Lecture Notes to Finansiella Derivat (5B1575) VT 2002

Harald Lang, KTH Matematik

Note 1: No Arbitrage Pricing

Let us consider a two period market model. A contract is defined by a stochastic payoff X – a bounded stochastic variable – at a future time T ("maturity") and a price of that contract p which is paid today.

We assume that one contract is a "bank account", a contract which gives a secure payoff of 1 at maturity, and whose price today $p = e^{-rT}$, where typically $r > 0$. Of course, r is interpreted as the interest rate from now to maturity.

The market consists of a set of contracts, and we assume that we can compose arbitrary portfolios of contracts, i.e., if we have contracts $1, \ldots, n$ with payoffs $X_1, \ldots, X_n$ and prices $p_1, \ldots, p_n$ then we can compose a portfolio consisting of λ_i units of contract i, $i = 1, \ldots, n$ where $\lambda_1, \ldots, \lambda_n$ are arbitrary real numbers. I.e., we assume that we can take a short position in any contract, and ignore any divisibility problems. The total cost of such a portfolio is of course $\sum_1^n \lambda_i p_i$ and the total payoff is $\sum_1^n \lambda_i X_i$. We call also such portfolios "contracts", so the set of contracts, defined as pairs (p, X), constitute a *linear space*.

We assume that the market is *arbitrage free* in the following sense:

1. There is no contract (p, X) such that $X \geq 0$ almost surely (a.s.) and $p < 0$.

2. There is no contract (p, X) such that $X \geq 0$ a.s., $\mathrm{E}[X] > 0$ and $p = 0$.

It is easy to see that condition 1 in particular implies *the law of one price:* if $X_1 = X_2$ a.s., then $p_1 = p_2$. We denote by $p(X)$ the unique price of a contract whose payoff is X.

Now we define an operator $\mathrm{E}^*[X]$ on all payoffs of contracts by $\mathrm{E}^*[X] \stackrel{\text{def}}{=} e^{rT} p(X)$. Now we establish the following properties of this operator:

Properties of operator E^*.

1. E^* is linear; i.e., $\mathrm{E}^*[\alpha X + \beta Y] = \alpha \mathrm{E}^*[X] + \beta \mathrm{E}^*[Y]$, where α and β are real numbers and X and Y are payoffs of contracts.

2. E^* is positive; i.e., if $X \geq 0$ a.s., then $\mathrm{E}^*[X] \geq 0$.

3. $\mathrm{E}^*[1] = 1$.

4. For any payoff X such that $X \geq 0$ a.s. $\mathrm{E}^*[X] = 0 \Leftrightarrow \mathrm{E}[X] = 0$.

Proof

1. The law of one price implies that $p(\alpha X + \beta Y) = \alpha p(X) + \beta p(Y)$. Now multiply both sides by e^{rT} to get the required relation.

2. Assume that $X \geq 0$ a.s. Then, by arbitrage condition 1, $p(X) \geq 0$, and hence $\mathrm{E}^*[X] = e^{rT} p(X) \geq 0$.

3. We have by definition $p(1) = e^{-rT}$, hence $\mathrm{E}^*[1] = e^{rT} p(1) = 1$.

4. Assume that $X \geq 0$ a.s. If $\mathrm{E}^*[X] = 0$ then $p(X) = 0$ and hence $\mathrm{E}[X]$ must be $= 0$, else $\mathrm{E}[X]$ would be > 0 which would be an arbitrage type 2. If, on the other hand, $\mathrm{E}[X] = 0$, then $X = 0$ almost surely, and since we don't have arbitrage type 1, $p(X) \geq 0$. But since also $-X = 0$ almost surely, $p(-X) \geq 0$, i.e., $p(X) \leq 0$, so in fact $p(X) = 0$. But then it follows that $\mathrm{E}^*[X] = e^{rT} p(X) = 0$, and the proof is complete.

Recall that the operator E^* is defined only on stochastic variables that are payoffs of contracts. But assume for a moment that *any* bounded stochastic variable X is the payoff of a contract on the arbitrage-free market under study. Then we could *define* a probability measure P^* by $\mathrm{P}^*(A) \stackrel{\text{def}}{=} \mathrm{E}^*[1_A]$ for any (measurable) set A, where 1_A denotes the indicator function of A. It is easy to verify, using the properties of E^*, that the axioms for a probability measure is satisfied. The value $\mathrm{E}^*[X]$ will then be the *expected value of X with respect to the measure* P^*. Property 4 of the operator E^* implies that $\mathrm{P}^*(A) = 0 \Leftrightarrow \mathrm{P}(A) = 0$, where P denotes the original probability measure, i.e., the two probability measures P^* and P are *equivalent*. This means, by the Radon-Nikodym theorem, that there is a stochastic variable $Z > 0$ such that $\mathrm{E}[Z] = 1$ and $\mathrm{E}^*[X] = \mathrm{E}[ZX]$ for any bounded stochastic variable X. For this reason, we will here call the operator $\mathrm{E}^*[X]$ the *equivalent risk-neutral expected value of X*, even if not all bounded stochastic variables are payoffs of contracts. We will come back to the issue of the existence of an equivalent probability measure later.

We have thus proved that if the market is arbitrage free, then there exists an equivalent expectation E^* such that $p = e^{-rT}\mathrm{E}^*[X]$ for any contract (p, X). Now we prove the opposite, which is quite easy: If there exists an an equivalent expectation E^*, i.e., an operator E^* satisfying $p = e^{-rT}\mathrm{E}^*[X]$ and properties 1 – 4, then the market is arbitrage free. Indeed, if $X \geq 0$ almost surely, then by property 2, $\mathrm{E}^*[X] \geq 0$, hence $p = e^{-rT}\mathrm{E}^*[X] \geq 0$, so we can not have an arbitrage type 1. Nor can we have an arbitrage type 2, since if $X \geq 0$ a.s. and $\mathrm{E}[X] > 0$, then, by property 2, $\mathrm{E}^*[X] \geq 0$, and by property 4, $\mathrm{E}^*[X] \neq 0$ and thus $p = e^{-rT}\mathrm{E}^*[X] > 0$.

In summary:

Theorem 1.

A two period market, with a bank account giving the fixed interest rate r, is arbitrage free if and only if there is an operator E^*, which we call an equivalent risk-neutral expectations operator, *which satisfies properties 1 – 4 and $p = e^{-rT}\mathrm{E}^*[X]$.*

An Application to Futures and Forwards

Let us divide up the time spell from now to maturity T in small time spells now $= t_0, t_1, \ldots, t_n = T$. A contract is any portfolio composed now, whose payoff is realized at maturity T, however, we allow re-allocating the portfolio at each time t_k. A re-allocation here means that we sell some assets and buy some assets such that the net cost is zero. No net cash flow thus takes place at other times than now and maturity.

