KTH Engineering Sciences

Lecture Notes on Financial Mathematics

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Preface

The aim of these notes is to present (some of) the fundamental features of financial mathematics in a rigorous way but avoiding stochastic calculus (stochastic differential equations.) This means that time is discrete, and the continuous case is considered as the limiting case when the time intervals $\to 0$.

These notes are not intended to be a stand alone text on financial mathematics; rather, they are intended to be a mathematical supplement to some text on financial economics. A good such text book is John C. Hull's: *Options, Futures, and Other Derivatives* (Prentice Hall.)

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Note 1: Introduction to Present-, Forward- and Futures Prices

Assume that we want to buy a quantity of coffee beans with delivery in nine months. However, we are concerned about what the (spot) price of coffee beans might be then, so we draw up a contract where we agree on the price today. There are now at least three ways in which we can arrange the payment: 1) we pay now, in advance. We call this price the *present* price of coffee beans with delivery in nine months time, and denote it by P. Note that this is completely different from the spot price of coffee beans, i.e., the price of coffee for immediate delivery. 2) we pay when the coffee is delivered, i.e., in nine months time. This price is the *forward* price, which we denote by G. 3) we might enter a *futures* contract with delivery in nine months time. A futures contract works as follows:

Let us denote the days from today to delivery by the numbers $0, 1, \ldots, T$, so that day 0 is today, and day T is the day of delivery. Each day n a futures price F_n of coffee beans with delivery day T is noted, and F_T equals the prevailing spot price of coffee beans day T. The futures price F_j is not known until day j; it will depend on how the coffee bean crop is doing, how the weather has been up to that day and the weather prospects up till day T, the expected demand for coffee, and so on. One can at any day enter a futures contract, and there is no charge for doing so. The long holder of the contract will each day j receive the amount $F_j - F_{j-1}$ (which may be negative, in which he has to pay the corresponding amount,) so if I enter a futures contract at day 0, I will day one receive $F_1 - F_0$, day two $F_2 - F_1$ and so on, and day T, the day of delivery, $F_T - F_{T-1}$. The total amount I receive is thus $F_T - F_0$. There is no actual delivery of coffee beans, but if I at day T buy the beans at the spot price F_T, I pay F_T, get my coffee beans and cash the amount $F_T - F_0$ from the futures contract. In total, I receive my beans, and pay F_0, and since F_0 is known already day zero, the futures contract works somewhat like a forward contract. The difference is that the value $F_T - F_0$ is paid out successively during the time up to delivery rather than at the time of delivery.

The simplest of these three contracts is the one when we pay in advance, at least if the good that is delivered is non-pecuniary, since in that case the interest does not play a part. For futures contrats, the interest rate clearly plays a part, since the return of the contract is spread out over time.

We will derive some book-keeping relations between the present prices, forward prices and futures prices, but first we need some interest rate securities.

Zero coupon bonds

A *zero coupon bond* with maturity T is a contract where the long holder pays Z_T day 0 and receives the amount 1 day T. One can take both long and short positions on zero coupon bonds. There is nothing random about a zero coupon bond, both the price Z_T paid today and the amount 1 received day T is known by day 0.

Money Market Account

A *money market account* is like a series of zero coupon bonds, maturing after only one day (or whatever periods we have in our futures contracts—it might be for example a week.) If I deposit the amount 1 day 0, the balance of my account day 1 is e^{r_1}, where r_1 is the short interest rate from day 0 to day 1. The rate r_1 is known already day 0. The next day, day 2, the balance has grown to $e^{r_1+r_2}$, where r_2 is the short interest rate from day 1 to day 2; it is random as seen from day 0, and its outcome is determined day 1. The next day, day 3, the balance has grown to $e^{r_1+r_2+r_3}$, where r_3 is the short interest rate from day 2 to day 3; it is random as seen from day 0 and day 1, and its ouctome occurs day 2, and so on. Day T the balance is thus $e^{r_1+\cdots+r_T}$ which is a *random variable.* In order to simplify the notation, we introduce the symbol $R(t,T) = r_{t+1} + \cdots + r_T$.

Relations between Present-, Forward- and Futures Prices

Let $P^{(T)}[X]$ be the present price of a contract that delivers the random value X (which may take negative values) at time T. Likewise, let $G^{(T)}[X]$ denote the forward price of the value X delivered at time T and $F_t^{(T)}[X]$ the futures price as of time t of the value X delivered at time T. We then have the following theorem:

Theorem

The following relations hold:

a) P, G *and* F_0 *are linear functions, i.e., if* X *and* Y *are random payments made at time* T*, then for any constants* a *and* b $P^{(T)}[aX+bY] = aP^{(T)}[X] + bP^{(T)}[Y]$*, and similarly for* G *and* F_0*.*

b) $G^{(T)}[1] = 1,\ F_0^{(T)}[1] = 1$ *and* $P^{(T)}[1] = Z_T$

c) $P^{(T)}[X] = Z_T G^{(T)}[X]$

d) $P^{(T)}[Xe^{R(0,T)}] = F_0^{(T)}[X]$

e) $P^{(T)}[X] = F_0^{(T)}[Xe^{-R(0,T)}]$

Proof

The proof relies on an assumption of the model: the *law of one price.* It means that there can not be two contracts that both yield the same payoff X at time T, but have different prices today. Indeed, if there were two such contracts, we would buy the cheaper and sell the more expensive, and make a profit today, and have no further cash flows in the future. But so would everyone else, and this is inconsistent with market equilibrium. In Lecture Note 4 we will extend this model assumption somewhat.

To prove c), note that if we take a long position on a forward contract on X and at the same time a long position of $G^{(T)}[X]$ zero coupon bonds, then we have a portfolio which costs $Z_T G^{(T)}[X]$ today, and yields the income X at time T. By the law of one price, it hence must be that c) holds.

To prove d), consider the following strategy: Deposit F_0 on the money market account, and take e^{r_1} long positions on the futures contract on X for delivery at time T.

The next day the total balance is then $F_0 e^{r_1} + e^{r_1}(F_1 - F_0) = F_1 e^{r_1}$. Deposit this on the money market account, and increase the futures position to $e^{r_1+r_2}$ contracts.

The next day, day 2, the total balance is then $F_1 e^{r_1+r_2} + e^{r_1+r_2}(F_2 - F_1) = F_2 e^{r_1+r_2}$. Deposit this on the money market account.

And so on, up to day T when the total balance is $F_T e^{r_1+\cdots+r_T} = X e^{r_1+\cdots+r_T}$. In this way we have a strategy which is equivalent to a contract where we pay F_0 day zero, and receive the value $F_T e^{r_1+\cdots+r_T} = X e^{r_1+\cdots+r_T}$ day T. This proves d).

Since relation d) is true for any random variable X whose outcome is known day T, we may replace X by $Xe^{-R(0,T)}$ in that relation. This proves e).

It is now easy to prove b). The fact that $P^{(T)}[1] = Z_T$ is simply the definition of Z_T. The relation $G^{(T)}[1] = 1$ now follows from c) with $X = 1$. In order to prove that $F_0^{(T)}[1] = 1$, note that by the definition of money market account, the price needed to be paid day zero in order to receive $e^{R(0,T)}$ day T is 1; hence $1 = P^{(T)}[e^{R(0,T)}]$. The relation $F_0^{(T)}[1] = 1$ now follows from d) with $X = 1$.

Finally, to prove a), note that if we buy a contracts which cost $P^{(T)}[X]$ day zero and gives the payoff X day T, and b contracts that gives payoff Y, then we have a portfolio that costs $aP^{(T)}[X] + bP^{(T)}[Y]$ day zero and yields the payoff $aX + bY$ day T; hence $P^{(T)}[aX + bY] = aP^{(T)}[X] + bP^{(T)}[Y]$. The other two relations now follow immediately employing c) and d). This completes the proof.

Comparison of Forward- and Futures Prices

Assume first that the short interest rates r_i are deterministic, i.e., that their values are known already day zero. This means that $e^{R(0,T)}$ is a constant (non-random,) so

$$1 = P^{(T)}[e^{R(0,T)}] = P^{(T)}[1]e^{R(0,T)} = Z_T e^{R(0,T)},$$

so

$$Z_T = e^{-R(0,T)}.$$

Hence

$$Z_T G^{(T)}[X] = P^{(T)}[X] = F_0^{(T)}[Xe^{-R(0,T)}] = F_0^{(T)}[X]e^{-R(0,T)} = F_0^{(T)}[X]Z_T$$

and hence

$$G^{(T)}[X] = F_0^{(T)}[X].$$

We write this down as a corollary:

Corollary

If interest rates are deterministic, the forward price and the futures price coincide: $G^{(T)}[X] = F_0^{(T)}[X]$

The equality of forward- and futures prices does not in general hold if interest rates are random, though. To see this, we show as an example that if $e^{R(0,T)}$ is random, then $F_0^{(T)}[e^{R(0,T)}] > G^{(T)}[e^{R(0,T)}]$.

Indeed, note that the function $y = \frac{1}{x}$ is convex for $x > 0$. This implies that its graph lies over its tangent. Let $y = \frac{1}{m} + k(x - m)$ be the tangent line through the point $(m, \frac{1}{m})$. Then $\frac{1}{x} \geq \frac{1}{m} + k(x - m)$ with equality only for $x = m$ (we consider only positive values of x.) Now use this with $x = e^{-R(0,T)}$ and $m = Z_T$. We then have

$$e^{R(0,T)} \geq Z_T^{-1} + k(e^{-R(0,T)} - Z_T)$$

where the equality holds only for one particular value of $R(0,T)$. In the absence of arbitrage (we will come back to this in Lecture Note 4), the futures price of the value of the left hand side is greater than the futures price of the value of the right hand side, i.e.,

$$F_0^{(T)}[e^{R(0,T)}] > Z_T^{-1} + k(F_0^{(T)}[e^{-R(0,T)}] - Z_T)$$

But $F_0^{(T)}[e^{-R(0,T)}] = P^{(T)}[1] = Z_T$, so the parenthesis following k is equal to zero, hence

$$F_0^{(T)}[e^{R(0,T)}] > Z_T^{-1}$$

On the other hand,

$$Z_T G^{(T)}[e^{R(0,T)}] = P^{(T)}[e^{R(0,T)}] = 1$$

so $G^{(T)}[e^{R(0,T)}] = Z_T^{-1}$, and it follows that $F_0^{(T)}[e^{R(0,T)}] > G^{(T)}[e^{R(0,T)}]$.

In general, if X is positively correlated with the interest rate, then the futures price tends to be higher than the forward price, and vice versa.

Spot prices, storage cost and dividends

Consider a forward contract on some asset to be delivered at a future time T. We have talked about the *forward* price, i.e., the price paid at the time of delivery for the contract, and the *present* price, by which we mean the price paid for the contract today, but where the underlying asset is still delivered at T. This should not be confused with the spot price today of the underlying asset. The present price should equal the spot price under the condition that the asset is an *investment asset* , and that there are no storage costs or dividends or other benefits of holding the asset. As an example: consider a forward contract on a share of a stock to be delivered at time T. Let r be the interest rate (so that $Z_T = e^{-rT}$) and S_0 the spot price of the share. Since 1 euro today is equivalent to e^{rT} euro at time T, the forward price should then be $G = S_0 e^{rT}$. But only if there is no dividend of the share between now and T, for if there is, then one could make an arbitrage by buying the share today, borrow for the cost and take a short position on a forward contract. There is then no net payment today, and none at T (deliver the share, collect the delivery price G of the forward and pay the $S_0e^{rT} = G$ for the loan.) But it would give the trader the dividend of the share for free, for this dividend goes to the holder of the share, not the holder of the forward. Likewise, the holder of the share might have the possibility of taking part in the annual meeting of the company, so there might be a *convenience yield.*

Note 2: Forwards, FRA:s and Swaps

Forward prices

In many cases the theorem of Lecture Note 1 can be used to calculate forward prices. As we will see later, in order to calculate option prices, it is essential to first calculate the forward price of the underlying asset.

