 4 = \partial^4 - \frac{40}{\cosh^2 x}.$$

A computation gives that

$$Q = \frac{2^8}{7}\cdot 229, \quad \tilde{Q} = \frac{2^6}{3}\cdot 100, \quad \delta Q = -\frac{2^{17}}{21}\left(= -2\,(\kappa = 2)^7\,\frac{2^9}{21}\right).$$

Example 3. The operator

$$L = \partial^4 + \left(45\,\Psi^4 - 40\,\Psi^2\right) = A^*A - 4$$

with

$$W = \Psi^2 := \frac{1}{\cosh^2 x}, \quad W_{12} = 2\,\Psi^2 - 3\,\Psi^4$$

and

$$A = -\partial^2 - 2\,\tanh x\,\partial - 2 + 3\,\Psi^2$$

has a doubly degenerate eigenvalue $E_0 = -4$. One easily verifies, that

$$\tilde{L} = \partial^4 - 8\,\partial\,\Psi^2\,\partial + 25\,\Psi^4 - 16\,\Psi^2.$$

Example 4. The operator

$$L = \partial^4 - \partial^2 + 4\,\partial\,\frac{1}{\cosh^2 x}\,\partial + \frac{6}{\cosh^2 x} - \frac{8}{\cosh^4 x} = A^*A$$

with

$$A = -\partial^2 - \tanh x\,\partial - \frac{1}{\cosh^2 x} = \partial\,(-\partial - \tanh x)$$

has a unique ground-state $E_0 = 0$ with eigenfunction

$$\psi(x) = \frac{1}{\cosh x}.$$

The second solution of $A\psi = 0$ is $\psi = \tanh x \notin L^2(\mathbb{R})$. One easily verifies, that

$$\tilde{L} = \partial^4 - \partial^2.$$

Example 5. For any $k > 0$, the operator

$$L = \partial^4 + \partial\, u\, \partial + v$$

with

$$\begin{cases} u(x) & = 2\left(1 + \frac{2}{k}\right)\Psi^2\left(\frac{x}{k}\right) \\ v(x) & = -4\left(1 + \frac{1}{k} - \frac{1}{k^3}\right)\Psi^2\left(\frac{x}{k}\right) + \left(1 - \frac{1}{k}\right)\left(1 + \frac{5}{k} + \frac{6}{k^2}\right)\Psi^4\left(\frac{x}{k}\right), \end{cases}$$

where

$$\Psi(x) := \frac{1}{\cosh x},$$

has a doubly degenerate ground-state, $E_0 = -4$, with eigenfunctions

$$\psi_{\pm}^{(k)}(x) = e^{\pm ix}\left(\frac{1}{\cosh\frac{x}{k}}\right)^k.$$

6. FOLLYTONS.

In order to find u and v such that $L = A^*A + E_0$ is 'conjugate' to the free operator $\tilde{L} = \partial^4 =: L_0$ one has to solve (1.5) with $u = v = 0$. Eliminating g and writing $E_0 = -4\kappa^4$ one obtains the ODE

$$2\, f''' + 6\, f\, f'' + 7\, f'^2 + 8\, f'\, f^2 + f^4 = 16\,\kappa^4.$$

One may reduce the order by taking f as the independent variable, and $F(f) := f'$ as the dependent one, yielding

$$2\left(F''\, F^2 + F'^2\, F\right) + 6\, F\, F'\, f + 7\, F^2 + 8\, F\, f^2 + f^4 = 16\,\kappa^4,$$

but both forms seem(ed) to be too difficult to solve. By using (1.14), however, it takes the form

$$16\kappa^4\, W^2 = 2\,(W''''\, W - W'''\, W') + W''^2;$$

a 4-parameter-class of solutions can be obtained via the ansatz

$$W = a + be^{2\kappa x} + ce^{-2\kappa x} + d\cos 2\kappa x + e\sin 2\kappa x$$

(yielding $4bc + d^2 + e^2 = a^2/2$). Let us take

$$\hat{W} = \text{const}\,\cdot\left(\sqrt{2} + \cosh\left(2\,\kappa\, x\right)\right)$$

as its 'prototypical' solution. Correspondingly,

$$\hat{f} := \frac{\hat{W}'}{\hat{W}} = 2\,\kappa\,\frac{\sinh\left(2\,\kappa\, x\right)}{\sqrt{2} + \cosh\left(2\,\kappa\, x\right)}.$$