Consider first a forward contract on some underlying asset whose spot price at maturity is S_T. A forward contract is a contract whose price (value) today is zero, and whose payoff at maturity is $S_T - G_0$ where G_0 is the forward price. Hence, we have the equality $0 = e^{-rT}\mathrm{E}^*[S_T - G_0]$, hence $0 = \mathrm{E}^*[S_T - G_0] = \mathrm{E}^*[S_T] - G_0$, since G_0 is a constant, so $\mathrm{E}^*[G_0] = G_0\,\mathrm{E}^*[1] = G_0$. The forward price is thus the same as the risk-neutral expected spot price: $G_0 = \mathrm{E}^*[S_T]$.

Next consider a futures contract on the same underlying asset. In this case the marking to market gives a cash-flow $F_k - F_{k-1}$ in each period t_k, where F_k is the futures price noted at t_k. Consider the following contract: at time t_{k-1} we enter a long position on a futures contract, and at time t_k we collect the cash flow $F_k - F_{k-1}$ (which may be negative), and close out the contract (for instance by taking an opposite short position.) The dividend $F_k - F_{k-1}$ is deposited (or borrowed) in a bank account until maturity. The final payoff at maturity is thus $(F_k - F_{k-1})e^{r(T-t_k)}$. Since the cost of this contract is zero, we have $0 = e^{-rT}\mathrm{E}^*[(F_k - F_{k-1})e^{r(T-t_k)}]$, i.e., $0 = \mathrm{E}^*[F_k - F_{k-1}]$, i.e., $\mathrm{E}^*[F_k] = \mathrm{E}^*[F_{k-1}]$. Thus we have

$$F_0 = \mathrm{E}^*[F_0] = \mathrm{E}^*[F_1] = \ldots = \mathrm{E}^*[F_T] = \mathrm{E}^*[S_T]$$

so the futures price F_0 is also equal to the risk-neutral expected spot price: $G_0 = \mathrm{E}^*[S_T]$. So, we have proved:

Theorem 2.

If the interest rate is deterministically constant, the futures price and the forward price coincide.

Note 2: The No Arbitrage Theorem

In lecture note 1 we proved the existence of an "equivalent expectations operator" E^* which is defined on the stochastic payoffs of contracts. Under "reasonable conditions" it is true that there is a probability measure P^*, defined on the given sample space and sigma algebra of measurable sets, such that P^* is equivalent to the original probability measure P and E^* is the expectations operator with respect to P^*. We will now prove that this is true under the much simplifying condition that the sigma-algebra of measurable sets is finitely generated, i.e., there is a finite number of subsets $A_1, \ldots, A_m$ such that the sigma algebra consists of arbitrary unions of such sets. This assumption is not "unrealistic", but for mathematical reasons too restrictive.

For the proof we need a

Lemma:

Assume that V is a subspace of $\mathbf{R}^n$ with the "alternating sign property:" *every non zero vector of V has both strictly negative and strictly positive entries. Then there is a vector $\bar{\lambda}$ with all entries strictly positive which is orthogonal to V with respect to the natural inner product on $\mathbf{R}^n$.*

Proof

Let K be the subset of $\mathbf{R}^n$ $K = \{\bar{u} \in \mathbf{R}^n \mid u_1 + \ldots + u_n = 1 \text{ and } u_i \geq 0 \text{ for all } i\}$. Obviously K and V have no vector in common. Let $\bar{\lambda}$ be the vector of shortest Euclidean length such that $\bar{\lambda} = \bar{k} - \bar{v}$ for some vectors $\bar{k}$ and $\bar{v}$ in K and V respectively. The fact that such a vector exists needs a proof, but we leve that out. We write $\bar{\lambda} = \bar{k}_0 - \bar{v}_0$ where $\bar{k}_0 \in K$ and $\bar{v}_0 \in V$.

Now note that for any $t \in [0,1]$ and any $\bar{k} \in K$, $\bar{v} \in V$, the vector $t\bar{k} + (1-t)\bar{k}_0 \in K$ and $t\bar{v} + (1-t)\bar{v}_0 \in V$, hence $|(t\bar{k} + (1-t)\bar{k}_0) - (t\bar{v} + (1-t)\bar{v}_0)|$ as a function of t on $[0,1]$ has a minimum at $t = 0$, by definition of $\bar{k}_0$ and $\bar{v}_0$, i.e., $|t(\bar{k}-\bar{v}) + (1-t)\bar{\lambda}|^2 = t^2|\bar{k}-\bar{v}|^2 + 2t(1-t)(\bar{k}-\bar{v})\cdot\bar{\lambda} + (1-t)^2|\bar{\lambda}|^2$ has minimum at $t = 0$ which implies that the derivative w.r.t. t at $t = 0$ is ≥ 0. This gives $(\bar{k}-\bar{v})\cdot\bar{\lambda} - |\bar{\lambda}|^2 \geq 0$ or, equivalently $\bar{k}\cdot\bar{\lambda} - |\bar{\lambda}|^2 \geq \bar{v}\cdot\bar{\lambda}$ for all $\bar{v} \in V$ and $\bar{k} \in K$. But since V is a linear space, it follows that we must have $\bar{v}\cdot\bar{\lambda} = 0$ for all $\bar{v} \in V$. It remains to prove that $\bar{\lambda}$ has strictly positive entries. But we have $\bar{k}\cdot\bar{\lambda} - |\bar{\lambda}|^2 \geq 0$ for all $\bar{k} \in K$, in particular we can take $k = (1, 0, \ldots, 0)$ which shows that the first entry of $\bar{\lambda}$ is > 0 and so on. This completes the proof of the lemma.

Proof of the main statement

A payoff X has constant values on each of the sets A_i, let us denote it $X(A_i)$. This means that $X(\omega) = \sum_1^m X(A_i) 1_{A_i}(\omega)$ where 1_A denotes the indicator function of the set A. Some of the sets A_i may have zero probability; for notational convenience let $\mathrm{P}(A_i) > 0$ for $i = 1, \ldots, n$ and $\mathrm{P}(A_i) = 0$ for $i = n+1, \ldots, m$.