Example 1.

Consider a share of a stock which costs S_0 today, and which gives a known dividend amount d in t years, and whose (random) spot price at time $T > t$ is S_T. Assume that there are no other dividends or other convenience yield during the time up to T. What is the forward price G on this stock for delivery at time T?

Assume that we buy the stock today, and sell it at time T. The present value of the dividend is $Z_t\, d$ and the present value of the income S_T at time T is $Z_T\, G$. Hence we have the relation

$$S_0 = Z_t\, d + Z_T\, G$$

from which we can solve for G

Example 2.

Consider a share of a stock which costs S_0 now, and which gives a known dividend yield $d\, S_t$ i t years, where S_t is the spot price immediately before the dividend is payed out. Let the (random) spot price at time $T > t$ be S_T. Assume that there are no other dividends or other convenience yield during the time up to T. What is the forward price G on this stock for delivery at time T?

Consider the strategy of buying the stock now, and sell it at time t immediately before the dividend is paid out. With the notation of Lecture Note 1, we have the relation

$$P^{(t)}(S_t) = S_0 \tag{1}$$

Consider now the strategy of buying the stock now, cash the dividend at time t, and eventually sell the stock at time T. The present value of the dividend is $d\, P^{(t)}(S_t)$ and the present value of the income S_T at time T is $Z_T\, G$. Hence we have the relation

$$S_0 = d\, P^{(t)}(S_t) + Z_T\, G$$

If we combine with (1) we get

$$(1 - d)S_0 = Z_T\, G$$

from which we can solve for G

Example 3.

With the same setting as in example 2, assume that there are dividend yield payments at several points in time $t_1, \ldots, t_n$, each time with the amount $d\, S_{t_j}$. As in the above example, we can buy the stock today and sell it just before the first dividend is paid out, so the relation

$$P^{(t_1)}(S_{t_1}) = S_0 \tag{2}$$

holds. For any $k = 2, 3, \ldots, n$ we can buy the stock at time t_{k-1} immediately before the payment of the dividend, collect the dividend, and sell the stock immediately before the dividend is paid out at time t_k. The price of this strategy today is zero, so we have

$$\begin{aligned} &0 = -P^{(t_{k-1})}(S_{t_{k-1}}) + d\,P^{(t_{k-1})}(S_{t_{k-1}}) + P^{(t_k)}(S_{t_k}), \text{ i.e.,} \\ &P^{(t_k)}(S_{t_k}) = (1-d)P^{(t_{k-1})}(S_{t_{k-1}}) \end{aligned}$$

and repeated use of this relation gives

$$P^{(T)}(S_T) = (1-d)^n P^{(t_1)}(S_{t_1}) = (1-d)^n S_0$$

where we have used (2) to obtain the last equality. Hence, by the theorem of Lecture Note 1, we have the relation

$$Z_T\, G = (1-d)^n S_0 \tag{3}$$

Example 4.

We now consider the setting in example 3, but with a continuous dividend yield ρ, i.e., for any small interval in time $(t, t + \delta t)$ we get the dividend $\rho S_t\, \delta t$. If we divide the time interval $(0, T)$ into a large number n of intervals of length $\delta t = T/n$, we see from (3) that

$$Z_T\, G = (1 - \rho\, \delta t)^n S_0$$

and when $n \to \infty$ we get

$$Z_T\, G = e^{-\rho T} S_0$$

Example 5.

Assume we want to buy a foreign currency in t years time at an exchange rate, the *forward rate,* agreed upon today. Assume that the interest on the foreign currency is ρ and the domestic rate is r per year. Let X_0 be the exchange rate now (one unit of foreign currency costs X_0 in domestic currency,) and let X_t be the (random) exchange rate as of time t. Let G be the forward exchange rate.

Consider now the strategy: buy one unit of foreign currency today, buy foreign zero coupon bonds for the amount, so that we have $e^{\rho t}$ worth of bonds at time t when we sell the bonds. Since the exchange rate at that time is X_t, we get $X_t\, e^{\rho t}$ in domestic currency. Since the price we have paid today is X_0 we have the relation

$$X_0 = P^{(t)}\big(X_t\, e^{\rho t}\big) = e^{-rt}\, G\, e^{\rho t} = G\, e^{(\rho - r)t}$$

Forward Rate Agreements

A *forward rate agreement* is a forward contract where the parties agree that a certain interest rate will be applied to a certain principal during a future time period. Let us say that one party is to borrow an amount L at time t and later pay back the amount $Le^{f(T-t)}$ at time $T > t$. The cash flow for this party is thus L at time t and $-Le^{f(T-t)}$ at time T. Since this contract costs nothing now, we have the relation

$$0 = Z_t L - Z_T L e^{f(T-t)}.$$

From this we can solve for f. The interest rate f is the *forward rate* from t to T.

Plain Vanilla Interest Rate Swap

The simplest form of an interest swap is where one party, say A, pays party B:

the floating interest on a principal L_1 from time t_0 to t_1 at time t_1
the floating interest on a principal L_2 from time t_1 to t_2 at time t_2
the floating interest on a principal L_3 from time t_2 to t_3 at time t_3
...
the floating interest on a principal L_n from time t_{n-1} to t_n at time t_n.

The floating rate between t_j and t_k is the zero coupon rate that prevails between these two points in time. The amount that A pays at time t_k is thus $L_k \cdot (1/Z(t_{k-1}, t_k) - 1)$, where $Z(t_{k-1}, t_k)$ of course is the price of the zero coupon bond at time t_{k-1} that matures at t_k. Note that this price is unknown today but known at time t_{k-1}. The total amount that B will receive, and A will pay is thus random.

On the other hand, party B pays A a fixed amount c at each of the times $t_1, \ldots, t_n$. The question is what c ought to be in order to make this deal "fair".

Consider the following strategy: Do nothing today, but at time t_{k-1} buy $1/Z(t_{k-1}, t_k)$ zero coupon bonds that mature at time t_k. This gives a cash flow of -1 at time t_{k-1} and $1/Z(t_{k-1}, t_k)$ at time t_k The price today of this cash flow is zero, so

$$\begin{aligned} 0 = -Z_{t_{k-1}} + P^{(t_k)}(1/Z(t_{k-1}, t_k)) \quad \text{i.e.,} \\ P^{(t_k)}\big(1/Z(t_{k-1}, t_k)\big) = Z_{t_{k-1}} \end{aligned}$$

We can now calculate the present value P_k of the payment of A to B at time t_k:

$$P_k = L_k \cdot P^{(t_k)}(1/Z(t_{k-1}, t_k) - 1) = L_k(Z_{t_{k-1}} - Z_{t_k})$$

So the total present value of A:s payments to B is

$$P_{AB} = \sum_1^n L_k(Z_{t_{k-1}} - Z_{t_k})$$

The present value of B:s payments to A is, on the other hand,

$$P_{BA} = c \sum_{1}^{n} Z_{t_k}$$

so we can calculate the fair value of c by solving the equation we get by setting $P_{BA} = P_{AB}$.

Note 3: Optimal Hedge Ratio

A company that knows that it is due to buy an asset in the future can *hedge* by taking a long futures position on the asset. Similarly, a firm that is going to sell may take a short hedge. But it may happen that there is no futures contract on the market for the exact product or delivery date. For instance, the firm might want to buy petrol or diesel, whereas the closest futures contract is on crude oil, or the delivery date of the futures contract is a month later than the date of the hedge. In this case one might want to use several futures on different assets, or delivery times, to hedge. Let S be the price of the asset to be hedged at the date t of delivery, and $F_t^1, \ldots, F_t^n$ the futures prices at date t of the contracts that are being used to hedge. All these prices are of course random as seen from today, but we assume that there are enough price data so that it is possible to estimate their variances and covariances.

Assume that for each unit of volume of S we use futures contracts corresponding to β_i units of volume for contract F^i. The difference between the spot price S and the total futures price at the date of the end of the hedge is $e = S - \sum_{i=1}^n \beta_i F_t^i$, or

$$S = \sum_{i=1}^n \beta_i F_t^i + e$$

The total price paid for the asset including the hedge is $S - \sum_{i=1}^n \beta_i (F_t^i - F_0^i) = e + \sum_{i=1}^n \beta_i F_0^i$, where only e is random, i.e., unknown to us today. The task is to choose β_i such that the variance of the residual e is minimised.

Lemma

If we choose the coefficients β_i such that $\mathrm{Cov}(F_t^i, e) = 0$ *for $i = 1, \ldots, n$, then the variance* $\mathrm{Var}(e)$ *is minimised.*

Proof

Assume that we have chosen β_i such that $\mathrm{Cov}(F_t^i, e) = 0$ for $i = 1, \ldots, n$, and consider any other choise of coefficients:

$$S = \sum_{i=1}^n \gamma_i F_t^i + f$$

For notational convenience, let $\delta_i = \beta_i - \gamma_i$. We can write the residual f as

$$\begin{aligned} f = S - \sum_{i=1}^n \gamma_i F_t^i &= \sum_{i=1}^n \beta_i F_t^i + e - \sum_{i=1}^n \gamma_i F_t^i \\ &= \sum_{i=1}^n \delta_i F_t^i + e \end{aligned}$$

Note that since $\mathrm{Cov}(F_t^i, e) = 0$ it holds that $\mathrm{Cov}(\sum_{i=1}^n \delta_i F_t^i, e) = 0$, hence

$$\mathrm{Var}(f) = \mathrm{Var}\left(\sum_{i=1}^n \delta_i F_t^i\right) + \mathrm{Var}(e) \geq \mathrm{Var}(e)$$

Q.E.D.

Theorem

The set of coefficients β_i that minimises the variance $\mathrm{Var}(e)$ *of the residual is the solution to the system*

$$\sum_{i=1}^{n} \mathrm{Cov}(F_t^j, F_t^i)\beta_i = \mathrm{Cov}(F_t^j, S) \quad j = 1, \ldots, n.$$

Proof

The condition that $\mathrm{Cov}(F_t^j, e) = 0$ means that

$$\begin{aligned} 0 &= \mathrm{Cov}(F_t^j, S - \sum_{i=1}^{n} \beta_i F_t^i) \\ &= \mathrm{Cov}(F_t^j, S) - \sum_{i=1}^{n} \beta_i \, \mathrm{Cov}(F_t^j, F_t^i) \end{aligned}$$

from which the theorem follows. *Q.E.D.*

If we regard S and F_t^i as any random variables, then the coefficients β_i are called the *regression coefficients* of S onto $F_t^1, \ldots, F_t^n$; in the context here they are the *optimal hedge ratios* when the futures $F_t^1, \ldots, F_t^n$ are used to hedge S: for each unit of volume of S we should use futures F_t^i corresponding to β_i units of volume in the hedge.

