KTH Engineering Sciences

Comments and Examples on “Lecture Notes on Financial Mathematics”

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These are comments on my *Lecture Notes on Financial Mathematics.* They are meant to accompany the notes and my intention is that they should be updated more often than the Lecture Notes, in particular each time a new edition of John Hull’s book *“Options, Futures, and Other Derivatives”* appears. I have also included some examples and exercises, most of them with solutions.

Harald Lang

Comments on Lecture Note 1

Here are timelines showing the cash flows for futures, forwards and "pay now" contracts:

Futures contract

day	0	1	2	3	⋯	$T-1$	T
cash flow	0	F_1-F_0	F_2-F_1	F_3-F_2	⋯	$F_{T-1}-F_{T-2}$	$X-F_{T-1}$

Forward contract

day	0	1	2	3	⋯	$T-1$	T
cash flow	0	0	0	0	⋯	0	$X-G_0$

Pay now

day	0	1	2	3	⋯	$T-1$	T
cash flow	$-P_0$	0	0	0	⋯	0	X

The current edition of John Hull's book *"Options, Futures, and other Derivatives"* is the seventh. You can read about forward and futures contracts in chapter 1. You may for the moment skip section 1.5 (options,) or at least not pay so much attention to it, since we will come to options later in this course. The same goes for the section "Hedging using Options".

Chapter 2 describes in detail how futures contracts work, and why they are specified in the somewhat peculiar way they are.

The *mathematical modeling* of a futures contract is described in Lecture Note 1. As you can see, the modeling is a slight simplification of the real contract. We disregard the maintenance account, thus avoiding any problems with interest on the balance. Furthermore, we assume that the delivery date is defined as a certain day, not a whole month. In this course we disregard the issues on accounting and tax (chapter 2.9.)

You may note that I use "continuous compounding" of interest rates (use of the exponential function) in Lecture Note 1. If you feel uncomfortable with this, you may want to read chapter 4.2 in Hull's book. I will use continuous compounding throughout the course, unless otherwise explicitly stated. It is the most convenient way to handle interest; for instance, if we get 1.5% interest per quarter with continuous compounding, it means that we get $1.5 \times 4\% = 6\%$ per year. Indeed, $e^{0.015}e^{0.015}e^{0.015}e^{0.015} = e^{0.06}$ which is simple enough, whereas if we get 1.5% interest per quarter with quarterly compounding, it means that we get 6.136% per year ($1.015 \times 1.015 \times 1.015 \times 1.015 \approx 1.06136$.) It is of course easy to convert between continuous compounded interest and any other compounding (see chapter 4.2 in Hull's book.)

I have introduced the *present price* P in this course. It is a concept I have found very convenient. It is similar to a forward price, but you pay *today,* and receive the value you pay for at a later date. I have introduced it as a convenient *theoretical* concept; special cases in real life are for instance options, where you pay for the contract today in order to receive some random value (which may be zero) at some later date. It is important that you don't confuse the *present price* with the *spot price* of the same type of good. The *spot price* is the price of the good for *immediate* delivery.

It is extremely important that you distinguish between *constants* (values that are currently known) and *random variables.* For instance, assume that $Z_T = e^{-rT}$ (where r hence is a number.) Then it is true that $P_0^{(T)}[X] = e^{-rT} G_0^{(t)}[X]$ (Theorem c,) however, the relation $P_0^{(T)}[X] = e^{-R(0,T)} G_0^{(t)}[X]$ **is invalid and nonsense!** Indeed, $R(0,T)$ is a random variable; its outcome is not known until time $T-1$, whereas G_0 and P_0 are known prices today.

The most relevant chapters in Hull's book are 1.1, 1.4, 5.8, 4.3.

Comments on Lecture Note 2

Here are timelines showing the cash flows in the examples given:

Example 1

day	0	t	T
cash flow	$-S_0$	d	S_T

The present value of the payments d and S_T are $Z_t d$ and $P_0^{(T)}(S_T) = Z_T G_0^{(T)}(S_T)$.

Example 2

day	0	t
cash flow	$-S_0$	S_t

and

day	0	t	T
cash flow	$-S_0$	dS_t	S_T

The present value of the payments dS_t and S_T are $dP_0^{(t)}(S_t)$ and $P_0^{(T)}(S_T) = Z_T G_0^{(T)}(S_T)$.

Example 3

day	0	t_{k-1}	t_k
cash flow	0	$-S_{t_{k-1}} + dS_{t_{k-1}}$	S_{t_k}

Example 4

Derivation of the limit: $\ln\big((1-\rho\,\delta t)^n\big) = n\,\ln(1-\rho\,\delta t) = n\big(-\rho\,\delta t+\mathcal{O}(\delta t^2)\big) = n\big(-\rho T/n + \mathcal{O}(1/n^2)\big) \to -\rho T$. Taking exponential gives the limit.

Example 5

day	0	t
cash flow	$-X_0$	$e^{\rho t}X_t$

The present value of the cash flow at time t is $P_0^{(t)}(e^{\rho t}X_t) = e^{-rt}G_0^{(t)}(e^{\rho t}X_t) = G_0\, e^{(\rho - r)t}$.

For a *forward rate agreement,* the cash flow is

day	0	t	T
cash flow	0	L	$-Le^{f(T-t)}$

Interest rate swaps

The notation here is a bit inconvenient. Let us introduce the *one period floating rate* $\hat{r}_j$: the interest from time t_{j-1} to time t_j, i.e., if I deposit an amount a on a bank account at time t_{j-1} the balance at time t_j is $a + a\hat{r}_j$. This is the same as to say that a zero coupon bond issued at time t_{j-1} with maturity at t_j is $1/(1+\hat{r}_j)$. Note that $\hat{r}_j$ is *random* whose value becomes known at time t_{j-1}. The cash flow that A pays to B is then

day	t_0	t_1	t_2	t_3	$\cdots$	t_n
cash flow	0	$\hat{r}_1 L_1$	$\hat{r}_2 L_2$	$\hat{r}_3 L_3$	$\cdots$	$\hat{r}_n L_n$

In order to calculate the present value of this cash flow, we first determine the present value $P_0^{(t_k)}(\hat{r}_k)$.

Consider the strategy: buy $1/(1+\hat{r}_j)$ zero coupon bonds at time t_{k-1} with maturity at t_k. This costs nothing today, so the cash flow is

day	t_0	$\cdots$	t_{k-1}	t_k
cash flow	0	$\cdots$	-1	$1+\hat{r}_k$

hence $0 = -P_0^{(t_{k-1})}(1) + P_0^{(t_k)}(1) + P_0^{(t_k)}(\hat{r}_k) = -Z_{t_{k-1}} + Z_{t_k} + P_0^{(t_k)}(\hat{r}_k)$ so we get

$$P_0^{(t_k)}(\hat{r}_k) = Z_{t_{k-1}} - Z_{t_k}.$$

It is now easy to calculate the present value of the cash flow from A to B: it is

$$P_{AB} = \sum_1^n L_k(Z_{t_{k-1}} - Z_{t_k})$$

You may read about swaps in Ch. 7 in Hull's book. An alternative way to present the valuation of a (plain vanilla) interest rate rate swap is to observe that it is nothing but a portfolio of FRA:s; see "Valuation in Terms of FRA:s" in Hull's book. As you see, there are many practical issues we leave aside in this course, such as day count conventions, credit risks, legal issues, and so on.

The most relevant chapters in Hull's book are 4.2, 5.1–5.7, 5.10, 5.11, 4.6, 4.7, 7.7–7.9

Exercises and Examples on Lecture Notes 1–2.

Interest rates always refer to continuous compounding. Answers are given in parenthesis; Solutions to some problems are given on the following pages.

1. A share is valued at present at 80 dollars. In nine months it will give a dividend of 3 dollars. Determine the forward price for delivery in one year given that the rate of interest is 5% a year. (81.06 dollars)

2. A share is valued at present at 80 dollars. In nine months it will give a dividend of 4% of its value at that time. Determine the forward price for delivery in one year given that the rate of interest is 5% a year. (80.74 dollars)

3. The current forward price of a share to be delivered in one year is 110 dollars. In four months the share will give a dividend of 2 dollars and in ten months will give a dividend of 2% of its value at that time. Determine the current spot price of the share given that the rate of interest is 6% a year. (107.67 dollars)

4. The exchange rate of US dollars is today 8.50 SEK per dollar. The forward price of a dollar to be delivered in six months is 8.40 SEK. If the Swedish six month interest rate is 4% a year, determine the American six month interest rate. (6.37%)

5. The forward price of a US dollar the first of August with delivery at the end of December is 0.94630 Euros. The forward price of a dollar to be delivered at the end of June next year is 0.95152 Euros. Assuming a flat term structure for both countries and that the Euro interest rate is 4% a year—what is the American rate of interest? (2.90%)

6. Determine the forward price of a bond to be delivered in two years. The bond pays out 2 Euros every 6-months during $4\frac{1}{2}$ years (starting in six months), and 102 Euros after five years. Thus the bond is, as of today, a 5-year 4%-coupon bond with a coupon dividend every six months with a 100 Euros face value. (94.05 Euros)

   The bond is to be delivered in two years immediately after the dividend has been paid. The present term structure is given by the following rates of interest (on a yearly basis)

6 months	5.0%	18, 24 months	5.6%
12 months	5.4%	30–60 months	5.9%

7. A one-year forward contract of a share which pays no dividend before the contract matures is written when the share has a price of 40 dollars and the risk-free interest rate is 10% a year.

   a) What is the forward price? (44.207 dollars)

   b) If the share is worth 45 dollars six months later, what is the value of the original forward contract at this time? If another forward contract is to be written with the same date of maturity, what should the forward price be? (2.949 dollars, 47.307 dollars)

8. Determine the forward price in Swedish crowns of a German stock which costs 25 euros today. The time of maturity is in one year, and the stock pays a dividend in nine months of 5% of the current stock price at that time. The interest rate of the euro is 4.5% per year, and the crown's rate of interest is 3% per year. One euro costs 9.40 crowns today. (230.05 crowns)

9. Let r_i be the random daily rate of interest (per day) from day $i-1$ to day i, and $R(0,t) = r_1 + \cdots + r_t$. The random variable X_t is the stock exchange index day t (today is day 0.) The random variables X_t and $R(0,t)$ are *not* independent.