We associate each contract with the $n+1$-vector $\big(-p, X(A_1), \ldots, X(A_n)\big)$, i.e., we ignore X:s values on the zero sets. The set of such vectors constitute a linear subspace of $\mathbf{R}^{n+1}$ and has the alternating sign property. Indeed; assume that at least one $X(A_j)$ is positive and none is negative; then $X \geq 0$ a.s. and $\mathrm{E}[X] > 0$, hence $p > 0$ by arbitrage condition 2, so the first entry of the vector is negative, and we have at least one negative and at least one positive entry.

If at least one $X(A_j)$ is negative and none is positive, we can look at the negative of the contract $\big(p, -X(A_1), \ldots, -X(A_n)\big)$ and the situation is brought back to the previous case.

Finally, if all $X(A_j)$:s are $= 0$, then $p = 0$ by arbitrage condition 1.

Hence, by the lemma, there are positive numbers $\lambda_0, \ldots, \lambda_n$ such that

$$-\lambda_0 p + \lambda_1 X(A_1) + \ldots + \lambda_n X(A_n) = 0$$

for all contracts. We can now write, for any contract

$$p = \sum_1^n \frac{\lambda_i}{\lambda_0} X(A_i) = d \sum_1^n q_i X(A_i) \tag{1}$$

where $d = \sum_1^n \lambda_i / \lambda_0$ and $q_i = \lambda_i / (d\lambda_0)$ for $i = 1, \ldots, n$. Note that $\sum_1^n q_i = 1$, so if we define $q_j = 0$ for $j = n+1, \ldots, m$, we can thus interpret q_i as new "artificial" probabilities $q_i = \mathrm{P}^*(A_i)$ of the events $A_i, i = 1, \ldots, m$. Now (1) can be written

$$p = d\,\mathrm{E}^*[X]$$

where E^* denotes expectation with respect to the P^*-measure. Note also that $\mathrm{P}^*(A_i) = 0 \Leftrightarrow \mathrm{P}(A_i) = 0$, so the two measures are equivalent. The Radon-Nikodym derivative of P^* with respect to P, $\dfrac{d\mathrm{P}^*}{d\mathrm{P}} = \dfrac{q_i}{\mathrm{P}(A_i)}$ on the sets $A_i, i = 1 \ldots, n$ and may be defined as any number on A_j for $j = n+1, \ldots, m$ which we may choose positive so as to get a strictly positive Radon-Nikodym derivative.

If there is a bank account with a fixed interest rate r, the discount factor d is identified as $d = e^{-rT}$. *Q.E.D.*

Note 3: Change of numéraire and martingale pricing

Now assume that there are several points in time $t_0, \dots, t_n = T$ when trading can take place. We consider first contracts (recall that portfolios are also contracts) which pay a stochastic payoff X at time T, and look at the market price p_k at t_k for $k = 1...T$. These prices are stochastic variables which are realized at t_k. [In a more fancy language: we have a filtration of sigma algebras $\dots \mathcal{F}_t \subset \mathcal{F}_{t+1} \dots$ representing the information available at time $\dots t, t+1 \dots$, and the price process $\{p_t\}$ is adapted to this filtration.]

We assume that the market is arbitrage free, and in view of earlier notes, that there is an equivalent probability measure P^* with expectations operator E^* such that $p_0 = d\,\mathrm{E}^*[X]$ for all contracts. Here d is the discount factor, which is equal to $e^{-\rho T}$ if there is a zero coupon bond with maturing at T and ρ is the corresponding zero rate.

We now assume that there is a *money market account,* a bank account (or bond) which pays a (continuous) interest rate — the short interest rate — r_t between two consecutive time periods $t-1$ and t. The short interest rate r_t is a stochastic variable which is realized in period $t-1$, i.e., when I invest an amount at time $t-1$ I know the interest I will receive in the next period, but the short rate at later time periods are stochastic. In other words, the short rate process r_t is *predictable.*

A *trading strategy* is a plan telling us what to invest at each period t of time, depending on the state of the world as of that time. For instance, a trading strategy might say that "if event A occurs at time t, then go long $£x$ in paper X and go short $£y$ in paper Y." A *self financing* trading strategy is one where after the initial investment, all further trading is self financing until the end date, i.e., in each of the periods $1, 2, \dots, T-1$, the net investment is zero. Thus, what we have called "contracts" are in fact portfolios following a self financing trading strategy.

Now we introduce a new measure by a *change of numéraire:* First, note that $\mathrm{E}^*[d\,\exp(\Sigma_1^T r_i)] = 1$, for an investment of 1 in the bank account will eventually end up with a wealth of $\exp(\Sigma_1^T r_i)$. Now define a new equivalent measure $\widetilde{\mathrm{P}}$ by $\mathrm{d}\widetilde{\mathrm{P}} = d \exp(\Sigma_1^T r_i)\,\mathrm{dP}^*$. This means that for any bounded stochastic variable Y, $\widetilde{\mathrm{E}}[Y] = d\,\mathrm{E}^*[\exp(\Sigma_1^T r_i) Y]$, hence, if (p, X) is any contract, we can let $Y = \exp(-\Sigma_1^T r_i)X$ to get $\widetilde{\mathrm{E}}[\exp(-\Sigma_1^T r_i)X] = d\,\mathrm{E}^*[X] = p$:

$$p = \widetilde{\mathrm{E}}[\exp(-\Sigma_1^T r_i)X]$$

for any contract. What we have done is called a *change of numéraire:* the measure P^* has the zero coupon bond as numéraire, whereas $\widetilde{\mathrm{P}}$ has the money market account (short rate) as a numéraire.

Now consider a contract whose price in each period t is a stochastic variable p_t, $t = 0, 1, \dots, T$ which is realized in period t. Consider the following trading strategy: If the event A has occurred in period t, then buy $e^{r_1 + \dots + r_t}$ units of the corresponding portfolio at the price p_t per unit, and finance the purchase by borrowing at the short rate r_{t+1} and roll the debt forward by renewing the debt in each period at the short interest rate. Then sell the securities at time s and invest the amount at the short rate and roll forward until time T. This means that at time T the net value is

$$-p_t e^{r_1+\ldots+r_T} + p_s e^{r_1+\ldots+r_t+r_{s+1}+\ldots+r_T}$$

if the event A did occur at time t, and 0 otherwise. We may write this as

$$(-p_t e^{r_1+\ldots+r_T} + p_s e^{r_1+\ldots+r_t+r_{s+1}+\ldots+r_T})1_A$$

where 1_A is the indicator function of the event A. Since there is no initial investment and all payoffs take place at time T, we have

$$\widetilde{\mathrm{E}}\left[(-p_t + p_s e^{-r_{t+1}-\cdots-r_s})\, 1_A\right] = 0.$$