Let us consider the case $n = 1$. In this case we have that

$$\beta = \frac{\mathrm{Cov}(S, F_t)}{\mathrm{Var}(F_t)}$$

Using the lemma, we have the following relation of variances:

$$\begin{aligned} \mathrm{Var}(S) &= \beta^2 \, \mathrm{Var}(F_t) + \mathrm{Var}(e) = \frac{\mathrm{Cov}^2(S, F_t)}{\mathrm{Var}(F_t)} + \mathrm{Var}(e) \\ &= \rho^2(S, F_t) \, \mathrm{Var}(S) + \mathrm{Var}(e) \end{aligned}$$

where $\rho(S, F_t) = \mathrm{Cov}(S, F_t)/\sqrt{\mathrm{Var}(S)\,\mathrm{Var}(F_t)}$ is the *correlation coefficient* between S and F_t.

Solving for $\mathrm{Var}(e)$ we get the plesant relation $\mathrm{Var}(e) = \mathrm{Var}(S)\big(1 - \rho^2(S, F_t)\big)$ which means that the *standard deviation* of the hedged position is $\sqrt{1 - \rho^2(S, F_t)}$ times that of the unhedged position.

Note 4: Conditions for No Arbitrage

Let us consider a two period market model of contracts where one party, the "long" holder, pays a sum P,—the present price—today when the contract is drawn up, and the other part delivers some value X at some later date t. A contract is defined by the random payoff X and the present price P; we describe it by the pair (P, X). The total cash flow for the long holder at time t is thus X, which is random, whereas the cash flow today is $-P$ and is deterministic. There is no other cash flow in relation to entering the contract.

The market consists of a set of contracts, and we assume that we can compose arbitrary portfolios of contracts, i.e., if we have contracts $1, \dots, n$ with payoffs $X_1, \dots, X_n$ and present prices $P_1, \dots, P_n$, then we can compose a portfolio consisting of λ_i units of contract i, $i = 1, \dots, n$ where $\lambda_1, \dots, \lambda_n$ are arbitrary real numbers. I.e., we assume that we can take a short position in any contract, and ignore any divisibility problems. The total present price of such a portfolio is of course $\sum_1^n \lambda_i P_i$ and the total payoff is $\sum_1^n \lambda_i X_i$. We call also such portfolios "contracts", so the set of contracts, defined as pairs (P, X), constitute a *linear space.*

We assume that the market is *arbitrage free* in the sense that the *law of one price* prevails (Lecture Note 1,) but also in the sense that

1. If $X \geq 0$, then $P(X) \geq 0$.
2. If $X = 0$ almost surely (abbr. "a.s.", i.e., with probability 1) then $P(X) = 0$.
3. If $X \geq 0$ and $P(X) = 0$, then $X = 0$ a.s.

These conditions avoid obvious arbitrage opportunities.

The operator $P(X)$ behaves very much like an expectations operator for a probability measure. We will now show that if the underlying sample space is finite, then there is a random variable U such that $U > 0$, and such that for any payoff X, $P(X) = \mathrm{E}[XU]$.

It is obvious that the opposite is true: If there is a random variable Q with the above properties, then the market satisfies the no arbitrage conditions 1) – 3).

Theorem

If the sample space is finite and the market is arbitrage free in the sense given above, then there exists a random variable U such that $U > 0$, and for any payoff X it holds that $P(X) = \mathrm{E}[XU]$.

Proof

Let $\Omega = \{\omega_1, \dots, \omega_n, \dots, \omega_N\}$ be the sample space and without loss of generality we may assume that ω_k have positive probabilities p_k for $k \leq n$ and zero probability for $k > n$.

We associate with each contract (P, X) the vector $\big(-P, X(\omega_1), \dots, X(\omega_n)\big)$ in $\mathbf{R}^{n+1}$ i.e., we ignore X:s values on events with probability zero. The set of such vectors constitute a linear subspace V of $\mathbf{R}^{n+1}$. We now prove that there is a vector $\bar{q}$ which is orthogonal to V all of whose coordinates are positive.

Let K be the subset of $\mathbf{R}^{n+1}$: $K = \{\bar{u} = (u_0, \dots, u_n) \in \mathbf{R}^{n+1} \mid u_0 + \cdots + u_n = 1 \text{ and } u_i \geq 0 \text{ for all } i\}$. Obviously K and V have no vector in common; indeed, it is easy to see that any such common vector would represent an arbitrage. Now let $\bar{q}$ be the vector of shortest Euclidean length such that $\bar{q} = \bar{u} - \bar{v}$ for some vectors $\bar{u}$ and $\bar{v}$ in K and V respectively. The fact that such a vector exists needs to be proved, however, we will skip that proof. We write $\bar{q} = \bar{u}^* - \bar{v}^*$ where $\bar{u}^* \in K$ and $\bar{v}^* \in V$.

Now note that for any $t \in [0,1]$ and any $\bar{u} \in K$, $\bar{v} \in V$, the vector $t\bar{u} + (1-t)\bar{u}^* \in K$ and $t\bar{v} + (1-t)\bar{v}^* \in V$, hence $|(t\bar{u} + (1-t)\bar{u}^*) - (t\bar{v} + (1-t)\bar{v}^*)|$ as a function of t on $[0,1]$ has a minimum at $t = 0$, by definition of $\bar{u}^*$ and $\bar{v}^*$, i.e., $|t(\bar{u} - \bar{v}) + (1-t)\bar{q}|^2 = t^2|\bar{u} - \bar{v}|^2 + 2t(1-t)(\bar{u} - \bar{v}) \cdot \bar{q} + (1-t)^2|\bar{q}|^2$ has minimum at $t = 0$ which implies that the derivative w.r.t. t at $t = 0$ is ≥ 0. This gives $(\bar{u} - \bar{v}) \cdot \bar{q} - |\bar{q}|^2 \geq 0$ or, equivalently, $\bar{u} \cdot \bar{q} - |\bar{q}|^2 \geq \bar{v} \cdot \bar{q}$ for all $\bar{v} \in V$ and $\bar{u} \in K$. But since V is a linear space, it follows that we must have $\bar{v} \cdot \bar{q} = 0$ for all $\bar{v} \in V$. It remains to prove that $\bar{q}$ has strictly positive entries. But we have $\bar{u} \cdot \bar{q} - |\bar{q}|^2 \geq 0$ for all $\bar{u} \in K$, in particular we can take $\bar{u} = (1, 0, \ldots, 0)$ which shows that the first entry of $\bar{q}$ is > 0 and so on.

Hence, we have found positive numbers $q_0, \ldots, q_n$ such that

$$-q_0 P + q_1 X(\omega_1) + \cdots + q_n X(\omega_n) = 0$$

for all contracts, and by scaling the q_i:s we can arrange that q_0 is equal to one, hence

$$P = \sum_{1}^{n} q_i X(\omega_i) \tag{1}$$

We now define $U(\omega_i) = \frac{q_i}{p_i}$ for $i = 1, \ldots, n$ and $U(\omega_i) =$ arbitrarily positive for $i = n+1, \ldots, N$. U is thus a positive random variable, and it is easy to verify that it satisfies the conditions of the theorem. *Q.E.D.*

The No Arbitrage Assumption

> *It is easy to see that also in the case of an infinite sample space, it is a* sufficient *condition for the absence of arbitrage that there exists a random variable U as in the theorem. In the sequel we will always assume the existence of such a random variable U.*

Note that by definition, the zero coupon price $Z_t = P(1) = \mathrm{E}[U]$. Define the random variable Q by $Q = Z_t^{-1} U$, then $Q > 0$, $\mathrm{E}[Q] = 1$ and

$$P(X) = Z_t \mathrm{E}[XQ]$$

which shows that the expectation is equal to the forward price:

$$G(X) = \mathrm{E}[XQ].$$

Note 5: Pricing European Derivatives

Black's Model

Let us compute the price of a European derivative on some underlying asset with value X at maturity t. We assume that there already is a forward contract on the market, or that we can compute the forward price, and denote the current forward price for delivery at t by G. Black's model assumes that the value X has a log-normal probability distribution:

$$X = Ae^{\sigma\sqrt{t}\,z} \quad \text{where} \quad z \in N(0,1) \tag{1}$$

A European derivative on X is a contract on a payoff of some function $f(X)$ of X. In accordance with Lecture Note 4 we assume the existence of a random variable Q such that $Q > 0$, $\mathrm{E}[Q] = 1$ and such that forward prices are set according to

$$G\big(f(X)\big) = \mathrm{E}\left[f(X)Q\right]$$

In this formula, Q can be replaced by $\mathrm{E}[Q \mid z]$ which is a function of z, so in the current context, we can regard Q to be a function of z. A natural choice is a log-normal specification: $Q = e^{a+bz}$ for some constants a and b. The condition $\mathrm{E}[Q] = 1$ imposes the restriction $a = -\frac{1}{2}b^2$. For notational convenience we choose $b = -\lambda\sqrt{t}$, i.e.,

$$Q = e^{-\frac{1}{2}\lambda^2 t - \lambda\sqrt{t}\,z}$$

Now we can express the forward value of the underlying asset as

$$G(X) = \mathrm{E}[XQ] = \mathrm{E}\left[Ae^{\sigma\sqrt{t}\,z}\,e^{-\frac{1}{2}\lambda^2 t - \lambda\sqrt{t}\,z}\right] = Ae^{\frac{1}{2}\sigma^2 t - \sigma\lambda t}$$

Hence $A = G(X)e^{\sigma\lambda t - \frac{1}{2}\sigma^2 t}$ which we substitute into (1):

$$X = G(X)e^{\sigma\lambda t - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,z} \tag{2}$$

The forward price of any derivative of X is thus

$$\begin{aligned}
G\big(f(X)\big) &= \mathrm{E}\left[f(X)Q\right] \\
&= \mathrm{E}\left[f\big(G(X)e^{\sigma\lambda t - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,z}\big)\,e^{-\frac{1}{2}\lambda^2 t - \lambda\sqrt{t}\,z}\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\big(G(X)e^{\sigma\lambda t - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,x}\big)\,e^{-\frac{1}{2}\lambda^2 t - \lambda\sqrt{t}\,x}\,e^{-\frac{1}{2}x^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\big(G(X)e^{\sigma\lambda t - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,x}\big)\,e^{-\frac{1}{2}(x+\lambda\sqrt{t})^2}\,dx \\
&= \big[\text{substitute } x + \lambda\sqrt{t} = y\big] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\big(G(X)e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,y}\big)\,e^{-\frac{1}{2}y^2}\,dy \\
&= \mathrm{E}\left[f\big(G(X)e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w}\big)\right] \quad \text{where} \quad w \in N(0,1)
\end{aligned}$$

Finally, if G denotes the forward price of the underlying asset, the present price of the derivative is:

$$p = Z_t \,\mathrm{E}\left[f\left(Ge^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w}\right)\right] \quad \text{where} \quad w \in N(0,1) \tag{3}$$

This is *Blacks pricing formula.*

The Black-Scholes Pricing Formula

Let us consider the case when the underlying asset is an investment asset with no dividends or convenience yield, for example, it might be a share of a stock which pays no dividend before maturity of the contract. In this case there is a simple relationship between the forward price G and the current spot price S_0 of the underlying asset. Indeed, $G = e^{rt} S_0$ where r is the t-year zero coupon interest rate. To see this, note that $S_0 = P^{(t)} = Z_t G$, i.e., $S_0 = e^{-rt} G$, see Lecture Note 2, example 1 with $d = 0$. Hence the present price of the derivative can be written

$$p = e^{-rt}\,\mathrm{E}\left[f\left(S_0 e^{rt - \frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w}\right)\right] \quad \text{where} \quad w \in N(0,1)$$

This is the *Black-Scholes pricing formula.*

Put and Call Options

A call option with strike price K is specified by $f(X) = \max[X - K, 0]$, and a put option by $f(X) = \max[K - X, 0]$. In these cases the expectation in Black's formula can be evaluated, and the result is that the call option price c and put option price p are

$$c = Z_t\big(G\Phi(d_1) - K\Phi(d_2)\big)$$
$$p = Z_t\big(K\Phi(-d_2) - G\Phi(-d_1)\big)$$

where

$$d_1 = \frac{\ln(G/K)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$
$$d_2 = d_1 - \sigma\sqrt{t}$$

and Φ is the cumulative normal distribution

The Interpretation of σ and λ

The parameter σ is called the *volatility* of the asset X. It is the standard deviation of the logarithm of X divided by the square-root of the time to maturity. If the value of X is some constant times the product of identically distributed independent random variables whose outcome occur every day (or hour, or any short time interval,) then $\ln X$ will be approximately normally distributed. Since the variance is proportional to the number of days (or whatever interval used,) the variance is proportional to the time interval until maturity. So the standard deviation is proportional to the square-root of the time interval to maturity.