   The forward price of a contract for delivery of the payment $X_t\, e^{R(0,t)}$ euros day t is 115 euros, a zero coupon bond which pays 1 euro day t costs 0.96 euro.

   Determine the futures price of a contract for delivery of X_t euro day t. The stock exchange index is today $X_0 = 100$.

10. A share of a stock currently costs 80 euros. One year from now, it will pay a dividend of 5% of its price at the time of the payment, and the same happens two years from now. Determine the forward price of the asset for delivery in 2.5 years. The interest rate for all maturities is 6% per year. (83.88 euros)

11. The 6 month zero rate is 5% per year and the one year rate is 5.2% per year. What is the forward rate from 6 to 12 months? (5.4% per year)

12. Show that the present value of a cash flow at time T that equals the floating interest from t to T ($t < T$) on a principal L is the same as if the floating rate is replaced by the current forward rate. (Note that this is an easy way to value interest rate swaps: just replace any floating rate by the corresponding forward rate.)

Solutions to Exercises on Lecture Notes 1–2

3. (The problem is admittedly somewhat artificial, but serves as an exercise.) If we buy the stock today and sell it after one year, the cash flow can be represented:

month	0	4	10	12
cash flow	$-S_0$	2	$0.02S_{10}$	S_{12}

i.e.,

$$S_0 = 2e^{-0.06\cdot 4/12} + 0.02\ P_0^{(10)}\big(S_{10}\big) + P_0^{(12)}\big(S_{12}\big). \tag{1}$$

On the other hand, if we sell the stock after 10 months, before the dividend, then the cash flow is

month	0	4	10
cash flow	$-S_0$	2	S_{10}

i.e.,

$$S_0 = 2e^{-0.06\cdot 4/12} + P_0^{(10)}\big(S_{10}\big)$$

If we here solve for $P_0^{(10)}\big(S_{10}\big)$ and substitute into (1) we get

$$0.98S_0 = 1.96e^{-0.06\cdot 4/12} + P_0^{(12)}\big(S_{12}\big)$$

But $P_0^{(12)}\big(S_{12}\big) = e^{-0.06}G_0^{(12)}\big(S_{12}\big) = e^{-0.06}110$, hence

$$0.98S_0 = 1.96e^{-0.06\cdot 4/12} + e^{-0.06}110$$

which yields $S_0 \approx 107.67$ dollars.

5. See *example 5* in Lecture Note 2. We have (X_0 = current exchange rate, r = American interest rate)

$$X_0 = e^{(r-0.04)5/12}\,0.94630$$

and

$$X_0 = e^{(r-0.04)11/12}\,0.95152$$

hence

$$e^{(r-0.04)5/12}\,0.94630 = e^{(r-0.04)11/12}\,0.95152$$

from which we can solve for r (take logarithms,) $r \approx 2.90\%$.

6. The present value P_0 of the cash flow of the underlying bond after 2 years (that is, after the bond has been delivered) is

$$P_0 = 2\,e^{-0.059\cdot 2.5} + 2\,e^{-0.059\cdot 3} + 2\,e^{-0.059\cdot 3.5} + 2\,e^{-0.059\cdot 4} + 2\,e^{-0.059\cdot 4.5} + 102\,e^{-0.059\cdot 5} \approx 84.0836$$

The forward price $G_0 = P_0/Z_2$, so

$$G_0 = 84.0836 \cdot e^{0.056\cdot 2} \approx 94.05$$

Note that we don't need the interest rates for shorter duration than two years!

7a. Since there are no dividends or other convenience yield, the present price P_0 is the same as the spot price: $P_0 = 40$ dollars. Hence $G_0 = Z_1^{-1} P_0 = e^{0.1} \cdot 40 \approx 44.207$ dollars.

b. By the same argument, six months later the present price $P_6\big(S_{12}\big) = 45$ dollars; S_{12} is the (random) spot price of the stock at 12 months. The cash flow of the forward contract is $S_{12} - 44.207$ dollars at 12 months, so the present value of this cash flow at 6 months is $P_6\big(S_{S_12 2}\big) - e^{-0.1\cdot 0.5}\, 44.207$ dollars ≈ 2.949 dollars.

If a forward contract were drawn up at six months, the forward price would be $G_0 = Z_{6,12}^{-1}\, P_0 = e^{0.1\cdot 0.5}\, 45 \approx 47.303$ dollars, given that the interest rate is still 10% per year.

9. $F_0^{(t)}(X) = P_0^{(t)}(Xe^R) = 0.96 \cdot G_0^{(t)}(Xe^R) = 0.96 \cdot 115$ euros $= 110.40$ euros.

12. The cash flow at time T is $\big(e^{r(T-t)} - 1\big)\, L$ where r is the zero coupon interest rate that pertain at time t for the duration up to time T. The present value today of this cash flow can be written as

$$P_0 = P_0^{(T)}\big[e^{r(T-t)} - 1\big]\, L = \big(P_0^{(T)}\big[e^{r(T-t)}\big] - Z_T\big)\, L \tag{1}$$

Now consider the strategy: we do nothing today, but at time t we deposit the amount 1 at the rate r up to time T when we withdraw the amount $e^{r(T-t)}$. The present value of this cash flow is zero, so $-Z_t + P_0^{(T)}\big[e^{r(T-t)}\big] = 0$ i.e., $P_0^{(T)}\big[e^{r(T-t)}\big] = Z_t$. Substitute this into the relation (1) above:

$$P_0 = (Z_t - Z_T)\, L$$

The deterministic cash flow at T when the forward rate is applied is $\big(e^{f(T-t)} - 1\big)\, L$ where f is the forward rate, and its present value is of course

$$\widetilde{P}_0 = Z_T\big(e^{f(T-t)} - 1\big)\, L = \big(Z_T e^{f(T-t)} - Z_T\big)\, L$$

But $Z_T e^{f(T-t)} = Z_t$ (see "Forward Rate Agreements" in Lecture Note 2), hence

$$\widetilde{P}_0 = (Z_t - Z_T)\, L = P_0$$

Q.E.D. ("Quad Erat Demonstrandum".)

Comments on Lecture Note 3

I have derived a formula where one can use more than one futures in the hedge. The formula is in fact just that of multivariate regression (Ordinary Least Squares, after inserting an intercept.) Perhaps in practice one uses only one futures; at least that is the only case discussed by Hull. The relevant chapter in Hull's book is chapter 3.

Exercises and Examples on Lecture Note 3.

1. (This is problem 3.6 in Hull's book) Suppose that the standard deviation of quarterly changes in the prices of a commodity is \$0.65, the standard deviation of quarterly changes in a futures price on the commodity is \$0.81, and the coefficient of correlation between the two changes is 0.8. What is the optimal hedge ratio, and what does it mean. (0.642)

   In addition: After the hedge, what is the standard deviation per unit of the commodity of the hedged position? (\$0.39)

2. (This is problem 3.18 in Hull's book.) On July 1, an investor holds 50'000 shares of a certain stock. The market price is \$30 per share. The investor is interested in hedging against movements in the market over the next months, and decides to use the September Mini S&P 500 futures contract. The index is currently 1'500 and one contract is for delivery of \$50 times the index. The beta of the stock is 1.3. What strategy should the investor follow? (short 26 futures)

   Comment. The *beta* of a stock is the regression coefficient of the return of the stock on the return of the index. The investor thinks that this particular share will do better than the index, but he wants to hedge against a general decline of the market. The strategy is to capitalize on this better performance, even if there is a general decline.

Solutions to Exercises on Lecture Note 3

1. The optimal hedge ratio β is the solution to

$$\operatorname{Cov}(F,F)\,\beta = \operatorname{Cov}(F,S)$$

where S = spot price and F = futures price at the time of the hedge. Since $\operatorname{Cov}(F,F) = \operatorname{Var}(F)$ we get (SD = standard deviation; ρ = correlation coefficient)

$$\beta = \frac{\operatorname{Cov}(F,S)}{\operatorname{Var}(F)} = \frac{\mathrm{SD(F)}\,\mathrm{SD}(S)\,\rho(S,F)}{\operatorname{Var}(F)} = \frac{\mathrm{SD}(S)}{\mathrm{SD(F)}}\rho(S,F) = \frac{0.65}{0.81}\,0.8 \approx 0.642$$

This means that for every unit of the commodity to be hedged we should take short futures positions on 0.642 units. The hedged position will have a standard deviation of $\$0.65\,\sqrt{1-\rho^2} = \$0.39$ per unit.

1. *Discussion.* I don't quite agree with Hull's presentation. S here is the commodity price at the time of the hedge, i.e., it is a random value. We need its standard deviation and its correlation with the futures price (we assume that either the futures is not on the exact same commodity, or that the maturity of the futures does not coincide with the date of the hedge, otherwise we could easily get a perfect hedge.) But Hull talks about the *price change* from today's spot price. Technically it may be correct, but conceptually the current spot price has nothing to do with the problem at hand. Assume for example that we in June want to hedge the price of Christmas trees in late December. Then it is relevant to estimate (or guess) the standard deviation of the spot price on Christmas trees in late December, but the spot price of Christmas trees in June is of course of no importance for the problem. We don't take today's spot price as a benchmark, rather our experience of prices of Christmas trees in late December.