Let $\widetilde{\mathrm{E}}_t$ denote expectation conditional on all information realized at t (i.e., conditional on the sigma algebra $\mathcal{F}_t$). Employing the "iterated expectations formula" and using the fact that A is realized at t, we get

$$0 = \widetilde{\mathrm{E}}\left[\widetilde{\mathrm{E}}_t[(-p_t + p_s e^{-r_{t+1}-\cdots-r_s})\, 1_A]\right] = \widetilde{\mathrm{E}}\left[1_A\, \widetilde{\mathrm{E}}_t[-p_t + p_s e^{-r_{t+1}-\cdots-r_s}]\right]$$

and since this is true for any A realized at t, we conclude that

$$\widetilde{\mathrm{E}}_t[-p_t + p_s e^{r_{t+1}-\cdots-r_s}] = 0$$

Since in particular p_t is realized at time t, we can re-write this as

$$p_t = \widetilde{\mathrm{E}}_t[p_s e^{-r_{t+1}-\cdots-r_s}]$$

in particular,

$$p_t = \widetilde{\mathrm{E}}_t[p_{t+1} e^{-r_{t+1}}]$$

This says that *if the market is arbitrage free, then there is an equivalent probability measure* $\widetilde{\mathrm{P}}$ *such that the discounted price process* $\{p_t \exp(-\Sigma_1^t r_i)\}$ *of any contract (i.e., any portfolio under self financing re-allocation) is a martingale.*

For this reason, we call $\widetilde{\mathrm{E}}$ the *Equivalent Martingale Measure*. It is trivial to see that the opposite is true: If there is a probability measure under which the discounted price process of any contract is a martingale, then there are no arbitrage possibilities.

Forward and futures prices.

As an example, we will describe the futures and forward prices of a security in terms of the Martingale measure. In order to ease notation somewhat we introduce $R(s,t) \stackrel{\text{def}}{=} e^{r_{s+1}+\cdots+r_t}$. First consider a zero coupon bond with par value 1 maturing at time T. The price $P(0,T)$ of this bond at time 0 is $P(0,T) = \widetilde{\mathrm{E}}\,[R(0,T)^{-1} \cdot 1]$.

Now consider a forward contract written today ($t = 0$), and let G_0 be the forward price and S_T the stochastic price of the underlying asset at time T. Since the price of the contract today is zero, we must have $0 = \widetilde{\mathrm{E}}\,[R(0,T)^{-1}(S_T - G_0)]$, i.e.,

$$G_0 = P(0,T)^{-1}\widetilde{\mathrm{E}}\,[R(0,T)^{-1}S_T]$$

Now consider a futures contract on the same underlying asset. In this case, the amount $F_t - F_{t-1}$ is paid to the long holder of the contract at time $t = 1, \ldots, T$. Here F_t is the futures price at time t. If we go long one such contract at time t and close it out (go short one contract) at the end of time period $t+1$, then the amount invested in period t is zero, and the payoff in period $t+1$ is $F_{t+1} - F_t$. Hence $0 = \widetilde{\mathrm{E}}_t[e^{-r_{t+1}}(F_{t+1} - F_t)]$. But r_{t+1} is known at time t (recall that r_t is predictable) so it can be regarded as a constant; hence, $\widetilde{\mathrm{E}}_t[F_{t+1} - F_t] = 0$, or, since also F_t is is realized in period t, $F_t = \widetilde{\mathrm{E}}_t[F_{t+1}]$. This means that the sequence $\{F_t\}_0^T$ is a martingale under the $\widetilde{\mathrm{P}}$-measure. Hence $F_0 = \widetilde{\mathrm{E}}\,[F_T]$, i.e., since by definition $F_T = S_T$,

$$F_0 = \widetilde{\mathrm{E}}\,[S_T]$$

Note that if the interest rate R(0,T) is deterministic, then $G_0 = F_0$, for then $P(0,T) = R(0,T)^{-1}$, so

$$\begin{aligned} G_0 &= P(0,T)^{-1}\widetilde{\mathrm{E}}\,[R(0,T)^{-1}S_T] \\ &= P(0,T)^{-1}P(0,T)\widetilde{\mathrm{E}}\,[S_T] \\ &= \widetilde{\mathrm{E}}\,[S_T] = F_0. \end{aligned}$$

Otherwise, we can do the following computation:

$$\begin{aligned} G_0 &= P(0,T)^{-1}\widetilde{\mathrm{E}}\,[R(0,T)^{-1}S_T] \\ &= P(0,T)^{-1}\big(\widetilde{\mathrm{E}}\,[R(0,T)^{-1}]\widetilde{\mathrm{E}}\,[S_T] + \widetilde{\mathrm{cov}}[R(0,T)^{-1}, S_T]\big) \\ &= F_0 + P(0,T)^{-1}\widetilde{\mathrm{cov}}[R(0,T)^{-1}, S_T] \end{aligned}$$

So $G_0 = F_0 +$ correction term, where the correction term is positive or negative depending on the sign of the correlation between the interest rate and the spot price of the underlying asset.

Note 4: Black's pricing model

Consider a derivative of a variable whose value is V. Let T be the maturity date of the derivative, G the *forward price* for a contract maturing at T, G_0 the value of G at time zero, e^{-rT} the discount factor from time zero to T (equivalently, e^{-rT} is the price of a zero coupon bond with face value 1 maturing at T). Black's model assumes that the value of V at time T, V_T, is stochastic, with a log-normal probability w.r.t. the risk-adjusted probability measure P^*, i.e., with the zero coupon bond as numeraire: $V_T = A\, e^{\sigma\sqrt{T}\, z}$, where A is some constant and σ is also a constant referred to as "the volatility" of V and z is a stochastic variable which has a normal (0,1) distribution w.r.t. the P^*-measure.