As interchanging A^* and A (as far as f is concerned) only changes the sign of f,

$$f(x) = -2\,\kappa\,\frac{\sinh(2\,\kappa\,x)}{\sqrt{2}+\cosh(2\,\kappa\,x)}.$$

The Wronskian of the two ground-states $\psi_\pm$ (of $L = \partial^4 + \partial\,u\,\partial + v$, conjugate to $L_0 = \partial^4$) is simply the inverse of $\hat{W}$, i.e. (choosing the constant in $\hat{W}$ to be 1),

$$W(x) = \frac{1}{\sqrt{2}+\cosh(2\,\kappa\,x)} =: \chi(2\,\kappa\,x)\,.$$

The function g is given by

$$g = \frac{1}{2}\left(3\,f' - f^2\right) = -2\,\kappa^2\left(1+\sqrt{2}\,W - 2\,W^2\right).$$

Insertion into equation (1.5) yields the reflectionless 'potentials'

$$\left\{\begin{array}{ll} u_\kappa & = 16\,\kappa^2\left(\sqrt{2}\,W - W^2\right) \\ v_\kappa & = 16\,\kappa^4\left(\sqrt{2}\,W - 12\,W^2 + 16\,\sqrt{2}\,W^3 - 8\,W^4\right)\end{array}\right. \tag{6.1}$$

with $L = \partial^4 + \partial\,u_\kappa\,\partial + v_\kappa$ having exactly one doubly degenerate negative eigenvalue $-4\,\kappa^4$. While in most other examples we scaled κ to be equal to 1 it is, in this case (due to the appearance of 2κ in W) easiest to choose $\kappa = \frac{1}{2}$, i.e. to take

$$\left\{\begin{array}{ll} u & = 4\left(\sqrt{2}\,\chi - \chi^2\right) \\ v & = \left(\sqrt{2}\,\chi - 12\,\chi^2 + 16\,\sqrt{2}\,\chi^3 - 8\chi^4\right)\end{array}\right. \tag{6.2}$$

and, when needed, use formulas like

$$\begin{array}{ll} \chi'' & = \chi\,\left(1 - 3\,\sqrt{2}\,\chi + 2\,\chi^2\right) \\ \chi'^2 & = \chi^2\,\left(1 - 2\,\sqrt{2}\,\chi + \chi^2\right) \\ \chi''' & = \chi'\,\left(1 - 6\,\sqrt{2}\,\chi + 6\,\chi^2\right) \\ \chi'''' & = \chi\,\left(1 - 15\,\sqrt{2}\,\chi + 80\,\chi^2 - 60\,\sqrt{2}\,\chi^3 + 24\,\chi^4\right). \end{array}$$

(Note that redefining χ by a factor of $-\sqrt{2}$ would make all the coefficients positive (integers)). These formulas are useful when checking that $u(x+4\,t)$ and $v(x+4\,t)$, with u and v given by (6.2), are exact solutions of the non-linear system of PDE's (3.2) (just as $u_\kappa(x+16\kappa^2\,t), v_\kappa(x+16\kappa^2\,t)$).

APPENDIX. $W \neq 0$

We shall prove here that the Wronskian type function defined in (1.3) never equals zero.