2. Let X be the value of one share some months from now, $X_0 = \$30$ is the current value. Likewise, let I be the value of the index some months from now, $I_0 = 1'500$ is the current value. The beta value β is by definition such that

$$\frac{X}{X_0} = \beta\,\frac{I}{I_0} + e$$

where the covariance $\operatorname{Cov}(I,e) = 0$. In other words,

$$X = \beta\,\frac{X_0}{I_0}\,I + X_0\,e$$

The idea of the strategy is to hedge the term containing I; this is the impact on the value of the share due to general market movements. We want to capitalize on the idiosyncratic term containing e.

Assume now that we own one share and short k futures. Each future is for delivery of $n = \$50$ times the index. The value of the hedged position after some months is then

$$X - knI = \big(\beta \frac{X_0}{I_0} - kn\big) I + X_0\, e$$

so, obviously, in order to get rid if the I-term we should have $\beta \frac{X_0}{I_0} - kn = 0$, i.e.,

$$k = \frac{1}{n}\, \beta \frac{X_0}{I_0} = 0.00052$$

If we own 50'000 shares, we should thus short 50'000 $\cdot$ 0.00052 = 26 futures.

Comments on Lecture Notes 4 and 5

The theorem in this note and the No Arbitrage Assumption are the most fundamental cornerstones of financial mathematics. The very elegant proof of the Theorem is essentially taken from a book or course notes I came across but, despite my efforts, I can not find the reference. The most common proof of the theorem employs “Farkas' Lemma” which is much more complicated.

Some details in Lecture Note 5 may need clarifications. First of all, where does the specification (1) come from? The idea is this. Let X be the value at some future time t of some asset, for example a share of a stock, or a commodity like a barrel of oil. There are several different random events which can influence this value. If G_0 is the forward price today, then tomorrow the forward price may change by a factor $(1 + w_1)$, i.e. tomorrow the forward price is $G_1 = G_0\,(1 + w_1)$. Here w_1 is a random variable with rather small standard deviation.

The day after tomorrow, the forward price has changed by a new factor $(1+w_2)$ such that the forward price then is $G_2 = G_1\,(1 + w_2) = G_0\,(1 + w_1)(1 + w_2)$ and so on. Eventually, at time t when the forward expires, the “forward” price is the same as the spot price, and we have

$$X = G_0\,(1 + w_1)(1 + w_2)\cdots(1 + w_t)$$

i.e.,

$$\ln X = \ln\big(G_0\big) + \ln(1 + w_1) + \ln(1 + w_2) + \cdots + \ln(1 + w_t)$$

If we assume that the random variables w_k are independent and identically distributed, then the Central Limit Theorem says that the sum in the right hand side is approximately Normally distributed, and the variance is proportional to the time interval up to t (the number of terms.) The standard deviation of $\ln X$ is thus Normally distributed with a standard deviation proportional to $\sqrt{t}$. This is the rationale for the specification (1).

Further down I say that “In this formula, Q can be replaced by $\mathrm{E}[Q \mid z]\ldots$”. If you have difficulties with this, you can read the section “Conditional Expectations and Martingales” in Lecture Note 7.

Somewhat later I say that “The condition $\mathrm{E}[Q] = 1$ imposes the restriction $a = -\frac{1}{2}b^2$.” This comes from the fact that if w has a Normal distribution with mean μ and standard deviation σ, then

$$\mathrm{E}[e^w] = e^{\mu+\frac{1}{2}\sigma^2}$$

I leave the derivation of this formula as an exercise! The computation is similar to, but simpler than, the one between (2) and (3) of Lecture Note 5.

Read about options in chapter 8 in Hull. You may also want to read chapter 9 and 10.

The approach taken by me is the “measure theoretic approach” which I find more appropriate than the older “replicating portfolio approach” which is used in many presentations. Hull uses the "replicating portfolio" approach, although the mathematics isn't treated rigorously; this is done in chapter 13.1–13.8 in Hull, which you can skip for the purpose of this course, but read 13.11, 15.3–15.5, 16.1–16.5 and 16.8.

Black's model is applicable on a variety of European options. It contains the well known "Black-Scholes" pricing formula as a special case, and it can be used to price options on shares of stocks which pay dividends, stock indices, currencies, futures and bonds. In all cases the procedure is in two steps: first we must find the forward price G_0 of the underlying asset. We have seen some examples of such computations in Lecture Notes 1–3. The next step is to decide on a volatility σ. There is no obvious way to do this. One way is to estimate it from historical data (see "Estimating Volatility from Historical Data" in chapter 13.4 in Hull's book,) another is to use "implied volatilities" (13.11.) The last method could be as follows: Assume we want to price a call option on a particular stock with strike price K. There is already an option on the market with a different strike price L. Since this option already has a price given by the market, we can use Black's model on that option to solve for σ; then use that same σ, the "implied volatility", for the option we want to price—see 13.11 i Hull.

Note, however, that Black's model (or Black-Scholes) doesn't perfectly conform with empirical data; see chapter 18 in Hull.

There is one problem with Black's model applied to futures options. The problem is that it is the *forward* price that should go into the formula, not the futures price.

However, it is reasonable to assume that for all assets except interest rate securities, the forward price equals the futures price (at least to a sufficient degree of accuracy,) which is true if the value f the asset is essentially independent of the interest rate.

In fact, under reasonable assumptions, it holds that

$$F_0^{(t)}[X] = G_0^{(t)}[X]\, e^c \tag{1}$$

where $c = \text{Cov}\big(\ln X, R(0,t)\big)$. Hence, if $c = 0$ then $F_0^{(t)} = G_0^{(t)}$.

You may want to prove this formula. The assumption is that $\ln X$ and $R(0,t)$ have a joint Normal distribution. But in this course you can also just accept (1) as a fact.

The last formula of Lecture Note 5 lends itself to an interpretation of the market price of risk, λ. If $\lambda > 0$, then the forward price is below the expected future spot price, which must mean that the demand for buying forwards (long positions) exceeds the supply (short positions.) Those who want to buy forwards must then bid down the forward price in order to draw speculators into the short side of the market to clear it. The reverse of course holds if $\lambda < 0$. In either case the word "market price of risk" is a bit misleading: rather the magnitude and sign of λ reflect the demand - supply balance on the forward market.

Exercises and Examples on Lecture Notes 4 and 5.

Interest rates always refer to continuous compounding. Answers are given in parenthesis; Solutions and hints are given on the following pages.

1. Determine the price of a European futures call option on a barrel of crude oil to be delivered in 4 months. This is also the time of maturity of the option. The futures price today is $F_0 = \$25.00$, the strike price is \$23.00 and the volatility of the futures price is estimated to be 25% for one year. The risk-free interest rate is 9% per year. (\$2.527)

2. Determine the price of a European call option on a share which does not pay dividends before the maturity of the option. The present spot price of the share is 45 SEK, the option matures in 4 months, the volatility is 25% in one year, the option's strike price is 43 SEK and the risk-free interest rate is 9% a year. (4.463 SEK)

3. The same question as above, but now we assume the share pays a dividend of 0.50 SEK in 3 months, all other assumptions are the same. (4.115 SEK)

4. Determine the price of a European put option on 1 GBP with a strike price of 14 SEK in 6 months. The exchange rate is 13 SEK for 1 GBP, and the pound's volatility is assumed to be 14% in one year. The pound's rate of interest is 11% and the Swedish crown's rate of interest is 7% a year. (1.331 SEK)

5. Determine the price of a European call option on an index of shares which is expected to give a dividend of 3% a year, continuously. The current value of the index is 93 SEK, the strike price is 90 SEK and the option matures in two months. The risk-free interest rate is 8% a year and the index has a volatility of 20% in one year. (5.183 SEK)

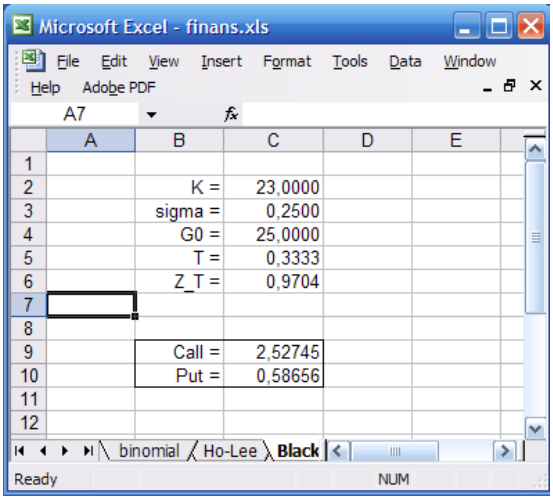
6. Let $S(t)$ be the spot price of a share at time t (year) which does not pay dividends the following year. Determine the price of a contract which after one year gives the owner $\dfrac{S(1)^2}{S(0)}$ SEK. The risk-free rate of interest is 6% a year, and the share's volatility is assumed to be 30% for one year. ($S(0)\, e^{0.15}$)

7. Let $S(t)$ be the spot price of a share at time t (year) which does not pay dividends the following year. Determine the price of a contract which after one year gives the owner \$100 if $S(1) > \$50$ and nothing if $S(1) \leq \$50$. The current spot price is $S_0 = \$45$, the share's volatility is assumed to be 30% for one year and the risk-free rate of interest is 6% a year. (\$35.94)

8. Show the *put-call parity:* If c and p are the prices of a call and a put option with the same strike price K, then

$$c - p = Z_t\,(G_0 - K)$$

Solutions to Exercises on Lecture Notes 4 and 5



I suggest that you make a spreadsheet in Excel (or some such) where you can calculate call- and put options with Black's formula. You input the parameters K, G_0, σ, time to maturity T and discount factor Z_T. You find the relevant formulas in Lecture Note 5. You can then use this spreadsheet to solve most of the problems below; the major problem is to find the correct value for G_0 to go into the formula.