Since a forward contract has a zero price when it is written, we have

$$0 = e^{-rT}\mathrm{E}^*[A\, e^{\sigma\sqrt{T}\, z} - G_0]$$

i.e.,

$$G_0 = \mathrm{E}^*[A\, e^{\sigma\sqrt{T}\, z}] = A\, e^{\sigma^2 T/2}$$

Assume that the value of the derivative at maturity is some function $\Phi(V_T)$ of V_T, e.g., if it is a European call option, then $\Phi(V_T) = \max[V_T - K, 0]$ where K is the strike price. The price of the derivative at time 0 must then be

$$p_0 = e^{-rT}\mathrm{E}^*[\Phi(A\, e^{\sigma\sqrt{T}\, z})]$$

Substituting $G_0 e^{-\sigma^2 T/2}$ for A from the relation above yields

$$\begin{aligned} P_0 &= e^{-rT}\mathrm{E}^*[\Phi(G_0\, e^{-\sigma^2 T/2 + \sigma\sqrt{T}\, z})] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(G_0\, e^{-\sigma^2 T/2 + \sigma\sqrt{T}\, z})\, e^{-z^2/2}\, dz \end{aligned}$$

This is Black's pricing formula. Note that the price is independent of the parameter A — it is already incorporated in the price G_0.

Black-Scholes pricing formula

Let us use Black's pricing formula to price a European option on a stock. We use the same notation and assumptions as in the previous section, and also that the underlying stock pays no dividend between now and matuirity of the option.

Let the price of the stock today be S and the forward price G_0 so that

$$S = e^{-rT}\mathrm{E}^*[S_T] \quad \text{and} \quad 0 = e^{-rT}\mathrm{E}^*[S_T - G_0]$$

from which we infer that $G_0 = Se^{rT}$

The value of the option at maturity is some function $\Phi(S_T)$ of the stock price, so the current price p of the option is

$$p = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(Se^{(r-\sigma^2/2)T + \sigma\sqrt{T}\, z})\, e^{-z^2/2}\, dz$$

which is the famous Black-Scholes pricing formula for European options. Note that the price of the option is independent of the expected growth rate (under the true probability measure) of the stock value. However, in order to compute the price, we need to know the volatility σ of the stock price under the *risk adjusted* measure P^*. If trading is possible in continuous time until maturity, and if $\log(S)$ follows an Itô process, it turns out that this volatility is the same as that under the true probability measure P; this is essentially a consequence of Girsanov's theorem, as we will see later.

Note 5: Binomial tree as a numerical solution to Black-Scholes' model

Assume that we want to calculate the price and hedging portfolio for a European option, using a binomial tree model. The underlying security is assumed to either go up by a factor $e^{x\Delta t+\sigma\sqrt{\Delta t}}$ or down by a factor $e^{x\Delta t-\sigma\sqrt{\Delta t}}$ in each period. The value of x depend on the exact model employed, and σ is the volatility of the underlying asset. In doing so, we come up with a value V_0 of the hedging portfolio in period 0. In the next period, period 1, we re-calculate the value of the hedging portfolio, again using the binomial tree. If the security price has indeed increased by the factor $e^{x\Delta t+\sigma\sqrt{\Delta t}}$ or $e^{x\Delta t-\sigma\sqrt{\Delta t}}$, then the re-balanced portfolio in that period will cost exactly the same as the currently held portfolio, i.e., the trading strategy is self-financing, since we will be using the same binomial tree. A problem arises if this doesn't happen, but the price dynamics has deviated from that of the binomial tree model. We then have to re-calculate the value of the hedging portfolio $V_1(S)$ in a new binomial tree, starting at the then current value S of the underlying security. $V_1(S)$ is thus the cost of the new hedging portfolio we want to hold in period 1.

Let us introduce some notation: The known period 0 price of the security is S_0, the stochastic price in period 1 is S. Now, whether or not S has followed the binomial dynamics, we can calculate the value $V_1(S)$ of the hedging portfolio in period 1 using the binomial tree model based on the then current security price S. Let $v = V_1(S) - V$ be the difference between the cost of the new hedging portfolio in period 1 and the value in period 1 of the portfolio held from the previous period. If the value S of the underlying security follows the binomial dynamics exactly, then $v = 0$, for the portfolio strategy is then self financing by construction. We now assume that the true dynamics of S is that of Black Scholes, i.e., $S = S_0 e^{\nu\Delta t+\sigma\sqrt{\Delta t}\,\theta}$ where θ is a N(0,1)-variable. The value of v will then depend on the movement of S: $v = v(\nu\Delta t + \sigma\sqrt{\Delta t}\,\theta)$, and $v = 0$ if the dynamics exactly matches that of the binomial tree model, i.e., if $\theta = \frac{x-\nu}{\sigma}\sqrt{\Delta t} \pm 1$.

We employ the following result from calculus: *if the function $f(x) = 0$ for two values x_1 and x_2 of x, then $f(x) = \frac{1}{2}f''(c)(x-x_1)(x-x_2)$ for some c in the smallest interval containing x, x_1 and x_2*. We conclude that

$$v(\nu\Delta t + \sigma\sqrt{\Delta t}\,\theta) = \frac{\sigma^2}{2}\,\Delta t\, v''(\nu\Delta t + \sigma\sqrt{\Delta t}\,\xi)(\theta - 1 - \tfrac{x-\nu}{\sigma}\sqrt{\Delta t})(\theta + 1 - \tfrac{x-\nu}{\sigma}\sqrt{\Delta t})$$

where ξ is a stochastic variable whose value is somewhere in the smallest interval containing $\frac{x-\nu}{\sigma}\sqrt{\Delta t} \pm 1$ and θ. We write this as

$$v = \frac{\sigma^2}{2}\,\Delta t\, v''(0)(\theta^2 - 1) + \eta$$

where η is a stochastic variable with small norm: $\|\eta\| = \mathcal{O}(\Delta t^{3/2})$. Note that $\mathrm{E}\,[\theta^2 - 1] = 0$, so we have

$$v = \chi + \eta$$

where $\mathrm{E}[\chi] = 0$ and $\mathrm{Var}[\chi] = \mathcal{O}(\Delta t^2)$. Now we add up all v:s for all periods $1 \ldots n$:

$$\| \sum_{i=1}^{n} v_i \| = \| \sum_{i=1}^{n} \chi_i + \eta_i \| \leq \| \sum_{i=1}^{n} \chi_i \| + \sum_{i=1}^{n} \| \eta_i \|$$

Note that the θ:s for different periods are assumed to be stochastically independent, hence the χ:s are uncorrelated. Thus:

$$\| \sum_{i=1}^{n} v_i \| = \sqrt{\sum_{i=1}^{n} \mathrm{Var}[\chi_i]} + \sum_{i=1}^{n} \| \eta_i \| = \sqrt{n\, \mathcal{O}(\Delta t^2)} + n\, \mathcal{O}(\Delta t^{3/2}) = \mathcal{O}(\sqrt{\Delta t}).$$

This proves the main result of this note:

Binomal tree and Blasc-Scholes dynamics:

> *Under the Black Scholes dynamics assumption, the binomial tree model will yield a trading strategy which is not fully self-financing, but the total cash flow is of the order of magnitude $\mathcal{O}(\sqrt{\Delta t})$. More precisely, the total cash flow is a stochastic variable whose norm is $\mathcal{O}(\sqrt{\Delta t})$. The relation is: the security price in the binomial model either goes up by a factor $e^{x\Delta t + \sigma\sqrt{\Delta t}}$ or down by the factor $e^{x\Delta t + \sigma\sqrt{\Delta t}}$, where σ is the volatility of the security price, and x can be chosen arbitrarily in order to achieve desirable values of the risk adjusted probabilities.*

We note in particular that, just as in Black Scholes' pricing formula, the growth rate ν of the Black Scholes security dynamics doesn't enter anywhere in the binomial computation.