If we compute the expected value in (2), we get

$$\mathrm{E}[X] = Ge^{\lambda\sigma t}$$

which means that the expected rate of return on a forward contract on X is $\lambda\sigma$. The expected rate of return is thus proportional to the standard deviation σ, and λ is called the *market price of risk.*

The situation when $\lambda > 0$ is called *backwardation,* and when $\lambda < 0$ the situation is known as *contango.*

Note 6: Yield and Duration

Consider at time $t = 0$ a zero coupon bond which matures at time $t = D$ with a face value of P_D. The present value of this bond is of course $P_0(y_0) = P_D e^{-y_0 D}$ where y_0 is the relevant zero coupon interest rate. Note that

$$D = -\frac{1}{P_0} P_0'(y_0) \tag{1}$$

$$P_0(y) = P_D e^{-yD} \tag{2}$$

$$P_t(y) = P_D e^{-y(D-t)} \tag{3}$$

where $P_t(y)$ is the value at time t, if the zero coupon interest rate at that time equals y, which may differ from y_0.

The idea of the concepts *yield* (y) and *duration* (D) is to define a number D for a coupon paying bond, or a portfolio of bonds, such that the relations (2) and (3) still hold approximately. Hence, let P_0 be the present value of a bond, or a portfolio of bonds:

$$P_0 = \sum_{i=1}^{n} Z_{t_i} c_i$$

where Z_s is the price today of a zero coupon bond maturing at time s and c_i is the payment received at time t_i. Now define the *yield* y_0 by the relation

$$P_0 = \sum_{i=1}^{n} e^{-y_0 t_i} c_i = P_0(y_0) \tag{4}$$

and the *duration* D by

$$D = -\frac{1}{P_0} P_0'(y_0) = \frac{1}{P_0} \sum_{i=1}^{n} t_i \, e^{-y_0 t_i} c_i \tag{5}$$

Now define $\hat{P}_D(y) = P_0(y) e^{yD}$ [see (4).] We then have

$$\hat{P}_D'(y_0) = P_0'(y_0) e^{y_0 D} + D P_0(y_0) e^{y_0 D} = 0,$$

hence we can consider $\hat{P}_D(y)$ to be constant regarded as a function of y, to a first order approximation. Henceforth we will treat $\hat{P}_D = \hat{P}_D(y_0) = P_0 e^{y_0 D}$ as constant. We now have

$$P_0(y) = \hat{P}_D e^{-yD} \tag{2'}$$

$$P_t(y) = \hat{P}_D e^{-y(D-t)} \tag{3'}$$

Here (2′) follows from the definition of $\hat{P}_D$ and (3′) holds if the yield is y at time t, as is seen as follows:

$$P_t(y) = \sum_{i=1}^{n} e^{-y(t_i - t)} c_i = e^{yt} \sum_{i=1}^{n} e^{-y t_i} c_i = e^{yt} P_0(y) = \hat{P}_D e^{-y(D-t)}$$

Summary

A bond, or a portfolio of bonds, has a yield y_0, defined by (4), and a duration D, defined by (5). At any time t before any coupon or other payments have been paid out, the value $P_t(y)$ of the asset is approximately equal to

$$P_t(y) = \hat{P}_D e^{-y(D-t)} \quad \text{where} \quad \hat{P}_D = P_0 e^{y_0 D}$$

and y is the yield prevailing at that time. In particular, the value of the asset at time $t = D$ is $\hat{P}_D = P_0 e^{y_0 D}$, and hence independent of any changes in the yield to a first order approximation.

Example

Consider a portfolio of bonds that gives the payment 1'000 after one year, 1'000 after two years and 2'000 after three years. Assume that $Z_1 = 0.945$, $Z_2 = 0.890$ and $Z_3 = 0.830$. The present value of the portfolio is then

$$P_0 = 0.945 \cdot 1\text{'}000 + 0.890 \cdot 1\text{'}000 + 0.830 \cdot 2\text{'}000 = 3\text{'}495$$

The yield y_0 is obtained by solving for y_0 in the relation

$$3\text{'}495 = P_0 = e^{-y_0} 1\text{'}000 + e^{-2y_0} 1\text{'}000 + e^{-3y_0} 2\text{'}000$$

which gives $y_0 = 0.06055$. Now

$$-P_0'(y_0) = e^{-y_0} 1\text{'}000 + 2\, e^{-2y_0} 1\text{'}000 + 3\, e^{-3y_0} 2\text{'}000 = 7\text{'}716.5$$

so the duration is $D = \frac{7\text{'}716.5}{3\text{'}495}$ years $= 2.208$ years. The value of the portfolio at time t, if the yield at that time is y and t is less than one year, can thus be approximated by

$$P_t(y) = 3\text{'}995\, e^{-y(2.208-t)}$$

(where $3\text{'}995 = P_0 e^{y_0 D}$.)

Forward Yield and Forward Duration

Consider a bond, or any interest rate security, that after time t gives the holder deterministic payments $c_1, c_2, \ldots, c_n$ at times $t_1 < t_2 < \cdots < t_n$ (where $t_1 > t$.) Today is time $0 < t$ and we want to compute the forward price of the security to be delivered at time t. Note that the security may give the holder payments also before time t, but this is of no concern to us, since we only are interested in the forward value of the security.

Let P_t be the (random) value of the security at time t and consider the strategy of buying the security at time t for this price and then cash the payments $c_1, \ldots, c_n$. The pricing methods of Lecture Note 2 shows that the *forward* price $G^{(t)}$ of the value P_t is given by

$$Z_t G^{(t)} = \sum_{i=1}^{n} Z_{t_i} c_i$$

The *forward yield* y_F is defined by

$$G^{(t)} = \sum_{i=1}^{n} c_i e^{-y_F(t_i - t)} \tag{6}$$

When we arrive at time t it may be that the prevailing yield y at that time is not the same as y_F. Let y be the prevailing yield at time t. Then the value of the security at that time is

$$P_t = P_t(y) = \sum_{i=1}^{n} c_i e^{-y(t_i - t)}$$

In particular, we see that $P_t(y_F) = G^{(t)}$. We define the *forward duration* D_F of the portfolio for time t as

$$D_F = -\frac{P_t'(y_F)}{P_t(y_F)} = \frac{1}{G^{(t)}} \sum_{i=1}^{n} c_i (t_i - t) e^{-y_F(t_i - t)}. \tag{7}$$

Now define P_F by $P_F(y) = P_t(y) e^{D_F\, y}$. The same calculation as in the previous section shows that P_F is independent of y, to a first order approximation, so we will regard $P_F = P_F(y_F) = G^{(t)} e^{D_F\, y_F}$ as a constant. So we have

$$P_t = P_F e^{-D_F\, y}$$

Summary

An interest rate security that after time t gives the deterministic payments $c_1, c_2, \ldots, c_n$ at times $t_1 < t_2 < \cdots < t_n$ has a forward yield y_F, defined by (6), and a forward duration D_F defined by (7). The value of the asset as of time t, $0 < t < t_1$ is then, to a first order approximation

$$P_t = P_F e^{-D_F\, y} \quad \text{where} \quad P_F = G^{(t)} e^{D_F\, y_F}$$

and y is the yield prevailing at that time.

Example

In the previous example, assume that $Z_{1.5} = 0.914$. The forward price $G_{1.5}$ of the portfolio for delivery in one and a half years is obtained from the relation (note that the first payment of 1'000 has already been paid out, and hence does not contribute to the forward value:)

$$0.914\, G_{1.5} = 0.890 \cdot 1'000 + 0.830 \cdot 2'000 \quad \Rightarrow \quad G_{1.5} = 2'790$$

The forward yield y_F is obtained from

$$2'790 = G_{1.5} = e^{-0.5 y_F} 1'000 + e^{-1.5 y_F} 2'000 \quad \Rightarrow \quad y_F = 0.06258$$

Now

$$-G'_{1,5}(y_F) = 0.5\, e^{-0.5 y_F}\, 1\text{'}000 + 1.5\, e^{-1.5 y_F}\, 2\text{'}000 = 3\text{'}215.8$$

so the forward duration is $D_F = \frac{3\text{'}215.8}{2\text{'}790}$ years $= 1.153$ years.

Black's Model for Bond Options

Consider again the bond or security in the section *"Forward Yield and Forward Duration"* above. In this section we want to price a European option on the value P_t, a *bond option.*

First we specify a random behaviour of y, the yield that will prevail for the underlying security at time t. We will assume that as seen from today, y is a normally distributed random variable:

$$y = \alpha - \sigma\sqrt{t}\, z, \quad \text{where } z \in \mathrm{N}(0,1)$$

The minus sign is there for notational convenience later on; note that z and $-z$ have the same distribution. Hence, the present value P_t at time t of the asset is

$$P_t = P_F e^{-D_F(\alpha - \sigma\sqrt{t}\, z)} = A\, e^{D_F\, \sigma\sqrt{t}\, z}$$

where $A = P_F e^{-D_F\, \alpha}$.

Note that this is exactly the same specification as (1) in Lecture Note 5 with σ replaced by $D_F\, \sigma$. We can thus use Black's formula for European options of Lecture Note 5, where $\sigma\sqrt{t}$ in that formula stands for the product of the standard deviation of the forward yield and the forward duration of the underlying asset.

Portfolio Immunising

Assume that we have portfolio P of bonds whose duration is D_P. The duration reflects its sensitivity to changes in the yield, so we might well want to change that without making any further investments. One way to do this is to add a *bond futures* to the portfolio (or possibly short such a futures.)

Consider again the security discussed under *"Forward Yield and Forward Duration"* above. We will look at a futures contract on the value P_t. Using the notation of Lecture Note 1, the futures price is $F_0 = F_0(P_t) = F_0(P_F e^{-D_F\, y})$, where the yield y at time t is random as seen from today. Assume now that the present yield y_0 of the portfolio P changes to $y_0 + \delta y$. The question arises what impact this has on y. If $y =$ (current yield) $+$ (random variable independent of current yield), then y should simply be replaced by $y + \delta y$. The new futures price should then be

$$F_0(P_F e^{-D_F\, (y+\delta y)}) = e^{-D_F\, \delta y} F_0(P_F e^{-D_F\, y}) = e^{-D_F\, \delta y} F_0$$

The "marking-to-market" is hence $(e^{-D_F\, \delta y} - 1)F_0$.