1. Since we assume that the price of oil is essentially unaffected by interest rates, we set $G_0 = F_0 = \$25$ in Black's formula. Of course $Z_T = e^{-0.09/3}$. In my spreadsheet the answer appears as in the figure above.

2. Since the share doesn't pay any dividend and there is no convenience yield, we get the forward price G_0 from the relation (see example 1 of Lecture Note 2 with $d = 0$) 45 SEK $= e^{-0.09/3}\, 45$ SEK, i.e., $G_0 \approx 46.37$ SEK.

3. See example 1 in Lecture Note 2. We get $G_0 \approx 45.867$ SEK.

4. See example 5 in Lecture Note 2. The forward price of 1 GBP is ≈ 12.7426 SEK. Note that $Z_T = e^{-0.035}$, i.e., the discounting should use the Swedish interest rate (why?)

5. See example 4 in Lecture Note 2. The index yields a continuous dividend of 3% per year, and $S_0 = 93$ SEK. We get $93\, e^{(0.08-0.03)/6}$ SEK ≈ 93.7782 SEK.

6. We determine the forward price G_0 as in problem 2: $G_0^{(1)}\big(S(1)\big) = S(0)\, e^{0.06}$. Hence Black's formula (3), Lecture Note 5, yields

$$p = e^{-0.06}\,\mathrm{E}\big[\tfrac{1}{S(0)}(G_0 e^{-\frac{1}{2}0.3^2+0.3\sqrt{1}\,w})^2\big] = e^{-0.06}\,\tfrac{1}{S(0)}\,G_0^2\,e^{-2\,\frac{1}{2}0.3^2}\,\mathrm{E}\big[e^{2\cdot 0.3\,w}\big]$$

$$= e^{-0.06}\,\tfrac{1}{S(0)}\,G_0^2\,e^{-2\,\frac{1}{2}0.3^2}\,e^{2\cdot 0.3^2} = S(0)\,e^{0.15}.$$

7. First we determine the forward price of the underlying asset as before: $G_0 = e^{0.06}\, S(0) = \$45 \cdot e^{0.06}$. Next, let A be the event $S > \$50$ and $1_A(S)$ the *index function* of this event, i.e.,

$$1_A(S) = \begin{cases} 1 & \text{if } S > \$50 \\ 0 & \text{if } S \leq \$50 \end{cases}$$

The payoff of the derivative is then $\$100 \cdot 1_A(S(1))$, hence the value of the contract is, with the usual notation

$$\begin{aligned} p &= Z_1\, \mathrm{E}\big[\$100 \cdot 1_A(G_0\, e^{-\frac{1}{2}\sigma^2 1 + \sigma\sqrt{1}\, w})\big] \\ &= e^{-0.06}\, \mathrm{E}\big[\$100 \cdot 1_A(45 \cdot e^{0.06-0.045+0.3w})\big] \\ &= e^{-0.06}\, \$100 \cdot \Pr\big(45 \cdot e^{0.06-0.045+0.3w} > 50\big) \\ &= e^{-0.06}\, \$100 \cdot \Pr\Big(w > \frac{1}{0.3}\, \ln\big(\frac{50}{45}\big) - 0.05\Big) \approx \$35.94 \end{aligned}$$

8. Let x^+ denote $\max(x, 0)$. Then the price of a call option with maturity t and strike price K on the underlying asset value X is $P_0^{(t)}[(X-K)^+]$, and similarly, the price of the corresponding put option is $P_0^{(t)}[(K-X)^+]$. Hence

$$\begin{aligned} c - p &= P_0^{(t)}[(X-K)^+] - P_0^{(t)}[(K-X)^+] \\ &= P_0^{(t)}[(X-K)^+ - (K-X)^+] \\ &= P_0^{(t)}[X-K] = P_0^{(t)}[X] - K\, P_0^{(t)}[1] \\ &= Z_t\, \big(G_0^{(t)}[X] - K\big) \end{aligned}$$

Remark: This is the put-call parity formula (15.10), ch. 15.4, in Hull.

Comments on Lecture Note 6

You can read about "duration" in ch. 4.8 in Hull. We don't need "Modified Duration". Duration is a measure of how long on average the holder of the portfolio has to wait before receiving cash payments, but more importantly, it is a measure of the sensitivity of the portfolio's value to changes in the rate of interest. It can also be thought of as a time when the value of the portfolio is nearly unaffected by the rate of interest: If the interest (yield) falls today, then the value of the portfolio immediately increases (if it is a portfolio of bonds,) but it's growth rate goes down, and at the time $t =$ duration the two effects nearly net out. Similarly, of course, if the rate of interest goes up. Note, however, that in the meantime coupon dividends have been payed out, and when these are re-invested, the duration goes up, so in order to keep a certain date as a target for the value of the portfolio, it has to be re-balanced. All this is captured in $(3')$ of Lecture Note 6.

In ch. 6.4 you can read about "Duration-Based Hedging Strategies". Note that Hull's formula (6.5) is a special case of the last formula in Lecture Note 6 (when $D_{tot} = 0$.) Hull, however, does not give any derivation of his formula. It is also somewhat unclear what the definition of D_F is in Hull's book. That is why I have introduced the concept "Forward Duration".

In chapter 28.1 Hull talks about "European Bond Options", corresponding to "Black's Model for Bond Options" in Lecture Note 6. Hull introduces the variable σ_B which of course corresponds to $D_F\,\sigma$ in my Lecture Note. Hull explains how σ_B is computed in the section "Yield Volatilities", but it is somewhat unclear, since he refers to the *"change Δy_F in the forward yield y_F ..."* but he never defines the concept "forward yield", as far as I can find. We use my definition in Lecture Note 6. Note that I avoid the concept "yield volatility", I prefer to talk about the standard deviation of the yield.

Exercises and Examples on Lecture Note 6

Interest rates always refer to continuous compounding. Answers are given in parenthesis; Solutions and hints are given on the following pages.

1. Determine the
   a) forward price (1993.29 SEK)
   b) forward yield (9.939%)
   c) forward duration (1.861 years)

   two years into the future for a bond that pays out 100 SEK in 2.5, 3 and 3.5 years and 2 100 SEK in 4 years. Zero-coupon interest rates are at present 6%, 6.5%, 7%, 7.5% and 8% for the duration of 2, 2.5, 3, 3.5 and 4 years.

2. Calculate approximately the duration of a portfolio containing a coupon bearing-bond which matures in two years with face value 100'000 SEK and pays a 6%-coupon (this means that the coupon is paid every six month at 3% of the face value,) plus a short position of a futures contract with maturity in two years on a three year (at the time of maturity of the futures) 6% coupon-bearing bond (the first coupon payment is six months after the maturity of the futures) with face value 50'000 SEK. Interest rates are today 5.5% a year with continuous compounding for any length of duration. (Approximate the futures price with the forward price.) (0.514 years)

3. Determine the price of a European put option with maturity in two years on a five year 6% coupon bond issued today (the first coupon payment after the maturity of the option occurs six months after the maturity of the option) with face value 50'000 SEK, which is also the option's strike price. The forward yield of the bond is 6.5% a year and the standard deviation of the yield is 0.012 in one year. The two year zero-coupon rate of interest is 5.5% a year. (1'253 SEK)

4. Calculate the value of a one-year put option on a ten year bond issued today. Assume that the present price of the bond is 1'250 SEK, the strike price of the option is 1'200 SEK, the one-year interest rate is 10% per year, the forward yield has a standard deviation of 0.013 in one year, the forward duration of the bond as of the time of maturity for the option is 6.00 years and the present value of the coupon payments which will be paid out during the lifetime of the option is 133 SEK. (20.90 SEK)

5. We want to determine the price of a European call bond option. The maturity of the option is in 1.5 years and the underlying bond yields coupon payments \$100 in 2 years, \$100 in 2.5 years and matures with a face value (including coupon) of \$2'000 3 years from today. We assume that the standard deviation of the change in the one-year yield is 0.01 in one year. Current zero coupon rates are (% per year)

1 yr	1.5 yr	2 yr	2.5 yr	3 yr
3.5	4.0	4.5	4.8	5.0

Use Black's model to find the price of the option when the strike price is \$1'950. ($\approx$ \$65.35)

Solutions to Exercises on Lecture Note 6

1. First we calculate the present price P_0 of the bond:

$$P_0 = e^{-2.5\cdot 0.065}\,100 + e^{-3\cdot 0.07}\,100 + e^{-3.5\cdot 0.075}\,100 + e^{-4\cdot 0.08}\,2'100 \approx 1'767.886$$

The forward price is thus $G_0 = z_2^{-1} \cdot P_0 = e^{2\cdot 0.06}\,1'767.886 \approx 1'993.29$ SEK.

Next we determine the forward yield y_F from the relation

$$1'993.29 = G_0 = 100\,e^{-y_F\cdot 0.5} + 100\,e^{-y_F\cdot 1} + 100\,e^{-y_F\cdot 1.5} + 2'100\,e^{-y_F\cdot 2}$$

which we solve numerically (trial and error) to $y_F = 0.09939$.