Note 6: Ho-Lee's Interest Rate Model in Discrete Time

Ho–Lee's model is a model of the short interest rate, and is a *no arbitrage* model; the parameters of the model can be chosen such that the current term structure is correctly represented.

Time is discrete: $t_0, t_1, \ldots, t_{n+1} = T$. The interest rate r_k from t_{k-1} to t_k is assumed to be

$$r_k = \theta_k + \sigma \sum_2^k b_j \quad k = 1, \ldots, n$$

where $\{b_k\}$ are stochastic variables, b_k is realized at time $k-1$ (fancy language: $\{b_k\}$ is a predictable stochastic process) and σ and θ_k are real numbers; i.e., they are parameters of the model. The probability distribution of b_k is given for the equivalent matringale measure $\widehat{P}$ with the money market account as a numeraire: $b_k = 1$ with (risk-adjusted) probability 0.5 and $b_k = -1$ with probability 0.5; and they are thus assumed to be statistically independent.

We can now compute the price Z_4 at t_0 of a zero coupon bond maturing at t_4 with face value 1:

$$\begin{aligned} Z_4 &= \widetilde{\mathrm{E}}\,[e^{-r_1-r_2-r_3-r_4} \cdot 1] = \widetilde{\mathrm{E}}\,[e^{-\theta_1-\theta_2-\theta_3-\theta_4-\sigma(3b_2+2b_3+b_4)}] \\ &= e^{-\theta_1-\theta_2-\theta_3-\theta_4}\widetilde{\mathrm{E}}\,[e^{-3\sigma b_2}]\,\widetilde{\mathrm{E}}\,[e^{-2\sigma b_3}]\,\widetilde{\mathrm{E}}\,[e^{-\sigma b_4}] \\ &= e^{-\theta_1-\theta_2-\theta_3-\theta_4}\cosh(3\sigma)\cosh(2\sigma)\cosh(\sigma) \end{aligned}$$

And, by the same token, in general

$$Z_k = e^{-\theta_1-\cdots-\theta_k}\cosh\big((k-1)\sigma\big)\cdot\ldots\cdot\cosh(\sigma)$$

Combining with the same expression for Z_{k-1} we get

$$\frac{Z_k}{Z_{k-1}} = e^{-\theta_k}\cosh\big((k-1)\sigma\big)$$

i.e.,

$$\theta_k = f_k + \ln\big[\cosh\big((k-1)\sigma\big)\big] \tag{1}$$

where $f_k \stackrel{\text{def}}{=} \ln\left[\dfrac{Z_{k-1}}{Z_k}\right]$ is the *forward rate* from t_{k-1} to t_k.

Thus, if the parameters θ_k are chosen according to (1), then the the model reflects the current term structure. The parameter σ is typically chosen to be the (estimated) volatility of the one period rate.

Once we have the parameters of the model, we can price any interest rate derivative in a binomial tree. We show the procedure by an example, where we want to price a European call option maturing at t_2 with strike price 86 on a zero coupon bond maturing at t_4 with face value 100 when the following parameters are given: $\theta_1 = 0.06$, $\theta_2 = 0.061$, $\theta_3 = 0.062$, $\theta_4 = 0.063$, $\sigma = 0.01$. We represent the interest rates in a binomial tree:

t_0	t_1	t_2	t_3
0.06	0.071	0.082	0.093
	0.051	0.062	0.073
		0.042	0.053
			0.033

The interest rate from one period to the next is obtained by going either one step to the right on the same line, or step to the right to the line below; each with a (risk adjusted) probability of 0.5. We can compute the value at t_2 of the bond: since its value at t_4 is 100, the value at t_3 and t_2 is obtained recursively backwards:

t_2	t_3
84.794	91.119
88.254	92.960
91.856	94.838
	96.754

The value of the option can now also be obtained by backward recursion:

t_0	t_1	t_2
2.309	1.050	0
	3.853	2.254
		5.856

The value of the option is thus 2.309. It is easy to price also American or other more exotic derivatives in this binomial tree model.

Note 7: The Black-Scholes-Merton Pricing Formula

I will derive the Black-Scholes-Merton Pricing Formula, using martingales and Itô calculus – however, I'm trying to use as little mumbo-jumbo as possible. This means that I have to invoke Itô's representation formula and some Itô calculus, but I avoid e.g. Girsanov's theorem and changes of measure. The reason for this is that I want to make the idea as transparent as possible; the cost is that the computations are more technically involved than proofs relying on change of measures and Girsanov's theorem that can be found in many text books. Of course, the proofs are the same, it is just the presentation that differ. A good way to understand what is going on is – I hope – to first read this presentation, and then the text-book.

Let W_t be a Wiener process, $0 \leq t \leq T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\left[\mathcal{F}_t\right]_{t=0}^T$ which is generated by the Wiener process W_t. Consider now a bounded stochastic variable X, measurable w.r.t. $\mathcal{F} = \mathcal{F}_T$ which is the payoff of a contract (or a self financing portfolio of contracts) written at time t. We will derive an expression for the price p of that contract in terms of the prices of two other contracts whose prices at time t are B_t and S_t respectively. The prices of these assets are assumed to be defined by the following Itô processes:

$$\begin{cases} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t \\ dB_t &= r_t B_t dt \end{cases}$$

Here μ_t, σ_t and r_t are bounded adapted processes and $\sigma_t > 0$. The asset B_t can be thought of as a bond, and S_t an asset whose value is underlying that of the derivative X.