Assume now that we add such a futures contract to our initial portfolio P. We let $P = P(y_0)$ denote its original value, and $P(y_0 + \delta y)$ its value after the yield has shifted by δy. The value of the total portfolio P_{tot} before the shift is equal to P, since the *value* of the futures contract is zero. However, after the yield shift, the marking-to-market is added to the portfolio, so after the shift the value of the total portfolio is

$$P_{tot}(y_0 + \delta y) = P(y_0 + \delta y) + (e^{-D_F\,\delta y} - 1)F_0$$

If we differentiate this w.r.t. δy we get for $\delta y = 0$

$$-P'_{tot}(y_0) = -P'(y_0) + D_F F_0 = D_P P + D_F F_0$$

The duration of the total portfolio is thus

$$D_{tot} = -\frac{1}{P} P'_{tot}(y_0) = D_P + D_F \frac{F_0}{P}$$

If we take N futures contracts (where negative N corresponds to going short the futures,) we get

$$D_{tot} = D_P + N D_F \frac{F_0}{P}$$

Note 7: Risk Adjusted Probability Measures

In Lecture Note 4 we introduced the notion that a forward price can be expressed as an expectation: $G^{(t)}(X) = \mathrm{E}\,[XQ]$ for some random variable $Q >$ with $\mathrm{E}\,[Q] = 1$. In probability theory, what we have done is called a *change of probability measure.* If Q is as above, then it is convenient to introduce a notation for the expected value $\mathrm{E}\,[XQ]$, for example

$$\mathrm{E}^*[X] = \mathrm{E}\,[XQ]$$

and say that $\mathrm{E}^*[X]$ is the expected value of X with respect to a *new probability measure* Pr^*, where the probability of an event A with this new probability measure is defined by

$$\mathrm{Pr}^*(A) = \mathrm{E}^*[I_A] = \mathrm{E}\,[I_A Q]$$

where I_A is the *indicator function* of A, i.e., $I_A(\omega)$ is equal to one if $\omega \in A$ and zero otherwise. It is easy to check that Kolmogorov's axioms for probabilities are satisfied for the probabilities $\mathrm{Pr}^*(A)$, and also that the expextation $\mathrm{E}^*[\cdot]$ satisfies the usual rules for an expectation:

$\mathrm{E}^*[X] \geq 0$ if $X \geq 0$,

$\mathrm{E}^*[1] = 1$,

$\mathrm{E}^*[\lambda_1 X_1 + \lambda_2 X_2] = \lambda_1 \mathrm{E}^*[X_1] + \lambda_2 \mathrm{E}^*[X_2]$,
if λ_1, λ_2 are real numbers and X_1, X_2 random variables.

The random variable Q is called the *Radon-Nikodym derivative* of the new probability measure w.r.t. the natural measure. When U is defined as in Lecture Note 4, we have $G_0^{(t)}(X) = \mathrm{E}\,[XQ]$ and we call the new probability measure the *forward measure* with maturity date t, and use the notation $\mathrm{E}^{(t)}$ for the expectation, i.e.,

$$G_0^{(t)}(X) = \mathrm{E}^{(t)}[X]$$

Choise of Numeraire

The forward probability measure introduced above is a very important example of a *risk adjusted probability measure,* but one can also use other *numeraires.* Let N be some random value whose outcome becomes known at some date t, and assume that $N > 0$. For instance, N may be the value of one barrel of crude oil at time t. Now define the random variable Q_N by

$$Q_N = \frac{1}{P_0^{(t)}(N)} UN$$

where U is defined as in Lecture Note 4. Then $Q_N > 0$ and $\mathrm{E}\,[Q_N] = 1$ (verification left to the reader) and we have for any random variable X whose outcome is known at time t

$$P_0^{(t)}(X) = P_0^{(t)}(N)\,\mathrm{E}\,\Big[\frac{X}{N} Q_N\Big].$$

The verification is left as an exercise for the reader. If we denote by $\mathrm{E}^N[\cdot]$ the expectation associated with Q_N, then

$$P_0^{(t)}(X) = P_0^{(t)}(N)\,\mathrm{E}^N\big[\frac{X}{N}\big] \tag{1}$$

Here N is the *numeraire.* For example, if X is the price of jet fuel at time t and the numeraire N is the price of crude oil at time t, the idea could be that we want to calculate the present price for jet fuel to be delivered at time t, but rather than creating a model for the price of jet fuel, we might find it easier to come up with a reasonable model for how the market perceives the ratio X/N of jet fuel to crude oil. We can then from that model compute $\mathrm{E}^N\big[\frac{X}{N}\big]$, whereas the present price $P_0^{(t)}(N)$ can be directly observed on the market (since there is a futures market for crude oil.)

If we choose $N =$ one unit of some currency , i.e., the numeraire asset is a zero coupon bond, then we have

$$P_0^{(t)}(X) = P_0^{(t)}(1)\,\mathrm{E}^N[X] = Z_t\,\mathrm{E}^N[X]$$

i.e., $\mathrm{E}^N = G_0^{(t)} = \mathrm{E}^{(t)}$, the forward mesaure with maturity date t.

Forward Measures for Different Maturities

The forward expectation $\mathrm{E}^{(t)}$ depends on the time of maturity t; i.e., in general, if $t < T$, then $\mathrm{E}^{(t)}(X) \neq \mathrm{E}^{(T)}(X)$ However, if interest rates are deterministic, then the two coincide; more specifically, we have the following theorem: let $Z(t,T)$ be the price at time t of a zero coupon bond maturing at T;

Theorem 1

Assume that $t < T$ and that the zero coupon bond price $Z(t,T)$ is known at the present time 0. Then, for any random variable X whose outcome is known at time t,

$$\mathrm{E}^{(t)}[X] = \mathrm{E}^{(T)}[X]$$

Proof:

Consider the following strategy: Enter a contract forward contract on X maturing at t (so $G^{(t)} = \mathrm{E}^{(t)}[X]$,) invest the net payment $X - G^{(t)}$ in zero coupon bonds maturing at T. In this way we have constructed a contract which pays $Z(t,T)^{-1}\,(X - G^{(t)})$ at time T and costs 0 today. Hence

$$\begin{aligned}
&0 = Z_T\,\mathrm{E}^{(T)}[Z(t,T)^{-1}\,(X - G^{(t)})] \quad \text{i.e.,}\\
&0 = \mathrm{E}^{(T)}[(X - G^{(t)})] \quad \text{i.e.,}\\
&\mathrm{E}^{(T)}[X] = G^{(t)} = \mathrm{E}^{(t)}[X]
\end{aligned}$$

which proves the theorem.

Conditional Expectations and Martingales

The outcome of random variables may occur at different points in time. For instance, if I toss a coin, the outcome X, which is *heads* or *tail*, is random before the coin is actually tossed, but at the time it has been tossed, the outcome is observed.

Let $t < T$ be two points in time, and X a random variable whose outcome is observed at time T, and Y a random variable whose outcome is observed at the earlier time t. Assume that for *any event A whose outcome can be observed at time t,* the relation

$$\mathrm{E}\left[I_A X\right] = \mathrm{E}\left[I_A Y\right] \tag{2}$$

holds. We then say that Y is the *conditional expectation at time t* of the variable X. It is easy to see that the variable Y is essentially uniquely determined by X, i.e., if Y_1 and Y_2 both satisfy $\mathrm{E}\left[I_A X\right] = \mathrm{E}\left[I_A Y_i\right]$ for all A, then $Y_1 = Y_2$ a.s.

The conditional expectation is denoted $\mathrm{E}_t[X]$, i.e., $Y = \mathrm{E}_t[X]$. Note in particular the *law of iterated expectations*

$$\mathrm{E}\left[\mathrm{E}_t[X]\right] = \mathrm{E}\left[X\right] \tag{3}$$

which is obtained by letting A be the entire sample space, so that $I_A = 1$. It is even true that

$$\mathrm{E}_s\left[\mathrm{E}_t[X]\right] = \mathrm{E}_s[X] \quad \text{if } s < t \tag{3'}$$

We leave the verification of this relation as an exercise for the reader.

The definition of $\mathrm{E}_t[X]$ may seem a bit strange, but the interpretation is as follows: if we at time t form the expectation of X, *using all relevant information available at the time,* then $\mathrm{E}_t[X]$ is the sensible choice. Note that as seen from a point in time earlier than t, this expectation is random, since we do not have all the information to be revealed at time t when the expectation is formed.

If "now" is time 0, then the conditional expectation $\mathrm{E}_0[X]$ is the same as the unconditional expectation $\mathrm{E}\left[X\right]$. To see this, note that I_A for an event whose outcome is observed now, is either the constant 1 or the constant 0.

Assume now that the time interval from t_0 to T is divided into discrete intervals $t_0, t_1, \ldots, t_n = T$, and that we have a sequence of random variables X_j, $j = 0, \ldots, n$, where the outcome of X_j is observed at time t_j. Such a sequence is a *stochastic process.* A stochastic process is a *martingale* if it is true that

$$X_j = \mathrm{E}_j[X_{j+1}] \quad \text{for } j = 0, \ldots, n-1. \tag{4}$$

It follows immediately from (3′) that in order for a stochastic process to be a martingale, it suffices that $X_j = \mathrm{E}_j[X_n]$ for $j < n$.

Martingale Prices

Consider a contract that gives the random payoff X at time T. The forward price $G_t^{(T)}$, $0 \le t < T$, of this contract at time t is a random variable whose outcome is determined at time t.

Theorem 2

For any t, $0 \leq t \leq T$, the forward price $G_t^{(T)}$ equals the conditional expectation

$$G_t^{(T)} = \mathrm{E}_t^{(T)}[X]$$

Hence, the forward process has the martingale property (4) w.r.t. the forward measure, i.e.,

$$G_j^{(T)} = \mathrm{E}_j^{(T)}[G_{j+1}]$$

Proof

Consider the following strategy: Let A be any event whose outcome occurs at time t. Wait until time t and then, if A has occurred, enter a forward contract on X maturing at T with forward price $G_t^{(T)}$, but if A has not occurred, do nothing. We have thus constructed a contract which gives payoff $I_A\,(X - G_t^{(T)})$ at time T which costs 0 today. Hence

$$Z_T\,\mathrm{E}^{(T)}[I_A(X - G_t^{(T)})] = 0 \quad \text{i.e.,} \quad \mathrm{E}^{(T)}[I_A X] = \mathrm{E}^{(T)}[I_A G_t^{(T)}]$$

Since this is true for any event A whose outcome is known at time t, we have by the definition of conditional expectation

$$G_t^{(T)} = \mathrm{E}_t^{(T)}[X]$$

Q.E.D.

Remark

From this we deduce (c.f. Lecture Note 1) that the present price p_t at time t of the contract yielding X at time T is

$$p_t = Z(t,T)\mathrm{E}_t^{(T)}[X]$$

where $Z(t,T)$ is the price at time t of a zero coupon bond maturing at time T. Especially, if the interest rate is *deterministic* and equal to r, then the present prices satisfy

$$p_j = e^{-r\,\Delta t}\,\mathrm{E}_j^*[p_{j+1}] \tag{5}$$

where p_j is the present price as of time t_j and $\Delta t = t_{j+1} - t_j$; E^* denotes expectation w.r.t. the forward measure (which is independent of maturity date, according to Theorem 1.)