Theorem. *Let $\psi_\pm$ be two linear independent eigenfunctions of the operator* (1.1) *corresponding to the lowest eigenvalue E_0 of double multiplicity. Then*

$$W[\psi_+,\psi_-](x) := \psi_+(x)\,\psi'_-(x) - \psi_-(x)\,\psi'_+(x) \neq 0, \quad x \in \mathbb{R}.$$

In order to prove this result we need

Lemma. *Let E_0 be the lowest eigenvalue of the operator L and let $\psi \in L^2(\mathbb{R})$ be a solution of the equation* (1.2) *satisfying $\psi(x_0) = \psi'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then $\psi(x) \equiv 0$.*

Proof. Indeed, the function

$$\tilde{\psi}(x) = \begin{cases} -\psi(x), & \text{if } x \le x_0, \\ \psi(x), & \text{if } x \ge x_0, \end{cases}$$

is linear independent with ψ. Obviously

$$\int_{\mathbb{R}} \left(|\tilde{\psi}''|^2 + u|\tilde{\psi}'|^2 + v|\tilde{\psi}|^2 \right) dx = E_0 \int_{\mathbb{R}} |\tilde{\psi}|^2 \, dx.$$

This implies that $\psi_1 = \tilde{\psi} + \psi$ is also an $L^2(\mathbb{R})$ eigenfunction of the operator L with the eigenvalue E_0 and therefore $\psi_1 = \tilde{\psi} = \psi \equiv 0$. □

Remark. In the last Lemma the conditions $\psi(x_0) = \psi'(x_0) = 0$ split the problem for the operator L in $L^2(\mathbb{R})$ into two Dirichlet boundary value problems on semiaxes $L^2((x_0, \infty))$ and $L^2((-\infty, x_0))$. Therefore, the lowest eigenvalue moves up.

Proof of Theorem.
Let $\psi_\pm$, be two linear independent eigenfunctions corresponding to the lowest eigenvalue E_0 of the operator L. Then

$$W[\psi_+, \psi_-] = \det \begin{pmatrix} \psi_+ & \psi_- \\ \psi'_+ & \psi'_- \end{pmatrix}.$$

If $W[\psi_+, \psi_-](x_0) = 0$ then there are constants α and β, not both zero, such that

$$\alpha \begin{pmatrix} \psi_+(x_0) \\ \psi'_+(x_0) \end{pmatrix} = \beta \begin{pmatrix} \psi_-(x_0) \\ \psi'_-(x_0) \end{pmatrix}.$$

Therefore the function $\psi_1(x) = \alpha\psi_+(x) - \beta\psi_-(x)$ and its derivative equal zero at x_0. By Lemma $\psi_1 \equiv 0$ which contradicts the fact that ψ_+ and ψ_- are linearly independent. The proof is complete. □

Acknowledgments. The authors would like to thank R. Benguria, H. Kalf, E. Langmann, A. Pushnitski, O. Safronov, R. Shterenberg and V. Sukhanov for useful discussions, as well as the ESF European programme SPECT and the EU Network: "Analysis & Quantum" for partial support. J. Östensson wishes to express his gratitude to the Wenner-Gren Foundation.

REFERENCES

[1] R. Beals, P. Deift and C. Tomei, *Direct and inverse scattering on the line*, Mathematical surveys and monographs (AMS), **28** (1988), 1–209.

[2] R. Beals, P. Deift and X. Zhou, *The inverse scattering transform on the line. Important developments in soliton theory*, Springer Ser. Nonlinear Dynam., Springer, Berlin, (1993), 7–32.

[3] R. Benguria, M. Loss, *A simple proof of a theorem of Laptev and Weidl*, Mathematical research letters, **7** (2000), 195–203.

[4] R. Benguria, I. Catto, J. Dolbeault and R. Monneau *Oscillating minimizers of a fourth order problem invariant under scaling*, Preprint.

[5] V.V Sukhanov, *An inverse problem for a selfadjoint differential operator on the line*, Mat. Sbornik **137** (1988), 242–259 (Russian); translation in Math. USSR Sbornik, **65** (1990), 249–266.

[6] J. Östensson, *Trace formulae for fourth order differential operators*, KTH, PhD-thesis (to appear).

[7] I. Schur, *Über vertauschbare lineare Differentialausdrücke*, Sitzungsbericht d. Berliner Mathematischen Gesellschaft (1905), 2-8.

E-mail address: hoppe@math.kth.se, laptev@math.kth.se, ostensson@cims.nyu.edu