First we introduce some notation:

$$\lambda_t = \frac{\mu_t - r_t}{\sigma_t} \quad \text{(the \textit{“market price of risk”})}$$

$$u_t = e^{-\int_0^t \lambda_s\, dW_s - \frac{1}{2}\int_0^t \lambda_s^2\, ds}$$

$$v_t = u_t^{-1} = e^{\int_0^t \lambda_s\, dW_s + \frac{1}{2}\int_0^t \lambda_s^2\, ds}$$

$$\widetilde{B}_t = e^{-\int_0^t r_s\, ds} B_t, \quad \widetilde{S}_t = e^{-\int_0^t r_s\, ds} S_t, \quad \widetilde{X} = e^{-\int_0^T r_s\, ds} X$$

Itô's formula now gives

$$\begin{aligned} du_t &= -\lambda_t u_t\, dW_t \\ dv_t &= \lambda_t^2 v_t\, dt + \lambda_t v_t\, dW_t \\ d\widetilde{S}_t &= (\mu_t - r_t)\widetilde{S}_t\, dt + \sigma_t \widetilde{S}_t\, dW_t \\ d\widetilde{B}_t &= 0 \end{aligned}$$

Now we employ Itô's representation formula on the stochastic variable $u_T\, \widetilde{X}$:

$$u_T\, \widetilde{X} = A + \int_0^T h_t\, dW_t$$

for some adapted process h_t and constant A. Define

$$Y_t = A + \int_0^t h_s \, dW_s$$

We are now in a position to create a self financing portfolio of the assets B_t and S_t which replicates X. The portfolio consists of φ_t of the asset B_t and ψ_t of the asset S_t, the value of the portfolio at any time t is thus

$$V_t = \varphi_t \, B_t + \psi_t \, S_t$$

i.e.,

$$\widetilde{V}_t = e^{-\int_0^t r_s \, ds} \, V_t = \varphi_t \, \widetilde{B}_t + \psi_t \, \widetilde{S}_t$$

We will chose φ_t and ψ_t such that two conditions are satisfied: first, $\widetilde{V}_t = v_t Y_t$, and second: the portfolio is self financing. The first condition then implies that the portfolio replicates X, indeed, $\widetilde{V}_T = v_T Y_T = \widetilde{X}$, so $V_T = X$. Now, by Itô's formula

$$\begin{aligned} d(v_t Y_t) &= (dv_t) Y_t + v_t \, dY_t + d\langle v_y, Y_t \rangle \\ &= \lambda_t v_t (\lambda_t Y_t + h_t) \, dt + v_t (\lambda_t Y_t + h_t) \, dW_t \end{aligned}$$

On the other hand,

$$\varphi_t \, d\widetilde{B}_t + \psi_t \, d\widetilde{S}_t = \psi_t (\mu_t - r_t) \widetilde{S}_t \, dt + \psi_t \sigma_t \widetilde{S}_t \, dW_t$$

So if we choose ψ_t such that $\psi_t \sigma_t \widetilde{S}_t = v_t (\lambda_t Y_t + h_t)$ then

$$d(v_t Y_t) = \varphi_t \, d\widetilde{B}_t + \psi_t \, d\widetilde{S}_t$$

and if we now choose φ_t such that $\varphi_t \widetilde{B}_t + \psi_t \widetilde{S}_t = v_t Y_t$, then

$$d\widetilde{V}_t = \varphi_t \, d\widetilde{B}_t + \psi_t \, d\widetilde{S}_t$$

which implies that the portfolio is self financing, and $\widetilde{V}_t = v_t Y_t$, so it replicates X.

The value of the derivative must hence – if there is no arbitrage – be that of the replicating portfolio. If p_t is the value at time t and $\tilde{p}_t = p_t e^{-\int_0^t r_s d\, s}$ the discounted value, then $\tilde{p}_t = v_t Y_t = v_t \mathrm{E}_t \left[Y_T \right] = v_t \mathrm{E}_t \left[u_T \widetilde{X} \right]$:

Theorem:

The discounted price $\tilde{p}_t$ of the derivative is given by

$$\tilde{p}_t = v_t \mathrm{E}_t \left[u_T \widetilde{X} \right]$$

It is common practice to introduce an equivalent probability measure whose Radon-Nikodym derivative w.r.t. the true measure is u_T. Note that $u_T > 0$ and $\mathrm{E}\left[u_T \right] = 1$, so it is a permissible Radon-Nikodym derivative. If we denote expectation w.r.t. this measure by $\widehat{\mathrm{E}}$ we can write the above

Corollary:

The discounted price $\tilde{p}_t$ of the derivative is given by

$$\tilde{p}_t = \widehat{\mathrm{E}}_t[\widetilde{X}]$$

Proof:

Note first that u_t is a martingale, since $du = -\lambda u\, dW$. Now, for any event A realized at t

$$\begin{aligned} &\mathrm{E}\{1_A \, \mathrm{E}_t[\widetilde{X} u_T]\} = \mathrm{E}\{\mathrm{E}_t[1_A \widetilde{X} u_T]\} = \mathrm{E}[1_A \widetilde{X} u_T] \\ \text{and} \quad &\mathrm{E}\{1_A u_t \, \widehat{\mathrm{E}}_t(\widetilde{X})\} = \mathrm{E}\{1_A \, \mathrm{E}_t(u_T) \, \widehat{\mathrm{E}}_t(\widetilde{X})\} = \mathrm{E}\{\mathrm{E}_t[1_A \widetilde{X} u_T]\} \\ &\qquad = \mathrm{E}[1_A \widetilde{X} u_T] \end{aligned}$$

Since A is any event realized at t this shows that $\mathrm{E}_t[\widetilde{X} u_T] = u_t \, \widehat{\mathrm{E}}_t(\widetilde{X})$, which proves the corollary.