Note 8: Asset Price Dynamics

In Black's model, there are only two relevant points in time: the time when the contract is written, and the time when it matures. In many cases we have to consider also what happens in between these two points in time.

First some notation. Time is discrete, $t = t_0, \ldots, t_n = T$, and we let for notational simplicity $t_0 = 0$ and all time spells $t_k - t_{k-1} = \Delta t$ be equally long; hence $t_k = k\Delta t$. Let X be some random value whose outcome becomes known at time T; we are interested in evaluating the prices of derivatives of X.

Black-Scholes Dynamics

The Black-Scholes assumption about the random behaviour of X is that

$$X = Ae^{\sigma\,\Sigma_1^n z_j} \tag{1}$$

where A is some constant, σ a parameter called the *volatility,* and the z_j:s are stochastically independent normal $\mathrm{N}(0, \Delta t)$ random variables (Δt is the variance; the standard deviation is thus $\sqrt{\Delta t}$) where the outcome of z_j occurs at time t_j.

This is the model for the price dynamics under the true probability measure. But in order to compute prices, we want the dynamics under the forward measure.

Creating the Forward Measure

Let E denote the expectations operator w.r.t. the true probability measure. Following the ideas of Lecture Note 5, we now specify a random variable Q such that $Q > 0$, $\mathrm{E}[Q] = 1$ and such that $\mathrm{E}[f(X)Q] = G_0^{(T)}\big(f(X)\big)$

Following the ideas of Lecture Note 5, a fairly natural guess to find such a function is to try the specification

$$Q = e^{\,\mu - \lambda \Sigma_1^n z_j} \tag{2}$$

for some constants μ and λ. In order for $\mathrm{E}[Q] = 1$ to be satisfied, we must choose

$$\mu = -\frac{1}{2}\Delta t\, n\, \lambda^2 = -\frac{1}{2}\lambda^2 T.$$

This follows immediately from the fact that

$$\mathrm{E}[e^y] = e^{\frac{1}{2}v} \tag{3}$$

for a normally distributed random variable $y \in \mathrm{N}(0, v)$, as is easily checked. Our suggestion thus looks like

$$Q = e^{-\frac{1}{2}\lambda^2 T - \lambda\,\Sigma_1^n z_i} \tag{4}$$

Now we prove

Theorem (Girsanov's theorem)

Let $h(z_1, \ldots z_n)$ be a function (of reasonable regularity) of the independent $N(0, \Delta t)$-variables $z_1, \ldots, z_n$. Then

$$\mathrm{E}\left[h(z_1, \ldots, z_n) e^{-\frac{1}{2}\Sigma_1^n \lambda_j^2 \Delta t - \Sigma_1^n \lambda_j z_j}\right] = \mathrm{E}\left[h(w_1 - \lambda_1 \Delta t, \ldots, w_n - \lambda_n \Delta t)\right]$$

where also w_j are independent $N(0, \Delta t)$-variables.

Proof

This is just a substitution of variables in a multiple integral:

$$\begin{aligned}
&\mathrm{E}\left[h(z_1, \ldots, z_n) e^{-\frac{1}{2}\Sigma \lambda_j^2 \Delta t - \Sigma \lambda_j z_j}\right] \\
&= \frac{1}{(2\pi\Delta t)^{n/2}} \int \ldots \int h(x_1, \ldots, x_n) e^{-\frac{1}{2}\Sigma \lambda_j^2 \Delta t - \Sigma \lambda_j x_j} \times \\
&\qquad\qquad \times\ e^{-\frac{1}{2\,\Delta t}\Sigma_1^n x_j^2}\, dx_1 \ldots dx_n \\
&= \frac{1}{(2\pi\Delta t)^{n/2}} \int \ldots \int h(x_1, \ldots, x_n) e^{-\frac{1}{2\,\Delta t}\Sigma (x_j + \lambda_j \Delta t)^2}\, dx_1 \ldots dx_n \\
&= [\text{change of variables: } x_j + \lambda_j \Delta t = y_j] \\
&= \frac{1}{(2\pi\Delta t)^{n/2}} \int \ldots \int h(y_1 - \lambda_1 \Delta t, \ldots, y_n - \lambda_n \Delta t) e^{-\frac{1}{2\,\Delta t}\Sigma y_j^2}\, dy_1 \ldots dy_n \\
&= \mathrm{E}\left[h(w_1 - \lambda_1 \Delta t, \ldots, w_n - \lambda_n \Delta t)\right]
\end{aligned}$$

Q.E.D.

Remark

Girsanov's theorem generally refers to the more general case when $\Delta t \to 0$ so the sum is replaced by an integral *("Itô integral".)*

We can now employ Girsanov's theorem in our case. In our case $\lambda_j = \lambda$, and the *Girsanov transformation* (4) hence defines a new measure, which will be our *forward measure* $\mathrm{E}^{(T)}$, such that under this measure (1) can be written

$$\begin{aligned}
X &= A\, e^{\sigma\, \Sigma_1^n (w_j - \lambda \Delta t)} \\
&= B\, e^{\sigma\, \Sigma_1^n w_j}
\end{aligned} \tag{5}$$

for some constant B and where w_j are independent $\mathrm{N}(0, \Delta t)$-variables under the forward measure. Since $G_0^{(T)} = \mathrm{E}^{(T)}[X]$, it follows that $B = G_0^{(T)} e^{-\frac{1}{2}\sigma^2 T}$.

Binomial Approximation

In almost all cases when the whole price path has to be taken into account, it is necessary to use some numerical procedure to calculate the price of the derivative. A useful numerical procedure is to approximate the specification (5) by a *binomial tree.* In (5), we replace w_j by a binary variable b_j which takes the value $-\sqrt{\Delta t}$ with probability (under the forward probability measure) 0.5 and the value $+\sqrt{\Delta t}$ with probability 0.5. For large values of n, and small Δt, this is a good approximation—in fact, as $n \to \infty$ and $\Delta t \to 0$ the price of a derivative calculated from the binomial tree (see next Lecture Note) will converge to the theoretical value it would have from the specification (5).

The binomial specification is thus

$$X = B\, e^{\sigma\, \Sigma_1^n b_j}$$

Note that $\mathrm{E}^{(T)}[e^{\sigma b_j}] = \cosh(\sigma\sqrt{\Delta t})$. We know from the previous Lecture Note that $G_k(X) = \mathrm{E}_k^{(T)}[X]$, so

$$G_k = B\, e^{\sigma\, \Sigma_1^k b_j} \cosh^{n-k}(\sigma\sqrt{\Delta t})$$

Combining this relation with the same for G_{k+1}, we get the relation

$$G_{k+1} = G_k \frac{e^{\sigma\, b_{k+1}}}{\cosh(\sigma\sqrt{\Delta t})} = G_k\big(1 \pm \varepsilon\big) \quad \text{where } \varepsilon = \tanh(\sigma\sqrt{\Delta t})$$

and the plus and minus sign occur with probability (under the forward probability measure) 0.5 each.

Remarks

It is worthwhile to note that (5) is identical to (1) except for the constant A; the volatility σ is the same. The parameter λ that appears in Q has a similar interpretation as in Lecture Note 5: $\mathrm{E}[X] = G_0\, e^{\lambda\sigma T}$ and is called the *market price of risk.*

Note 9: The Binomial Model

In the previous Lecture Note we derived the binomial specification of the forward price dynamics of the asset that will be underlying a derivative. The forward price refers to a forward contract that matures at the same time T as the derivative, or later. If we write the dynamics in the usual tree style, we have for the forward price dynamics

$$\begin{array}{cccccc} G_0 & G_0\,u & G_0\,u^2 & G_0\,u^3 & G_0\,u^4 & \cdots \\ & G_0\,d & G_0\,du & G_0\,du^2 & G_0\,du^3 & \cdots \\ & & G_0\,d^2 & G_0\,d^2u & G_0\,d^2u^2 & \cdots \\ & & & G_0\,d^3 & G_0\,d^3u & \cdots \\ & & & & G_0\,d^4 & \cdots \end{array}$$

In this diagram, each column represents an instant of time, and moving from one instant to the next (one step right) means that the forward price either moves "up" or "down". In this diagram "up" means going one step to the right on the same line, and "down" going one step to the right on the line below (this configuration is convenient in a spread sheet.) Under the forward measure the probabilities of moving up and down are both $\frac{1}{2}$. The specification of u and d are as follows:

$$\begin{cases} u = 1 + \varepsilon \\ d = 1 - \varepsilon \end{cases} \quad \text{where } \varepsilon = \tanh(\sigma\sqrt{\Delta t})$$

where Δt is the time spell between columns, and σ is the volatility of the forward price.

Pricing an American Futures Option

As long as we assume that interest rates are deterministic, the forward prices in the binomial tree equals the corresponding futures prices. Hence we can use this tree for calculating the price of an *American futures option.* In order to do so, we construct a tree of prices of the futures option. In this tree the prices are calculated from right to left. If the time for maturity of the option is at $T = t_4$, then we construct the following tree of prices for the futures option:

$$\begin{array}{ccccc} p & p_{1,0} & p_{2,0} & p_{3,0} & p_{4,0} \\ & p_{1,1} & p_{2,1} & p_{3,1} & p_{4,1} \\ & & p_{2,2} & p_{3,2} & p_{4,2} \\ & & & p_{3,3} & p_{4,3} \\ & & & & p_{4,4} \end{array}$$

Here the prices in the rightmost column are known, since they represent the value of the option at maturity when the underlying futures price is that in the corresponding position of the tree of futures prices. Then option prices are calculated backwards:

$$\hat{p}_{j,k} = e^{-r\Delta t}\big(0.5\,p_{j+1,k} + 0.5\,p_{j+1,k+1}\big) \tag{1}$$

$$p_{j,k} = \max[\hat{p}_{j,k}, \text{value if exercised}] \tag{2}$$

Indeed, $\hat{p}_{j,k} = e^{-r\Delta t}\mathrm{E}^*[p_{j+1,\bullet}]$, which is just another way of writing (1), is the value if the option is not exercised (see (5) of Lecture Note 7,) and the highest value of this and the value if exercised is the present value. In order to calculate the value if exercised, we employ the original tree of futures prices, of course. The end result p is the price of the option today.

American Call Option on a Share of a Stock

If we want to price an *American* call option on a share of a stock which pays no dividend before maturity, we need a binomial tree for the current stock price, not the futures price of the stock. The assumption is still that the forward price dynamics is as above, and we derive the stock prices from there. This is easy; according to Lecture Note 2, example 1 with $d = 0$, the relation between the current stock price S_t and the forward price at the same date G_t is $S_t = G_t\, e^{-r(T-t)}$. The relevant binomial tree for the stock price is thus

$$\begin{array}{lllll} S_0 & S_0\, e^{r\Delta t}\, u & S_0\, e^{2r\Delta t}\, u^2 & S_0\, e^{3r\Delta t}\, u^3 & \cdots \\ & S_0\, e^{r\Delta t}\, d & S_0\, e^{2r\Delta t}\, du & S_0\, e^{3r\Delta t}\, du^2 & \cdots \\ & & S_0\, e^{2r\Delta t}\, d^2 & S_0\, e^{3r\Delta t}\, d^2u & \cdots \\ & & & S_0\, e^{3r\Delta t}\, d^3 & \cdots \end{array}$$

The price of the option is then calculated in the same way as above.