Finally, the forward duration D_F is obtained from

$$1'993.29 \cdot D_F = G_0\,D_F = 0.5 \cdot 100\,e^{-y_F\cdot 0.5} + 1 \cdot 100\,e^{-y_F\cdot 1} + 1.5 \cdot 100\,e^{-y_F\cdot 1.5} + 2 \cdot 2'100\,e^{-y_F\cdot 2}$$

and gives $D_F = 1.861$ years.

2. First we determine the present value P_0 of the portfolio. Note that the *value* of the futures contract is zero.

$$P_0 = 3'000\,e^{-0.055\cdot 0.5} + 3'000\,e^{-0.055\cdot 1} + 3'000\,e^{-0.055\cdot 1.5} + 103'000\,e^{-0.055\cdot 2} \approx 100'791.43$$

We get the duration of the bond D_P in the portfolio from the relation

$$P_0 D_P = 0.5 \cdot 3'000\,e^{-0.055\cdot 0.5} + 1 \cdot 3'000\,e^{-0.055\cdot 1} + 1.5 \cdot 3'000\,e^{-0.055\cdot 1.5} + 2 \cdot 103'000\,e^{-0.055\cdot 2} \approx 192'984.25$$

which yields $D_P = 1.915$ years.

Next we calculate the present value of the cash flow from the bond underlying the futures:

$$p = 1'500\,e^{-0.055\cdot 2.5} + 1'500\,e^{-0.055\cdot 3} + 1'500\,e^{-0.055\cdot 3.5} + 1'500\,e^{-0.055\cdot 4} + 1'500e^{-0.055\cdot 4.5} + 51'500\,e^{-0.055\cdot 5} \approx 45'309.35$$

so the futures price, which we approximate with the forward price, is

$$F_0^{(2)} = Z_2^{-1} p = e^{0.055\cdot 2} \cdot 45'309.35 \approx 50'577.84$$

The forward yield y_F is determined by

$$F_0 = 1'500\,e^{-y_F\cdot 0.5} + 1'500\,e^{-y_F\cdot 1} + 1'500\,e^{-y_F\cdot 1.5} + 1'500\,e^{-y_F\cdot 2} + 1'500\,e^{-y_F\cdot 2.5} + 51'500\,e^{-y_F\cdot 3}$$

which yields $y_F = 0.055$, which we should have anticipated: if the term structure is flat, all yields, also the forward yield, are equal to the prevailing interest rate. The forward duration D_F of the bond underlying the futures is now determined from

$$\begin{aligned} F_0 D_F &= 0.5 \cdot 1\text{'}500\, e^{-y_F \cdot 0.5} + 1 \cdot 1\text{'}500\, e^{-y_F \cdot 1} + 1.5 \cdot 1\text{'}500\, e^{-y_F \cdot 1.5} \\ &\quad + 2 \cdot 1\text{'}500\, e^{-y_F \cdot 2} + 2.5 \cdot 1\text{'}500\, e^{-y_F \cdot 2.5} + 3 \cdot 51\text{'}500\, e^{-y_F \cdot 3} \end{aligned}$$

which gives $D_F = 2.791$ years. Finally, we can calculate the duration of the portfolio:

$$D_{tot} = D_P - D_F \frac{F_0}{P_0} = 1.915 - 2.791 \frac{50\text{'}577.84}{100\text{'}791.43} \text{ years} \approx 0.514 \text{ years.}$$

3. We will employ Black's model for put options—see Lecture Note 5—so we need the forward price G of the underlying asset, and the relevant volatility σ.

   Since we know the forward yield $y_F = 0.065$ we can easily calculate G:

$$\begin{aligned} G &= 1\text{'}500 e^{0.5 \cdot 0.065} + 1\text{'}500 e^{1 \cdot 0.065} + 1\text{'}500 e^{1.5 \cdot 0.065} \\ &\quad + 1\text{'}500 e^{2 \cdot 0.065} + 1\text{'}500 e^{2.5 \cdot 0.065} + 51\text{'}500 e^{3 \cdot 0.065} \\ &\approx 49\text{'}186.44 \end{aligned}$$

   The forward duration D_F is obtained from

$$\begin{aligned} G\, D_F &= 0.5 \cdot 1\text{'}500 e^{0.5 \cdot 0.065} + 1 \cdot 1\text{'}500 e^{1 \cdot 0.065} + 1.5 \cdot 1\text{'}500 e^{1.5 \cdot 0.065} \\ &\quad + 2 \cdot 1\text{'}500 e^{2 \cdot 0.065} + 2.5 \cdot 1\text{'}500 e^{2.5 \cdot 0.065} + 3 \cdot 51\text{'}500 e^{3 \cdot 0.065} \\ &\approx 137122.40 \end{aligned}$$

   and we get $D_F = 2.788$ years. The relevant value of σ in Black's model is thus $0.012 \cdot D_F = 0.0335$. Finally, the price of the out option is 1'252.76 SEK.

4. Again we need the forward price of the underlying asset and the appropriate value of σ. The value of G is easy; the present price of the cash flow from the bond after the maturity of the option is $1\text{'}250 - 133 = 1\text{'}117$ SEK, hence $G = 1117e^{0,1} \approx 1\text{'}234.48$ SEK. The relevant value of σ is $\sigma = 0.013 \cdot 6 = 0.0780$. We get the option's price 20.90 SEK from Black's formula.

5. The forward price of the underlying bond is

$$
\begin{aligned}
G &= e^{1.5\cdot 4}\big(100\, e^{-0.045\cdot 2} + 100\, e^{-0.048\cdot 2.5} + 2\text{'}000\, e^{-0.050\cdot 3} \\
&= 2\text{'}019.08 \\
&= 100\, e^{-y_F\cdot 0{,}5} + 100\, e^{-y_F\cdot 1} + 2\text{'}000\, e^{-y_F\cdot 1.5} \text{ for } y_F = 0.06
\end{aligned}
$$

The forward duration D_F is calculated from

$$
\begin{aligned}
G\, D_F &= 0.5\cdot 100\, e^{-y_F\cdot 0{,}5} + 1\cdot 100\, e^{-y_F\cdot 1} + 1.5\cdot 2\text{'}000\, e^{-y_F\cdot 1.5} \\
&= 2\text{'}891.00,
\end{aligned}
$$

hence $D_F = 1.429$ years. Now $G = \$2\text{'}019.08$ and $\sigma = 0.01\cdot D_F = 0.01429$. The option's price is \$65.35.

Comments on Lecture Note 7

The concept "Risk Adjusted Probability Measure" is at the very core of Financial Mathematics. Hull talks about the *"traditional risk neutral world"* (see e.g. ch. 27.0) by which he means the probability measure where *the Money Market Account* is used as numeraire (see ch. 27.4.) We will come to that measure in Lecture Note 10, where I call it *the Futures Measure* for a reason that will be apparent. For now, we will use zero coupon bonds as numeraire (ch. 27.4)—the *forward measure.*

Hull's description of these different measures in ch. 27 and mine, in the Lecture Notes, differ a lot, and I think you should stay away from Hull on this issue for this course, otherwise you may get confused. The difference stems from the fact I mentioned before: Hull uses the "replicating portfolio" approach, whereas I use the "measure theoretic" approach. The theoretical foundation for this latter approach is given in Lecture Note 4. The advantage of this approach is, in my opinion, that it is more general and at the same time requires less fancy mathematics (if done rigorously, which doesn't happen Hull's book.)

Intuitively we can think about these probability measures as follows. Let us consider the event A : *it will rain in London at today's date next year.* What is the probability of this event occurring? Is there even an "objective" probability of this event? If so, how do we assess it?

Assume that we put forth the following contract on the market: *If the event A occurs, then the holder of this contract will get one kilo of gold next year.* Anybody can buy or sell such contracts, and the payment to the seller is another contract: *the holder of this contract will receive p kilos of gold in one year.* The value of p is then determined by market equilibrium: demand equals supply. Obviously, if the market is rational, p will be a number between zero and one, and intuitively p should reflect the market's assessment of the probability of A occurring. Indeed, we can take p to be the *definition* of the probability of the event A. However, the number p may depend on choice of gold as the *numeraire.* If we use US$ instead of gold, it might happen that we get a different number for p, so we have different probabilities for different numeraires.

What is it that can cause the p:s to differ depending on numeraires? It is because the market can have different attitudes towards risk pertaining to different numeraires. Think for example of a situation where there is a shortage of water and water is the numeraire instead of gold. For the same event A as above, the contract would say *If the event A occurs, then the holder of this contract will get one cubic metre of water next year,* and the payment would be a contract *the holder of this contract will receive p cubic metres of water in one year.* Now the equilibrium value of p is presumably less than in the previous example, since people (at least London residents) might argue as follows: "If A occurs, and it hence rains, then I don't need extra water, so why buy this contract and risk to lose water if it *doesn't* rain?" Few people are willing to take long positions on the contract, which means that p will be bid down. With gold as numeraire, the situation is different.

In Financial Mathematics we are often better served by some "risk adjusted" probability measure than by an "objective" probability measure. The latter is even a bit slippery conceptually in many contexts.

Exercises and Examples on Lecture Note 7

Interest rates always refer to continuous compounding. Answers are given in parenthesis; Solutions to some problems are given on the following pages.

1. Show that

$$G_0^{(t)}(X) = G_0^{(t)}(N)\,\mathrm{E}^N\,[\frac{X}{N}]$$

2. It is now May 1:st. Let N be the value of 1 kg of gold August 31:st. We now write a forward contract on $X = 100$ barrels of crude oil to be delivered August 31:st where the payment should be payed in gold. Show rigorously that the forward price should be

$$\frac{G_0(X)}{G_0(N)} \text{ kg of gold.}$$

3. Derive an expression for Black's formula, similar to (3) in Lecture Note 5, for a European call option on 100 barrels of crude oil when the strike price is K kilos of gold.

