Open problems: Suppose that $D$ is a bounded open set in $\mathbb{R}^n$ and $k > 0$ is a fixed integer. Consider partitions $\{D_j\}_{1 \leq j \leq k}$ of $D$, i.e., families of disjoint open sets $D_j \subset D$ such that $D \setminus \bigcup_{1 \leq j \leq k} D_j$ has an empty interior. Let $\lambda_j > 0$ denote the first eigenvalue for the Laplacian with Dirichlet boundary conditions in $D_j$. A particle model considered in mathematical physics literature ([BHIM] K. Burdzy, R. Holyst, D. Ingerman and P. March “Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of Laplacian eigenfunctions” J. Phys. A 29, 1996, 2633–2642; [CBH] O. Cybulski, V. Babin and R. Holyst “Minimization of the Renyi entropy production in the space-partitioning process” Phys. Rev. E 71, 046130, 2005) gives rise to the following problems.

(i) Does there exist a partition which minimizes $\sum_{1 \leq j \leq k} \lambda_j$? The answer is positive—this follows from the results of D. Bucur, G. Buttazzo and A. Henrot “Existence results for some optimal partition problems” Adv. Math. Sci. Appl. 8 (1998), no. 2, 571–579.

(ii) The partition that minimizes $\sum_{1 \leq j \leq k} \lambda_j$ is not always unique, for example, if $D$ is a ball. Characterize domains using geometric conditions where the partition minimizing $\sum_{1 \leq j \leq k} \lambda_j$ is unique.

(iii) Characterize pairs $(D, k)$ such that the partition that minimizes $\sum_{1 \leq j \leq k} \lambda_j$ forms the nodal domains of an eigenfunction of the Laplacian. See [BHIM] and [CBH] for specific conjectures when $D$ is a rectangle.

(iv) Is it true that for a fixed $D$ and $k$ sufficiently large, the partition that minimizes $\sum_{1 \leq j \leq k} \lambda_j$ looks like honeycomb, i.e., most of the sets $D_j$ have an approximately hexagonal shape?