# Two problems which were presented in Matrei.

*

T. Hoffmann-Ostenhof[1,2]
Institut für Theoretische Chemie, Universität Wien[1]
International Erwin Schrödinger Institute for Mathematical Physics[2]

## 1 Problem 1

I start with a general problem in spectral theory, which has been also described in [4]. Consider on a Hilbert space $\mathcal{H}$ an essentially selfadjoint positive definite operator $L$ with compact resolvent.

Suppose there is a finite group $\mathfrak{G}$ whose actions commute with $L$. Let $\{D_i\}_{i=1}^h$ denote the irreducible representations of $\mathfrak{G}$ and their degree $d_i$. Corresponding to the irreducible representations we can decompose $\mathcal{H}$ into $\mathcal{H} = \oplus \mathcal{H}_i$. Associated to the $\mathcal{H}_i$ we have essentially selfadjoint operators $L_i$ such that

$$L = \bigoplus_{i=1}^{h} L_i.$$

Let $\lambda_i$ be the lowest eigenvalue of $L_i$ and denote by $m_i$ its multiplicity, i.e. the dimension of the associated eigenspace.

For a large class of selfadjoint second order elliptic operators one knows that the lowest eigenvalue is simple. This motivates the following **questions**: *When is it possible to give upper bounds to the $m_i$ in terms of the $d_i$, in particular for which cases $m_i = d_i$?*

*When is it possible to find a labeling of the $h$ irreducible representations such that*

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_h?$$

In [4] these two questions were completely answered affirmatively for the

*Supported by Ministerium für Bildung, Wissenschaft und Kunst der Republik Österreich

1991 Mathematics Subject Classification 35B05 case that $L$ is a Schrödinger operator $L = -\Delta + V$ defined on a bounded domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition. Thereby $L$ was assumed to commute the the actions of the dihedral group $\mathbb{D}_{2n}$, i.e. the group of the regular n-gon. Note that these results are in some sense "universal", hence basically of topological nature. Indeed, the proof constructs via perturbation theory and the analysis of nodal sets a simple topological obstruction. Thereby the fact that the elements of $\mathbb{D}_{2n}$ include not only rotations but also reflections are important.

In [5] these results were generalized to periodic problems in 2 dimensions and in [6] the first question was answered for periodic problems in higher dimensions. Also there the reflection planes played an essential role.

One can of course take the two questions as a starting point for a whole series of problems. This is probably much too ambitious. Here I want to state one problem explicitely since we, this is the authors of [4], worked hard on this without success.

Replace the dihedral group by the rotation group $\mathbb{Z}_n$ in $\mathbb{R}^2$. Hence the actions are just the rotations by $2k\pi/n$ and we have no reflection lines. One can analyze fairly easily $\mathbb{Z}_2$, i.e. the case of a center of inversion and also $\mathbb{Z}_4$ is doable. Here comes the problem which is just the simplest one.

Suppose $H = -\Delta + V$ is a Schrödinger operator defined on a bounded domain $\Omega$ with Dirichlet boundary condition so that $H$ commutes with the actions of $\mathbb{Z}_3$. Then, just considering the real case, we can write $H = H_s \oplus H_a$ where $H_s$ is the operator restricted to the subspace which is invariant with respect to rotations by $2\pi/3$. $H_a$ is then the operator related to the other representation which has degree 2 for the real case (rotation by $\pm 2\pi/3$). Obviously the groundstate eigenvalue $\lambda_s$ is simple and the lowest eigenvalue of $H_a$, $\lambda_a$ must have an even multiplicity $m_a$.

**Problem:**

*Prove or disprove that* $m_a = 2$.

If it turns out that $m_a$ can be greater than 2, the following seems to be interesting.

**Problem:**

*Prove or disprove the existence of a universal constant* $m$ *such that* $m_a \leq m$.