Solutions to Exercises on Lecture Note 7

1. This follows immediately from (1) Lecture Note 7, if we divide both sides of the equality with Z_t (see Lecture Note 1.)

2. Let $g(X)$ be the forward price of X in terms of gold. For the long holder of the contract, the cash flow at maturity is then $X - g(X)N$. The present value of this cash flow is zero (= value of *contract*.) Hence

$$0 = P_0^{(t)}\big(X - g(X)N\big) = P_0^{(t)}(X) - g(X)P_0^{(t)}(N).$$

Hence

$$g(X) = \frac{P_0^{(t)}(X)}{P_0^{(t)}(N)} = \frac{G_0^{(t)}(X)}{G_0^{(t)}(N)}$$

(see example 1.)

3. We model the price of 100 barrels of oil X relative to the price of 1 kg of gold N at the date of maturity in accordance with Black's model ((1), Lecture Note 5:)

$$\frac{X}{N} = Ae^{\sigma\sqrt{t}\,z} \quad \text{where} \quad z \in N(0,1)$$

The value of the option at maturity is $\max[\frac{X}{N} - K, 0] = (\frac{X}{N} - K)^+$ kilos of gold. We now employ Black's price formula (3) of Lecture Note 5. This formula is valid independent of numeraire: there is nothing magical about US$ or any other currency, we can just as well use for instance gold as currency. However, we must figure out what Z_t corresponds to in this case. We have $Z_t = P_0^{(t)}(1)$ where the 1 is one unit of the curreny, i.e., one kilo of gold, in our case. Hence, we use $Z_t(\text{gold}) = P_0^{(t)}(N)$ and we get

$$P_0(\text{option}) = P_0^{(t)}(N)\mathrm{E}\left[\Big(g(X)e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w} - K\Big)^+\right] \tag{1}$$

where $w \in N(0,1)$ and $g(X)$ is the forward price in kilos of gold of 100 barrels of crude oil, i.e., according to example 2

$$g(X) = \frac{G_0(X)}{G_0(N)} \text{ kg of gold.}$$

Here G_0 denotes forward price in any unit, for example US$. If we now express values in US$, we have $P_0^{(t)}(N) = Z_t G_0^{(t)}(N)$ where now Z_t denotes the usual zero coupon price in US$ for 1 US$. Now (1) can be written as

$$\begin{aligned}
P_0(\text{option}) &= Z_t G_0(N)\mathrm{E}\left[\Big(\frac{G_0(X)}{G_0(N)}e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w} - K\Big)^+\right] \\
&= Z_t\mathrm{E}\left[\Big(G_0(X)e^{-\frac{1}{2}\sigma^2 t + \sigma\sqrt{t}\,w} - G_0(N)K\Big)^+\right]
\end{aligned}$$

which is the formula we are looking for.

Comments on Lecture Note 8

The binomial approximation: Binomial trees are treated in Lecture Note 9. It was introduced by J. Cox, S. Ross, and M. Rubenstein in "Option pricing: A simplified approach", *Journal of Financial Economics* 7 (1979), 87–106, and has been a very extensively used numerical procedure. The idea is that the binary variable

$$b = \begin{cases} \sqrt{\Delta t} & \text{with probability } 0.5 \\ -\sqrt{\Delta t} & \text{with probability } 0.5 \end{cases}$$

has the same first and second moments (expectation and variance) in common with a normal $N(0, \Delta t)$ variable. Hence, a sum of many independent such variables will have almost the same distribution as a corresponding sum of $N(0, \Delta t)$ variables (Bernulli's theorem, or the Central Limit Theorem.)

In the last formula of the Lecture Note I have used the equality
$e^{\sigma\, b_{k+1}} = \cosh(\sigma\, b_{k+1}) + \sinh(\sigma\, b_{k+1})$.

Comments on Lecture Note 9

You can read about Binomial Trees in Hulls book ch. 11 and 19. Note, however, that I employ a slightly different method than Hull; I use the "Alternative Procedure" Hull describes in ch. 19.4. I find that method more appealing, and also easier to generalise to different situations (see for example Lecture Note 12, where we discuss an interest rate model.) Make sure you don't mix up the two approaches!

In chapter 19.2–19.3 Hull constructs binomial trees for various underlying assets: indices, currencies, futures, and shares paying various forms of dividends. In order to make these appear less ad-hoc, I do this in two steps in Lecture Note 9: First we construct a tree over the *forward* prices of the underlying asset. These trees are constructed the same way independent of underlying asset. Then we construct a tree over the relevant spot prices (unless the underlying is a futures contract,) where we use the techniques from Lecture Note 2. Finally, we construct the tree over the derivative price.

Hull's approach is at first a bit unconvincing to me: it is not immediately obvious that, for example, the tree described in figure 19.7 p.418 is consistent with the fact that the corresponding forward prices must have the Martingale property. It *is* true, but to me it isn't obvious. Therefore I prefer to *first* make a tree over the forward prices such that the Martingale property is satisfied. Now I know that there are no arbitrage possibilities lurking in the model; thereafter I derive the relevant spot prices, and I can feel safe that the model is sound.

One can also use *trinomial* trees, where the $N(0, \Delta t)$ variable is approximated by the "trinary" variable

$$u = \begin{cases} \sqrt{3\Delta t} & \text{with probability } \frac{1}{6} \\ 0 & \text{with probability } \frac{2}{3} \\ -\sqrt{3\Delta t} & \text{with probability } \frac{1}{6} \end{cases}$$

This variable shares the same *five* first moments with the $N(0, \Delta t)$ variable, so the approximation is better in a corresponding trinomial tree compared to a binomial tree.

Trinomial trees are also used in many interest rate models (Hull and White;) if you are interested you may read ch. 19.4 and 30.6ff.

Exercises and Examples on Lecture Notes 8 and 9

Interest rates always refer to continuous compounding. Answers are given in parenthesis; Solutions to some problems are given on the following pages.

In these examples we use binomial trees with very few time steps. In a "real" situation one would of course use many more, maybe 50–100.

1. The futures price of coffee (100 kg) to be delivered in four weeks is \$70. The volatility is 2% in a week. The risk-free rate of interest is 0.1% per week. In the following cases, use a binomial tree with time interval of one week.

   a) Determine the price of a European futures put option on coffee (100 kg) with strike price \$72. The option matures in four weeks (which is also the maturity of the futures.) (\$2.4255)

   b) The same question, though for an American option, ceteris paribus (latin: "everything else being the same"). (\$2.4287)

2. A share costs today \$13.28. The volatility is 2% in a week, and the share pays no dividends the coming month. The risk-free rate of interest is 0.1% a week. In the following cases, use a binomial tree with time interval one week.

   a) Determine the price of an American put option with maturity in four weeks at a strike price of \$13.70. (\$0.464143)

   b) The same question, but the share pays a dividend of \$0.27 in just under two weeks (that is just before time 2 in the binomial tree.) Also, the present price of the share is \$13.55, ceteris paribus. (\$0.462168)

3. Determine the price of an American option to buy 100 GBP at 13 SEK per pound, which is the value of the pound today. The time to maturity is one year, the pound's volatility relative to the Swedish crown is 10% in one year. The rate of interest of the Swedish crown is 3% a year, and the one year forward exchange rate is 12.60 SEK for one GBP. Use a binomial tree to solve the problem, with time interval tree months. (37.70 SEK)

4. Determine the price of an American call option on a share whose present price is 100 SEK with maturity in one year with strike price 98 SEK. The share's volatility is 20% in one year, the rate of interest is 6% a year, and the share will pay a dividend in 5.5 months of 4% of the value of the share at that time. Use a binomial tree with time interval three months. (9.345 SEK)

5. Calculate the price of an eight month American call option on a maize futures when the present futures price is \$26.4, the strike price of the option is 26.67, the risk-free rate of interest is 8% and the volatility of the futures price is 30% over one year. Use a binomial tree with time interval two months. (\$2.3515)

6. (Hull problem 19.14) A two month American put option on an index of shares has the strike price 480. The present value of the index is 484, the risk-free rate of interest is 10% per year, the dividend of the index is 3% per year and the volatility is 25% over a year. Determine the value of the option by using a binomial tree with time interval of half a month. (15.2336)

7. (Hull problem 19.19) The spot price of copper is \$0.60 per pound. Assume that, at present, the futures price of copper is

maturity	futures price
3 months	0.59
6 months	0.57
9 months	0.54
12 months	0.50

The volatility of the price of copper is 40% during one year and the risk-free interest rate is 6% a year. Use a suitable binomial tree with time interval three months to estimate the price of an American call option on copper with strike price \$0.60 and maturity in one year. (\$0.063167.)

Solutions to Exercises on Lecture Notes 8 and 9

1.

4	75,7695	72,7985	69,9440	67,2015	64,5665
3	74,2840	71,3713	68,5728	65,8840	
2	72,8276	69,9720	67,2284		
1	71,3998	68,6002			
0	70,0000				

4	0,0000	0,0000	2,0560	4,7985	7,4335
3	0,0000	1,0270	3,4238	6,1099	
2	0,5130	2,2232	4,7621		
1	1,3667	3,4891			
0	2,4255				

4	0,0000	0,0000	2,0560	4,7985	7,4335
3	0,0000	1,0270	**3,4272**	**6,1160**	
2	0,5130	2,2249	**4,7716**		
1	1,3676	3,4948			
0	2,4287				

Here are three binomial trees (the display shows decimal *commas,* which is Swedish standard) cut from an Excel spread sheet. Note that time goes from bottom to top; I think this is the easiest way to arrange binomial trees in a spreadsheet.