Options on Assets Paying Dividends

Known Dividend Amount

Assume we want to price an option on a share of a stock when just before time i the stock pays a known dividend d. We need to set up a binomial tree for the current price S_t of the stock. Let $n > i$ be the period at which time T the option matures.

For any time $j \le n$ we have the relation

$$S_j = \begin{cases} e^{-r(i-j)\Delta t}\, d + e^{-r(n-j)\Delta t}\, G_j & \text{if } j < i \\ e^{-r(n-j)\Delta t}\, G_j & \text{if } i \le j \le n \end{cases} \tag{3}$$

This follows as in Lecture Note 2, example 1.

In particular, we have

$$G_0 = S_0\, e^{rn\Delta t} - d\, e^{r(n-i)\Delta t} \tag{4}$$

It is now easy to build the tree of stock prices: First compute G_0 from (4), then construct the tree for G_j, and from these prices, compute S_j from (3).

Known Dividend Yield

Now assume instead that at some time between times $i-1$ and i the stock will pay a dividend $d\,S_i'$ where S_i' is the stock price the moment before the dividend is paid out.

The relation between the stock price S_j and the futures price G_j for times j later or at time i is as before

$$S_j = e^{-r(n-j)\Delta t}\,G_j \quad j \geq i \tag{5}$$

For times j before i we can compute the stock price in accordance with example 2 of Lecture Note 2:

$$S_j(1-d) = e^{-r(n-j)\Delta t}\,G_j$$

from which it follows that

$$S_j = \frac{1}{1-d}G_j\,e^{-r(n-j)\Delta t} \quad j < i \tag{6}$$

In particular,

$$G_0 = (1-d)S_0\,e^{rn\Delta t} \tag{7}$$

It is now easy to build the tree of stock prices: First compute G_0 from (7), then construct the tree for G_j, and from these prices, compute S_j from (5) and (6).

Note 10: Random Interest Rates: The Futures Measure

We have assumed interest rates to be for the most part deterministic. In order to study interest rate derivatives, or other situations where interest rates are assumed to be random, we need a different probability measure than the forward measure. One reason for this is that when interest rates are random, forward measures for different maturity dates are not the same. Rather than having zero coupon bonds as numeraires (see Lecture Note 7) we use the money market account (see Lecture Note 1) as a numeraire. Hence, let the numeraire N be $N = e^{R(0,t)}$ where we have used the notation introduced in Lecture Note 1 and 7. Then (see (1) of Lecture Note 7)

$$P_0^{(t)}(X) = P_0^{(t)}(e^{R(0,t)})\,\widehat{\mathrm{E}}[Xe^{-R(0,t)}] = \widehat{\mathrm{E}}[Xe^{-R(0,t)}]$$

where $\widehat{\mathrm{E}}[\cdot]$ denotes expectation w.r.t. the probability measure associated with the numeraire $N = e^{R(0,t)}$. We see from this relation and Lecture Note 1 that

$$\widehat{\mathrm{E}}[X] = P_0^{(t)}(Xe^{R(0,t)}) = F_0^{(t)}(X) \tag{1}$$

so it seems natural to call the probability measure associated with this numeraire the *futures measure*. In the litterature it is often called the *Equivalent Martingale Measure* (Hull uses the term "traditional risk neutral measure.")

It is important to note that the forward measures $\mathrm{E}^{(T)}$ differ for different maturities T when interest rates are random. This is not the case for the futures measure, though. Indeed, we will now prove two theorems on the futures measure:

Theorem 1

The futures measure as defined above is independent of the date of maturity T in the following sense: if the outcome of X is known at time t_n then $\widehat{\mathrm{E}}^{(n)}[X] = \widehat{\mathrm{E}}^{(m)}[X]$ if $m > n$, where we denote by $\widehat{\mathrm{E}}^{(k)}$ the futures measure for contracts maturing at time t_k.

Hence there is no need to index the futures measure by maturity date.

Theorem 2

The futures prices $\{F_j\}$ have the martingale property w.r.t. the futures measure: $F_j = \widehat{\mathrm{E}}_{t_j}[F_k]$ for $j < k$. In particular, the futures price F_0 is the expected value w.r.t. the futures measure of X, the spot price at delivery.

Proof of Theorem 1

Let P be the present price of the value $Xe^{R(0,n)}$ to be delivered at time t_n. We can convert this contract to one where instead the value $Xe^{R(0,m)}$ is delivered at time t_m by simply depositing the payoff $Xe^{R(0,n)}$ in the money market account up to time t_m. The present price is of course the same. Using the theorem in Lecture Note 1, we have the following equalities from the two contracts:

$$P = P^{(t_n)}[Xe^{R(0,n)}] = F_0^{(t_n)}[X] \quad \text{and} \quad P = P^{(t_m)}[Xe^{R(0,m)}] = F_0^{(t_m)}[X]$$

By (1) this implies that

$$\widehat{\mathrm{E}}^{(n)}[X] = \widehat{\mathrm{E}}^{(m)}[X]$$

Q.E.D.

Proof of theorem 2

Consider the following strategy: Let A be any event whose outcome is known at time $t_j < t_k$. At time t_{k-1} enter a long position of $e^{R(0,k)}$ futures contracts if A has occurred, otherwise do nothing. At t_k collect $I_A(F_k - F_{k-1})e^{R(0,k)}$ (which may be negative) and close the contract. This gives the payment $I_A(F_k - F_{k-1})e^{R(0,k)}$ at time t_k and no other cash flow. The present price is zero, hence

$$0 = P^{(t_k)}[I_A(F_k - F_{k-1})e^{R(0,k)}] = F_0[I_A(F_k - F_{k-1})] = \widehat{\mathrm{E}}[I_A(F_k - F_{k-1})]$$

Hence,

$$\widehat{\mathrm{E}}[I_A F_k] = \widehat{\mathrm{E}}[I_A F_{k-1}]$$

Employing this equality repeatedly gives, for $j < k$:

$$\widehat{\mathrm{E}}[I_A F_j] = \widehat{\mathrm{E}}[I_A F_k]$$

Since A can be any event whose outcome is known at time t_j, this means that (see "Conditional Expectations and Martingales" in Lecture Note 7)

$$F_j = \widehat{\mathrm{E}}_{t_j}[F_k] \quad \text{for } j < k.$$

This proves theorem 2.

Theorem 3

The following formula for the present prices p_j as of time t_j for the asset X to be delivered at time T holds:

$$p_j = \widehat{\mathrm{E}}_{t_j}[p_k e^{-R(j,k)}], \quad j < k.$$

In particular,

$$p_j = \widehat{\mathrm{E}}_{t_j}[Xe^{-R(j,T)}]$$

Proof

With the notation of Lecture Note 1,

$$p_j = F_j^{(T)}[Xe^{-R(j,T)}] = \widehat{\mathrm{E}}_{t_j}[Xe^{-R(j,T)}]$$

and likewise

$$p_k = \widehat{\mathrm{E}}_{t_k}[Xe^{-R(k,T)}]$$

Now we can employ the iterated expectations rule to get

$$\begin{aligned} p_j &= \widehat{\mathrm{E}}_{t_j}[Xe^{-R(j,T)}] = \widehat{\mathrm{E}}_{t_j}\left[\widehat{\mathrm{E}}_{t_k}[Xe^{-R(j,T)}]\right] \\ &= \widehat{\mathrm{E}}_{t_j}\left[e^{-R(j,k)}\widehat{\mathrm{E}}_{t_k}[Xe^{-R(k,T)}]\right] = \widehat{\mathrm{E}}_{t_j}[e^{-R(j,k)}p_k] \end{aligned}$$

Q.E.D.

Note 11: A Model of the Short Interest Rate: Ho-Lee

We divide time into short time intervals $t_0, \ldots, t_n = T$, $t_k - t_{k-1} = \Delta t$. Ideally, the points in time t_k should coincide with the times of settlement of futures contracts.

At t_{k-1} there is an interest rate $r_k \Delta t$ prevailing from t_{k-1} to t_k. This interest rate is random, but its value is known at time t_{k-1}. The *Ho-Lee model* of the interest rate is:

$$r_k = a_k + \sigma\sqrt{\Delta t}\,(z_1 + \cdots + z_{k-1})$$

where a_k are some numbers, the factor σ is the *volatility* of the short interest rate. The z_j:s are independent N(0,1)-variables and the outcome of each z_j occurs at time t_j. This is under the true probability measure.

Under the *futures* measure, the model looks the same, with σ maintained, but with new terms a_k; the reasoning is the same as in Lecture Note 8 ("remarks".) Thus, under the futures measure

$$r_k = \theta_k + \sigma\sqrt{\Delta t}\,(z_1 + \cdots + z_{k-1})$$

It remains to compute the exact value of θ_k, but first a notational simplification: multiplying by Δt yields

$$r_k \Delta t = \theta_k \Delta t + \sigma \Delta t^{\frac{3}{2}}\,(z_1 + \cdots + z_{k-1})$$

We now normalise $\Delta t = 1$. This means that r_k is the one period interest rate—it is proportional to Δt whereas σ is proportional to $\Delta t^{3/2}$. Now the model reads

$$r_k = \theta_k + \sigma(z_1 + \cdots + z_{k-1}) \tag{1}$$

Let Z_{t_k} be the price of a zero coupon bond maturing at t_k with face value 1. Then, according to Theorem 3, Lecture Note 10,

$$Z_{t_k} = \widehat{\mathrm{E}}[1\, e^{-r_1 - \cdots - r_k}]$$

In order for the model (1) to correctly represent the current term structure, we must thus have

$$\begin{aligned} Z_{t_k} &= \widehat{\mathrm{E}}[e^{-\theta_1 - \cdots - \theta_k - \sigma((k-1)z_1 + (k-2)z_2 + \cdots + 1\, z_{k-1})}] \\ &= e^{-\theta_1 - \cdots - \theta_k}\, \widehat{\mathrm{E}}[e^{-\sigma((k-1)z_1 + (k-2)z_2 + \cdots + 1\, z_{k-1})}] \end{aligned}$$

i.e.,

$$Z_{t_k} = e^{-\theta_1 - \cdots - \theta_k}\, e^{\frac{\sigma^2}{2}((k-1)^2 + \cdots + 1^2)}$$

Since this must hold for any k, this is equivalent to

$$\theta_k = \ln\left(\frac{Z_{t_{k-1}}}{Z_{t_k}}\right) + \frac{\sigma^2}{2}\,(k-1)^2$$

The Ho-Lee model thus reads

$$r_t = \ln\Big(\frac{Z_{t_{k-1}}}{Z_{t_k}}\Big) + \frac{\sigma^2}{2}\,(t-1)^2 + \ \sigma(z_1 + \cdots + z_{t-1}) \tag{2}$$

under the futures measure.