## 2 Problem 2.

Consider again a selfadjoint Schrödinger operator on a bounded domain $\Omega \subset \mathbb{R}^d$, say, with Dirichlet boundary conditions and bounded potential $V$. $H$ has compact resolvent and the eigenvalues orderd increasingly so that

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \leq \lambda_k \leq \ldots$$

tend to infinity. Without loss we can assume the eigenfunctions $u_k$ to be real valued. We define the nodal set of a $u_k$ by

$$N(u_k) = \overline{\{x \in \Omega \mid u_k(x) = 0\}}.$$

The nodal domains of $u_k$ are then the connected components of $\Omega \setminus N(u_k)$. The number of nodal domains of $u_k$ will be denoted by $\mu_k$. One of the classical results in spectral theory is Courant's nodal theorem ‡:

$$\mu_k \leq k. \tag{2.1}$$

This is in fact an extension of Sturm's oscillation theorem which says for a Schrödinger operator on an interval, i.e. a Sturm Liouville problem, that $\mu_k = k$. In 1956 Pleijel [8] showed that in (2.1) equality holds only finitely many times. In 1976, H. Lewy [7] showed for spherical harmonics that

$$\liminf_{k\to\infty} \mu_k = 2.$$

Later on, [2], related results were obtained for eigenfunctions of Laplacians on specific manifolds. The result by Lewy came quite as a surprise since for the typical examples, rectangles, or other problems which can be constructed from one-dimensional problems the number of nodal domains tends to infinity.

Here I want to pose a problem which could have been asked quite a long time ago, for instance by Prleijel about 50 years ago.

**Problem.**
Can it happen that for some Schrödinger operator

$$\limsup_{k\to\infty} \mu_k < \infty?$$

Of course one might pose the same problem for Schrödinger operators on compact manifolds. It might make a difference whether one considers the 2-dimensional case or higher dimensions.

‡see also [1] for some recent results which generalize Courant's theorem.

Once I mentioned this problem, David Jerison and independently Nikolai Nadirashvili came up with a related, presumably, simpler question. Consider for simplicity a Schrödinger operator $H$ on the sphere $S^2$. Let $\mathcal{C}(k)$ denote the number of critical points of $u_k$. Is it true that

$$\limsup_{k\to\infty} \mathcal{C}(k) = \infty?$$

There are many related problems. Smilansky and coworkers, [3], recently investigated in some very interesting work $\mu_k$ for specific cases in relation with the famous problem "can you hear the shape of a drum?".

# References


[1] A. Ancona, B. Helffer, T. Hoffmann-Ostenhof. Nodal domain theorems á la Courant. *Documenta Math.* 9: 283–299, 2004.

[2] L. Bérard-Bergery, L. Bourguignon. Laplacians and Riemannian submersions with totally geodesic fibers. *Illinois J. Math.* 26: 181–200, 1982.

[3] S. Gnutzmann, U. Smilansky, N. Sondergaard. Resolving isospectral "drums" by counting nodal domains. arXiv:nlin.CD/0504050.

[4] B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, N. Nadirashvili. Spectral theory for the dihedral group *GAFA* 12: 989–1017, 2002.

[5] B. Helffer, T. Hoffmann-Ostenhof, N. Nadirashvili. Periodic Schrödinger operators and Aharonov Bohm Hamiltonians. *Mosc. Math. J.* 258: 45–61, 2003.

[6] B. Helffer, T. Hoffmann-Ostenhof. Spectral theory for periodic Schrödinger operators with reflection symmetries. *Comm. Math. Phys.* 242: 501–529, 2003.

[7] H. Lewy. On the minimum number of domains in which the nodal lines of the spherical harmonics divide the sphere. *Comm. Partial Differential Equations* 2: 1233-1244, 1977.

[8] A. Pleijel. Remarks on Courant's nodal theorem. *Comm. Pure. Appl. Math.* 9: 543–550, 1956.



T. Hoffmann-Ostenhof: Institut für Theoretische Chemie, Universität Wien, Währinger Strasse 17, A-1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Wien, Austria.

email:thoffman@esi.ac.at