The first tree displays the futures (=forward) prices of coffee. Week 0 (today) it is \$70, week 1 it is either \$71.3998 or \$68.6002, each with probability 0.5 *under the forward probability measure.* Week two, the futures price is \$72.8276 or \$69.9720 if the futures price were \$71.3998 week 1, and \$69.9720 or \$76.2284 if the futures price were \$68.6002 week 1; and so on. This tree is generated in accordance with the first tree (forward prices) appearing in Lecture Note 9.

The second tree displays the value of the European put option. It is generated from top to bottom: week 4 the option is worth 0 if the futures price is less than or equal to \$72, otherwise it is worth \$72 − futures price (where the futures price equals the spot price of coffee, since the futures contract matures week 4.) Hence we get the first row (week 4) easily from the previous tree's week 4. Next we compute the week 3 option prices, employing (5) of Lecture Note 7: the price week 3 is the discounted value of the expected value week 4 under the forward probability measure. Hence, each price week 3 is $e^{-0.001}$ times $(0.5 \cdot p_1 + 0.5 \cdot p_2)$ where p_1 and p_2 are the prices above and right-above respectively, For example, $3.4238 = e^{-0.001}(0.5 \cdot 2.0560 + 0.5 \cdot 4.7985)$.

Going down through the tree, we finally find the price week 0 to be \$2.4255.

The third tree displays the values of the corresponding American put option. It is obtained as in the tree on the European option, except that the price is replaced by the exercise value if that is higher. These numbers are in boldface. For instance, the value \$3.4272 is the value if the option is exercised: $3.4272 = 72 - 68.5728$ which is more than $e^{-0.001}(0.5 \cdot 2.0560 + 0.5 \cdot 4.7985)$.

Going down through this tree, we end up with the price \$2.4287 for the American option.

2a.

4	14,4322	13,8663	13,3226	12,8002	12,2983
3	14,1492	13,5944	13,0614	12,5492	
2	13,8718	13,3279	12,8053		
1	13,5999	13,0666			
0	13,3332				
4	14,4322	13,8663	13,3226	12,8002	12,2983
3	14,1351	13,5808	13,0483	12,5367	
2	13,8441	13,3013	12,7797		
1	13,5591	13,0275			
0	13,2800				
4	0,000000	0,000000	0,377435	0,899820	1,401723
3	0,000000	0,188529	**0,651683**	**1,163315**	
2	0,094170	0,419686	**0,920285**		
1	0,256671	**0,672544**			
0	0,464143				

First we compute the forward price of the share according to Lecture Note 2: $G_0 = S_0\, e^{0.004} = \$13.3332$. Starting out with this value, we construct the first tree of forward prices. Now we know that the forward prices of the underlying share indeed have the Martingale property under the forward probability measure.

Next we construct the tree of spot prices of the share: for example, week 2 the spot price $S_2 = G_2\, e^{-2 \cdot 0.001}$, so we get the spot prices by just discounting the values in the forward price tree. Of course, week 0 we get the initial spot price of the share.

Now we construct the tree of option prices. We start with the prices week 4. Next week three: these prices are computed as the mean of the two prices above and right-above, discounted by the one period interest rate. However, we must check that it isn't profitable to exercise the option. This happens in a few places; these are shown in boldface. Finally, the option price is found to be \$0.464143.

2b.

4	14,4327	13,8668	13,3231	12,8007	12,2988
3	14,1498	13,5950	13,0619	12,5497	
2	13,8724	13,3284	12,8058		
1	13,6004	13,0671			
0	13,3338				
4	14,4327	13,8668	13,3231	12,8007	12,2988
3	14,1356	13,5814	13,0488	12,5372	
2	13,8447	13,3018	12,7802		
1	13,8294	13,2977			
0	13,5500				
4	0,000000	0,000000	0,376894	0,899300	1,401223
3	0,000000	0,188259	**0,651153**	**1,162805**	
2	0,094035	0,419286	**0,919766**		
1	0,256404	0,668857			
0	0,462168				

First we compute the forward price of the underlying share according to example 1, Lecture Note 2. Starting from this value, we construct the first tree which contains the forward prices.

Next we compute the tree of spot prices, again using example 1 of Lecture Note 2. For the time periods 2 and 3 this means just discounting the forward price (example 1 with d=0.)

Finally, we compute the tree of option prices in the same way as in part a. Early exercise is shown in boldface. The present price of the option is \$0.462168.

3.

4	15,3130	13,8557	12,5372	11,3441	10,2646
3	14,5843	13,1965	11,9406	10,8043	
2	13,8904	12,5686	11,3725		
1	13,2295	11,9705			
0	12,6000				
4	15,3130	13,8557	12,5372	11,3441	10,2646
3	14,6987	13,3000	12,0343	10,8891	
2	14,1092	12,7665	11,5516		
1	13,5432	12,2544			
0	13,0000				
4	231,2950	85,5730	0,0000	0,0000	0,0000
3	**169,8736**	42,4668	0,0000	0,0000	
2	**110,9159**	21,0748	0,0000		
1	65,5022	10,4586			
0	37,6966				

First we construct the three of forward exchange rates.

According to example 5, Lecture Note 2, the relation between forward exchange rate G and the spot exchange rate X is $X = G\,e^{\Delta r/12\,\tau}$ where τ is the number of months to maturity and Δr is the foreign interest rate minus the domestic rate. For $\tau = 12$ we have $13 = 12.60\,e^{\Delta r}$ which gives $\Delta r = 0.03125$. We use this value to create the second tree containing the spot exchange rates.

Finally, we compute the last tree, the tree of option prices in the usual way. Here we discount with the SEK interest rate, of course (3% per year.) Early exercise is labeled in boldface. The price of the option is 37.70 SEK.

4.

4	149,0648	122,0440	99,9212	81,8085	66,9792
3	135,5544	110,9826	90,8648	74,3938	
2	123,2685	100,9237	82,6293		
1	112,0961	91,7765			
0	101,9363				
4	149,0648	122,0440	99,9212	81,8085	66,9792
3	133,5363	109,3303	89,5120	73,2863	
2	119,6254	97,9410	80,1873		
1	111,6287	91,3939			
0	100,0000				
4	51,0648	24,0440	1,9212	0,000000	0,000000
3	36,9953	12,7893	0,9463	0,000000	
2	24,5217	6,7655	0,4661		
1	15,4107	3,5620			
0	9,3451				

The forward price of the underlying share is $100\,(1 - 0.04)\,e^{0.06} = 101.9363$ SEK (example 2, Lecture Note 2.) We construct the first tree: forward prices of the underlying asset.

From the forward prices we compute the spot prices by just discounting (example 1 with d=0, Lecture Note 2) for times later than 5.5 months. For times previous to 5.5 months we discount and divide by 0.96 (example 2, Lecture Note 2.) Thus we obtain the second tree of spot prices.

Finally, we construct the tree of option prices in the usual way. Early exercise is never profitable. Note that for non-paying dividend stocks, early exercise is never profitable for a call option, but in this case we couldn't know, since there is a dividend payment. The option price is 9.3451 SEK.

6.

4	597,4494	539,4810	487,1370	439,8718	397,1926
3	568,4652	513,3090	463,5044	418,5322	
2	540,8871	488,4067	441,0183		
1	514,6469	464,7125			
0	489,6797				
4	597,4494	539,4810	487,1370	439,8718	397,1926
3	566,8096	511,8141	462,1545	417,3133	
2	537,7411	485,5660	438,4532		
1	510,1634	460,6640			
0	484,0000				
4	0,0000	0,0000	0,0000	40,1282	82,8074
3	0,0000	0,0000	19,9807	**62,6867**	
2	0,0000	9,9488	**41,5468**		
1	4,9537	25,6407			
0	15,2336				

The forward value of the index is $484\,e^{(0.10-0.03)\cdot 2/12)} \approx 489,6797$ (Lecture Note 2, example 4.) We use this value to generate the tree of forward values of the index.

Next we construct the tree of spot index values, again using example 5 of Lecture Note 2.

Finally we construct the tree of option prices; early exercise is as usual shown in boldface. The present option value is 15.2336

7.

4	1,027759	0,688927	0,461802	0,309555	0,207501
3	0,927011	0,621394	0,416533	0,279210	
2	0,817213	0,547795	0,367198		
1	0,706451	0,473549			
0	0,600000				
4	0,427759	0,088927	0,000000	0,000000	0,000000
3	**0,327011**	0,043802	0,000000	0,000000	
2	**0,217213**	0,021575	0,000000		
1	0,117617	0,010627			
0	0,063167				

The first tree is a tree on spot prices on copper. We know from Lecture Note 7 that the forward price is the expected value w.r.t. the forward probability measure of the spot price at the time of maturity: $G_0^{(t)} = \mathrm{E}^*[S_t]$. In order to achieve this, we construct the tree like this (t=quarters:)

$G_0^4\, u^4$	$G_0^4\, u^3 d$	$G_0^4\, u^2 d^2$	$G_0^4\, u d^3$	$G_0^4\, d^4$
$G_0^3\, u^3$	$G_0^3\, u^2 d^1$	$G_0^3\, u d^2$	$G_0^3\, d^3$	
$G_0^2\, u^2$	$G_0^2\, u d$	$G_0^2\, d^2$		
$G_0^1\, u$	$G_0^1\, d$			
S_0				

You can check that in this tree, $G_0^{(t)} = \mathrm{E}^*[S_t]$ (see notation in Lecture Note 9.)

The tree of option prices is now calculated as usual. Early exercise in boldface.