The Price of a Zero Coupon Bond

Consider a zero coupon bond with face value 1 which matures at time T. We will now compute its price $Z(t,T)$ at an earlier time $t < T$. At this time, the bond's value is, by Theorem 3, Lecture Note 10 (the sub index t on $\widehat{\mathrm{E}}$ indicates conditional expectation as of time t, i.e., the outcomes of z_j for $j \le t$ are already known and are regarded as constants):

$$\begin{aligned}
Z(t,T) &= \widehat{\mathrm{E}}_t[e^{-r_{t+1}-\cdots-r_T}] \\
&= \frac{Z_T}{Z_t}\widehat{\mathrm{E}}_t[e^{-\frac{\sigma^2}{2}(t^2+\cdots+(T-1)^2)-\sigma(T-t)(z_1+\cdots+z_t)} \\
&\quad \times e^{-\sigma((T-t-1)z_{t+1}+(T-t-2)z_{t+2}+\cdots+1\,z_{T-1})}] \\
&= \frac{Z_T}{Z_t}e^{-\frac{\sigma^2}{2}(t^2+\cdots+(T-1)^2)}\,e^{-\sigma(T-t)(z_1+\cdots+z_t)} \\
&\quad \times \widehat{\mathrm{E}}_t[e^{-\sigma((T-t-1)z_{t+1}+(T-t-2)z_{t+2}+\cdots+1\,z_{T-1})}] \\
&= \frac{Z_T}{Z_t}e^{-\sigma(T-t)(z_1+\cdots+z_t)}\,e^{-\frac{\sigma^2}{2}(t^2+\cdots+(T-1)^2)}\,e^{\frac{\sigma^2}{2}((T-t-1)^2+\cdots+1^2)}
\end{aligned}$$

We compute the sums

$$\begin{aligned}
-\frac{\sigma^2}{2}(t^2 + \cdots + (T-1)^2) &+ \frac{\sigma^2}{2}((T-t-1)^2 + \cdots + 1^2) \\
&= -\frac{\sigma^2}{2}\sum_{j=0}^{T-t-1}(t+j)^2 - j^2 \\
&= -\frac{\sigma^2}{2}t\sum_{j=0}^{T-t-1}(2j+t) \\
&= -\frac{\sigma^2}{2}(T-1)(T-t)t
\end{aligned}$$

Hence,

$$Z(t,T) = \frac{Z_T}{Z_t}e^{-\frac{\sigma^2}{2}(T-1)(T-t)t}e^{-\sigma(T-t)(z_1+\cdots+z_t)} \tag{3}$$

Forward and Futures on a Zero Coupon Bond

We will now compute the futures and forward prices of a contract maturing at t on a zero coupon bond maturing at $T > t$ with face value 1. The forward price is easy, it is

$$G = \frac{Z_T}{Z_t}$$

regardless of the interest rate model. We leave the proof to the reader. The futures price F_0 is

$$\begin{aligned} F_0 = \widehat{\mathrm{E}}[Z(t,T)] &= \widehat{\mathrm{E}}[\frac{Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-1)(T-t)t} e^{-\sigma(T-t)(z_1+\cdots+z_t)}] \\ &= Ge^{-\frac{\sigma^2}{2}(T-1)(T-t)t}\, \widehat{\mathrm{E}}[e^{-\sigma(T-t)(z_1+\cdots+z_t)}] \\ &= Ge^{-\frac{\sigma^2}{2}(T-1)(T-t)t}\, e^{\frac{\sigma^2}{2}(T-t)^2 t} = Ge^{-\frac{\sigma^2}{2}(T-t)(t-1)t} \end{aligned}$$

Recall that we have normalised $\Delta t = 1$. With no normalisation, we must replace σ by $\sigma \Delta t^{\frac{3}{2}}$ and t and T by $t\Delta t^{-1}$ and $T\Delta t^{-1}$ respectively, hence

$$F_0 = Ge^{-\frac{\sigma^2}{2}(T-t)(t-\Delta t)t} \longrightarrow Ge^{-\frac{\sigma^2}{2}(T-t)t^2} \quad \text{when } \Delta t \to 0.$$

We see that in contrast to the situation when interest rates are deterministic, the forward price G and the futures price F_0 differ. Since the price of the underlying asset (the bond) is negatively correlated with the interest rate, the futures price is lower; in the Ho-Lee model by factor $e^{-\frac{\sigma^2}{2}(T-t)t^2}$.

The Forward Measure

The computation of prices on bond options, for example, is simpler if we use the forward measure rather than the futures measure. We will see that in the next section. Let us therefore derive the forward measure for a certain maturity t. We have:

$$\begin{aligned} \mathrm{E}^{(t)}[X] &= Z_t^{-1} P^{(t)}(X) = Z_t^{-1} F_0(Xe^{-\Sigma_1^t r_j}) \\ &= Z_t^{-1}\widehat{\mathrm{E}}(Xe^{-\Sigma_1^t r_j}) = \widehat{\mathrm{E}}[XW] \end{aligned}$$

where $$W = Z_t^{-1} e^{-\Sigma_1^t r_j}$$

Substituting (2) we get

$$W = e^{\mu - \sigma\, \Sigma_1^t (t-j) z_j}$$

for a constant μ such that $\widehat{\mathrm{E}}[W] = 1$. Note that this is a specification in accordance with Girsanov's theorem (Lecture Note 8) with $\lambda_i = -\sigma(t-i)$. We can thus employ Girsanov's theorem to get

Theorem

The variable z_j is $N(-\sigma(t-j), 1)$ under the forward measure with maturity at time $t \geq j$. Thus we may write $z_j = w_j - \sigma(t-j)$ where $w_j \in N(0,1)$ under the forward (t) measure.

In particular, we have from (3) that under the $\mathrm{E}^{(t)}$-measure,

$$
\begin{aligned}
Z(t,T) &= \frac{Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-1)(T-t)t} e^{-\sigma(T-t)(w_1-\sigma(t-1)+\cdots+w_{t-1}-\sigma\cdot 1+w_t)} \\
&= \frac{Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-t)^2 t} e^{-\sigma(T-t)(w_1+\cdots+w_t)}
\end{aligned}
\tag{4}
$$

Pricing a European Option on a Zero Coupon Bond

It is now easy to calculate the price of a European option maturing at t on a zero coupon bond maturing at time $T > t$. Let $F(Z(t,T))$ be the value of the option at maturity. Then its price today is, using (4) and the fact that $w_1+\cdots+w_t$ is a normal distributed variable with standard deviation $\sqrt{t}$,

$$
p = Z_t \mathrm{E}^{(t)}\big[F\big(Z(t,T)\big)\big] = \frac{Z_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\Big(\frac{Z_T}{Z_t} e^{-\frac{\sigma^2}{2}(T-t)^2 t} e^{\sigma(T-t)\sqrt{t}\,x}\Big) e^{-\frac{1}{2}x^2}\, dx
$$

But $\frac{Z_T}{Z_t} = G_0$, the forward price of the underlying bond, hence, (also with no normalisation of time,)

$$
\begin{aligned}
p &= Z_t \mathrm{E}^{(t)}\big[F\big(Z(t,T)\big)\big] \\
&= \frac{Z_t}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\Big(G_0\, e^{-\frac{\sigma^2}{2}(T-t)^2 t} e^{\sigma(T-t)\sqrt{t}\,x}\Big) e^{-\frac{1}{2}x^2}\, dx
\end{aligned}
$$

Note that the same formula may be obtained by Black's model for bond options (Lecture Note 6.)

Note 12: Ho-Lee's Binomial Interest Rate Model

When we want to price other than European interest rate derivatives, we need a numerical procedure, and one such is again a binomial tree. This is constructed by replacing z_k in Lecture Note 11 by binomial variables b_j (see Lecture Note 8, "Binomial Approximation".) Hence, time is discrete: $t_0, t_1, \ldots, t_n = T$, $t_k - t_{k-1} = \Delta t$. The interest rate r_k from t_{k-1} to t_k is assumed to be

$$r_k = \theta_k + \sigma \sum_2^k b_j \quad k = 1, \ldots, n$$

where $\{b_k\}$ are binomial random variables which takes the values ± 1, each with probability 0.5 under the futures measure; they are thus assumed to be statistically independent. The outcome of the variable b_k occurs at time $k-1$. Since we don't need the dynamics under the true probability measure, we model the dynamics from the start under the futures measure. However, as in Lecture Note 11, σ represents the volatility of the interest rate under either measure, so it can be measured from real data; σ is proportional to $\Delta t^{\frac{3}{2}}$ just as in Lecture Note 11.

We will choose the terms θ_j:s such that the model is consistent with the current term structure—it will be close to, but not exactly, the same as in the Normal distribution case.

Let us compute the price Z_4 at time t_0 of a zero coupon bond maturing at time t_4 with face value 1:

$$\begin{aligned}
Z_4 &= \widehat{\mathrm{E}}[1 \cdot e^{-r_1 - r_2 - r_3 - r_4}] = \widehat{\mathrm{E}}[e^{-\theta_1 - \theta_2 - \theta_3 - \theta_4 - \sigma(3b_2 + 2b_3 + b_4)}] \\
&= e^{-\theta_1 - \theta_2 - \theta_3 - \theta_4} \widehat{\mathrm{E}}[e^{-3\sigma b_2}]\, \widehat{\mathrm{E}}[e^{-2\sigma b_3}]\, \widehat{\mathrm{E}}[e^{-\sigma b_4}] \\
&= e^{-\theta_1 - \theta_2 - \theta_3 - \theta_4} \cosh(3\sigma) \cosh(2\sigma) \cosh(\sigma)
\end{aligned}$$

And, by the same token, in general

$$Z_k = e^{-\theta_1 - \cdots - \theta_k} \cosh\big((k-1)\sigma\big) \cdot \ldots \cdot \cosh(\sigma)$$

Since this must hold for all k, we must have

$$\theta_k = \ln\Big(\frac{Z_{k-1}}{Z_k}\Big) + \ln\big[\cosh\big((k-1)\sigma\big)\big]$$

and the dynamics under the futures measure is thus described by

$$r_k = \ln\Big(\frac{Z_{k-1}}{Z_k}\Big) + \ln\big[\cosh\big((k-1)\sigma\big)\big] + \sigma \sum_2^k b_j \quad k = 1, \ldots, n$$

$$\text{where } \begin{cases} b_j = 1 & \text{with probability } \frac{1}{2} \\ b_j = -1 & \text{with probability } \frac{1}{2} \end{cases}$$

Once we have the parameters of the model, we can price interest rate derivatives in the binomial tree. We show the procedure by an example, where we want to price a European call option maturing at t_2 with strike price 86 on a zero coupon bond maturing at t_4 with face value 100 when the following parameters are given: $f_1 = 0.06$, $f_2 = 0.06095$, $f_3 = 0.06180$, $f_4 = 0.06255$, $\sigma = 0.01$, where $f_k = \ln\left(Z_{k-1}/Z_k\right)$ are the forward rates. We represent the interest rates in a binomial tree:

t_0	t_1	t_2	t_3
0.06	0.071	0.082	0.093
	0.051	0.062	0.073
		0.042	0.053
			0.033

The interest rate from one period to the next is obtained by going either one step to the right on the same line, or step to the right to the line below; each with a (risk adjusted) probability of 0.5. We can compute the value at t_2 of the bond: since its value at t_4 is 100, the value at t_3 and t_2 is obtained recursively backwards by use of the formula of theorem 3 in Lecture Note 10:

t_2	t_3
84.794	91.119
88.254	92.960
91.856	94.838
	96.754

The value of the option can now also be obtained by backward recursion:

t_0	t_1	t_2
2.309	1.050	0
	3.853	2.254
		5.856

The value of the option is thus 2.309. It is also easy to price other more exotic derivatives in this binomial tree model.