Comments on Lecture Note 10

The term *"futures measure"* is an invention by me. In the literature it is called the *"Equivalent Martingale Measure"* (EMM) or the *"Q-measure"* and expectation w.r.t. this measure is often written as $\mathrm{E}^Q[\,\cdot\,]$. A big advantage of this measure compared to the forward measure is that it is unique; it doesn't depend on the date of maturity, even if interest rates are random (as long as we use the same time intervals.) A big drawback is that the discounting is done inside the expectations operator: $P_0^{(t)}(X) = \widehat{\mathrm{E}}[Xe^{-R(0,t)}]$. This implies that the stochastic dependence between X and R has to be taken into account.

Comments on Lecture Note 11 and 12

The model presented here is Ho-Lee's interest rate model. It was originally presented as a binomial model—see ch. 30.3 in Hull's book. It is the simplest interest rate model, and it can be calibrated so as to fit the current term structure perfectly. There are two features with this model that many people feel unhappy with: (1) the short rate will eventually far into the future take negative values with a probability that approaches one half, and (2) there is no "mean reversion" which means that all shocks to the short rate are permanent; they don't wear off with time.

A modification of Ho-Lee's model is Hull-White's model, which is a combination of Ho-Lee's model and Vasicek's model. In a time-discrete version it looks like this:

$$r_k = \theta_k + \sigma\sqrt{\Delta t}\,(\beta^{(k-2)\Delta t} z_1 + \cdots + \beta^{\Delta t} z_{k-2} + z_{k-1}),$$

where $0 < \beta \leq 1$ is the mean-reverting factor. As you can see, the shock z_1, for instance, wear off exponentially with time, since $\beta^{(k-2)\Delta t} \to 0$ as $k \to \infty$ if $\beta < 1$. The Ho-Lee model is the special case $\beta = 1$ (compare with the first expression in Lecture Note 11.) In Hull-White's model, the standard deviation of the short rate is always bounded by $\dfrac{\sigma}{\sqrt{-2\ln\beta}}$, so if β is small, the probability of negative interest rates can stay small.

This model can not be represented by a binomial tree (if $\beta \neq 1$,) but it can be implemented as a *trinomial* tree; see Hull ch. 30.6–8. In this course we confine ourselves to the Ho-Lee model, the difference to the Hull-White model is solely on technicalities concerning the construction of the tree of interest rates; the conceptual ideas are the same.

In Lecture Note 11 I show how one can analytically compute some prices using the Ho-Lee model with normally distributed increments. If you are interested, you can read the technical details, but otherwise you can skip those and just appreciate the following two results:

1. The relation between the futures price F_0 and the forward price G_0 for a zero coupon bond in Ho-Lee's model is $F_0 = G_0 e^{-\frac{1}{2}\sigma^2(T-t)t^2}$ where (T-t) is the time to maturity of the bond at the maturity of the futures / forward contract, and t is the time to maturity of the futures / forward contract. In general, we can write with our usual notation:

$$F_0 = G_0 e^{-\frac{1}{2}\sigma^2 D_F t^2}$$

   for any underlying bond (or portfolio of bonds.)

2. The last section, "Pricing a European Option on a Zero Coupon Bond": the Ho-Lee model produces exactly the same price as Black's model we have encountered earlier.

Examples and Exercises on Lecture Notes 10–12.

Interest rates always refer to continuous compounding. Answers are given in parenthesis; Solutions to some problems are given on the following pages.

1. We have the following Ho-Lee binomial tree of the interest rate in % per time-step:

period	0	1	2	3
	3.0	3.3	3.5	3.8
		2.9	3.1	3.4
			2.7	3.0
				2.6

The interest from period 0 to period 1 is hence 3.0%, and so on. An interest rate security is in period 3 worth $400\,r$ where r is the interest (if $r = 3.8\%$ the value is \$1'520, if $r = 2.6$ the value is \$1'040 etc.)

a) Determine today's futures price of the security to two decimal places. (\$1'280.00)

b) Determine today's forward price of the security to two decimal places. (\$1'279.52)

2. The following zero-coupon rates of interest holds: 1-year: 8%, 2-year: 8.25%, 3-year: 8.5%, 4-year: 8.75%. The volatility of the one-year rate is assumed to be 1.5% during one year. Determine the price of a European call option with maturity in two years on a (at maturity of the option) zero coupon bond with face value 10'000 SEK (thus the bond matures in four years time from the present.) The strike price of the option is 8'000 SEK.

a) Use a binomial tree based on Ho-Lee's model with a time interval of one year. (302.05 SEK.)

b) Calculate the value using Black's model. (293.24 SEK.)

3. Determine the price of a "callable bond" by use of a Ho-Lee tree. A callable bond is a bond where the issuer have the option to buy back the bond at certain points in time at predetermined prices (see ch. 28.1 in Hull's book.) The topical bond is a coupon paying bond which pays \$300 each half year and matures with face value \$10'000 (i.e., \$10'300 is payed out including the coupon) after 2.5 years.

The issuer has the option to buy back the bond for \$9'950 plus the coupon dividend after 18 months.

Use the following Ho-Lee tree:

0-6	6-12	12-18	18-24	24-30 months
3.0	3.3	3.5	3.8	4.0
	2.9	3.1	3.4	3.6
		2.7	3.0	3.2
			2.6	2.8
				2.4

The numbers are the interest rate in percent per 6 months, i.e., per time step in the tree. ($9'896.85)

Remark. This problem can also be solved "analytically" with Black's model. Note that the callable bond is equivalent to the corresponding non-callable bond and a short position of a European call option on the bond with maturity 18 months. See Lecture Note 6, "Black's Model for Bond Options".

4. Calculate the value of a callable zero-coupon bond with maturity in 10 years and strike price price 100 SEK. The bond can be exercised after 3 years at 70 SEK, after 6 years at 80 SEK and after 8 years at 90 SEK. The volatility of the one-year rate of interest is assumed to be 1.5% during one year. The present zero coupon rates of interest are (% per year with continuous compounding):

maturity	interest	maturity	interest
1 year	4.0	6 year	5.0
2 year	4.2	7 year	5.2
3 year	4.4	8 year	5.4
4 year	4.6	9 year	5.6
5 year	4.8	10 year	5.8

Use a binomial tree with a time interval of one year. (53.1950)

5. Calculate the futures price of a ten-year zero-coupon bond with face value 100 and maturity in six years. The volatility of the one year rate of interest is assumed to be 1.5% during one year, and the zero-coupon rate of interest is the same as in the previous question.

   a) Use a binomial tree with time interval of one year based on Ho-Lee's model. (74.5689)

   b) Do the calculation analytically using the continuous time Ho-Lee model (see the last formula in section "Forward and Futures on a Zero Coupon Bond" in Lecture Note 11.) (74.3639)

Solutions to Exercises on Lecture Notes 10–12

1. Here is first the tree of interest rates:

3,8	3,4	3	2,6
3,5	3,1	2,7	
3,3	2,9		
3			

The futures prices of the security is calculated simply as the average 0.5(number above + number above-right), since the futures prices are a Martingale under the futures measure; see Theorem 1 Lecture Note 7:

1 520	1 360	1 200	1 040
1 440	1 280	1 120	
1 360	1 200		
1 280			

The futures price is thus \$1'280.

In order to calculate the forward price G_0, we first calculate the present price P_0. This is done with discounting; the values in a row is 0.5(number above + number above-right) × discount;

1 520,00	1 360,00	1 200,00	1040
1 390,47	1 240,93	1 090,16	
1 272,99	1 132,23		
1 167,07			

The present price is thus \$1167.07. In order to calculate the forward price $G_0 = Z_{30}^{-1}$ we need the zero coupon price Z_{30}. But $Z_{30} = P_0^{30}(1)$:

1	1	1	1
0,965605	0,969476	0,97336	
0,936133	0,943652		
0,912114			

We have thus G_0 = \$1'167.07/0.912114 = \$1'279.52. The difference is small but in accordance with theory; the security is positively correlated with the interest rate, so the futures price is higher than the forward price.

2a. The interest rate tree:

3	0,141012	0,111012	0,081012	0,051012
2	0,120450	0,090450	0,060450	
1	0,100112	0,070112		
0	0,080000			

The value of the bond is calculated backwards in time: $\big(0.5$(number above + number above-right) × discount where we get the discounting rate from the above tree:

4	10 000,00	10 000,00	10 000,00	10 000,00	10 000,00
3	8 684,79	8 949,28	9 221,82	9 502,67	
2	7 816,49	8 299,83	8 813,07		

The value of the option is calculated in the same way:

2	0,00	299,83	813,07
1	135,64	518,77	
0	302,05		

The option price is thus 302.05 SEK.

b. This is an ordinary call option where the forward price of the underlying asset is 8'311.04 SEK and the volatility is $0.015 \cdot 2 = 0.03$. The factor 2 is the forward duration (time to maturity of the underlying bond when the option is exercised.)

3. Here is the interest rate tree, now with time going from bottom to top:

30					
24	4,00	3,60	3,20	2,80	2,40
18	3,80	3,40	3,00	2,60	
12	3,50	3,10	2,70		
6	3,30	2,90			
0	3,00				

The values of the bond can be arranged in various ways in a tree. Here I have written the values with the dividend excluded in each period.

30	10 000,00	10 000,00	10 000,00	10 000,00	10 000,00	10 000,00
24	9 896,13	9 935,80	9 975,62	10 015,60	10 055,74	
18	9 835,04	9 912,87	**9 950,00**	**9 950,00**		
12	9 824,03	9 919,13	9 976,95			
6	9 841,39	9 955,12				
0	9 896,85					

After 30 months, the value is \$10'000 plus dividend. In each subsequent row, the value, net of dividend, is $\big(0.5(\text{number above} + \text{number above-right}) + 300\big) \times \text{discount}$, except i two places after 18 months (boldface) where the bond is retracted.