I.e., the discounted prices $\{\tilde{p}_t\}_{t=1}^T$ is a *martingale* under the new probability measure; the *equivalent martingale measure.*

The Black-Scholes Pricing Formula for a European Option

Assume that μ, σ and r are constants. The price p of a European option on the underlying asset S_T, giving the return $F(S_T)$ at time T is then

$$p = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\Big(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x}\Big) e^{-x^2/2}\, dx$$

Proof:

We define $W_0 = 0$. Now $u_T = e^{-\lambda W_T - \frac{1}{2}\lambda^2 T}$. Hence

$$\begin{aligned} p &= \mathrm{E}\Big[e^{-\lambda W_T - \frac{1}{2}\lambda^2 T} e^{-rT} F\big(S_0 e^{(\mu-\sigma^2/2)T + \sigma W_T}\big)\Big] \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda\sqrt{T}z - \frac{1}{2}\lambda^2 T} \, F\Big(S_0 e^{(\mu-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z}\Big) e^{-z^2/2}\, dz \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\Big(S_0 e^{(\mu-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}z}\Big) e^{-\frac{1}{2}(z+\lambda\sqrt{T})^2}\, dz \\ &= [\text{make change of variable } z + \lambda\sqrt{T} = x \text{ in integral}] \; \ldots \\ &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F\Big(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x}\Big) e^{-x^2/2}\, dx \end{aligned}$$

Quad Erat Demonstrandum

Note 8: Ho-Lee's Interest Rate Model in Continuous Time

This is a continuation on Lecture Note 5. Let $r(t)$ denote the stochastic instantaneous short rate at time t as seen from now. The Ho-Lee model in continuous time is

$$r(t) = \theta(t) + \sigma W(t), \quad W(0) = 0$$

where $W(t)$ is a standard Wiener process (Brownian motion) under the martingale measure ("the risk adjusted probability measure") with the money market account (short rate) as numéraire. The function $\theta(t)$ is deterministic, and chosen so that the model correctly reflects the current term structure.

Let $Z(t)$ denote the current price of a zero coupon bond maturing at t with par value 1. We denote by $\widetilde{\mathrm{E}}$ expectation w.r.t. the martingale measure. We have

$$Z(t) = \widetilde{\mathrm{E}}\,[e^{-\int_0^t r(s)\,ds}]$$

so we start by computing the integral. Using "integration by parts" of Itô calculus, we get

$$\int_0^t r(s)\,ds = \int_0^t \theta(s)\,ds + \sigma \int_0^t W(s)\,ds = \int_0^t \theta(s)\,ds + \sigma \int_0^t (t-s)\,dW(s)$$

and hence $\qquad \widetilde{\mathrm{Var}}\Big(\int_0^t r(s)\,ds\Big) = \sigma^2 \int_0^t (t-s)^2\,ds = \dfrac{\sigma^2}{3} t^3$

Since the $\mathrm{E}[z] = e^{a + \frac{\sigma^2}{2}}$ if $z \sim \mathrm{N}(a, \sigma^2)$ we get

$$Z(t) = \widetilde{\mathrm{E}}\,[e^{-\int_0^t r(s)\,ds}] = e^{-\int_0^t \theta(s)\,ds}\,\widetilde{\mathrm{E}}\,[e^{-\sigma \int_0^t (t-s)\,dW(s)}] = e^{-\int_0^t \theta(s)\,ds + \frac{\sigma^2}{6} t^3}$$

from which follows

$$\theta(t) = -\frac{d}{dt} \ln Z(t) + \frac{\sigma^2}{2} t^2 \tag{1}$$

Equation (1) defines the function θ such that the model is calibrated with the current term structure. Note that the term $-\dfrac{d}{dt} \ln Z(t) = f(t)$ is defined as the short *forward rate*, the the interest rate for a infinitesimally short forward rate agreement (FRA) maturing at t.

Forward and futures prices of a zero coupon bond

As an example we will compute the forward price and the futures price of a zero coupon bond using Ho-Lee's model. The forward price G_0 of a zero coupon bond with par value 1 to be bought at t and maturing at $T > t$ is, in any model, $G_0 = \dfrac{Z(T)}{Z(t)}$. We now proceed to compute the futures price F_0. If p_t denotes the (stochastic) price of the bond at t, We have

$$p_t = \widetilde{\mathrm{E}}_t[e^{-\int_t^T r(s)\,ds}]$$

and

$$F_0 = \widetilde{\mathrm{E}}\,[p_t]$$

where $\widetilde{\mathrm{E}}_t$ denotes expectation conditional on the information available at time t. Combining these and using the "law of iterated expectations" we get

$$F_0 = \widetilde{\mathrm{E}}_t[e^{-\int_t^T r(s)\,ds}] = e^{-\int_t^T \theta(s)\,ds}\,\widetilde{\mathrm{E}}\,[e^{-\sigma\int_t^T W(s)\,ds}]$$

Employ integration by parts of Itô calculus on the last integral to get

$$\begin{aligned}
F_0 &= e^{-\int_t^T \theta(s)\,ds}\,\widetilde{\mathrm{E}}\,[e^{-\int_t^T W(s)\,ds}] \\
&= e^{\int_t^T (\frac{d}{ds}\ln Z(s) - \frac{\sigma^2}{2}s^2)\,ds}\,\widetilde{\mathrm{E}}\,[e^{-\sigma\int_t^T (T-s)\,dW(s) - \sigma(T-t)W(t)}] \\
&= \frac{Z(T)}{Z(t)} e^{-\frac{\sigma^2}{6}(T^3-t^3)}\, e^{\frac{\sigma^2}{2}\int_t^T (T-s)^2\,ds + \frac{\sigma^2}{2}(T-t)^2 t} \\
&= \frac{Z(T)}{Z(t)} e^{-\frac{\sigma^2}{2}t^2(T-t)} = G_0 e^{-\frac{\sigma^2}{2}t^2(T-t)}
\end{aligned}$$

We see that, as anticipated, the futures price is lower than the forward price (the bond price is negatively correlated with the interest rate), with a conversion factor $e^{-\frac{\sigma^2}{2}t^2(T-t)}$.

Note 9: Ho-Lee's Interest Rate Model in Continuous Time, continued

This is a continuation of Lecture Note 8. The aim is to compute the price of a European option on a zero coupon bond. Let $Z(t,T)$ be the price at time t of a zero coupon bond maturing at T with face value 1. In order to compute the expression for $Z(t,T)$ we first note that

$$\begin{aligned}
\int_t^T r(s)\,ds &= \int_t^T \theta(s)\,ds + \sigma \int_t^T W(s)\,ds \\
&= \int_t^T \Big(-\frac{d}{ds}\ln Z(s) + \frac{\sigma^2}{2}s^2\Big)\,ds + \sigma \int_t^T W(s)\,dt \\
&= \ln\Big(\frac{Z(t)}{Z(T)}\Big) + \frac{\sigma^2}{6}(T^3 - t^3) + \sigma \int_t^T (T-s)\,dW(s) + \sigma(T-t)W(t)
\end{aligned}$$

Hence

$$\begin{aligned}
Z(t,T) &= \widetilde{\mathrm{E}}_t[e^{-\int_t^T r(s)\,ds}] \\
&= \tfrac{Z(T)}{Z(t)} e^{-\frac{\sigma^2}{6}(T^3-t^3)}\, e^{-\sigma(T-t)W(t)}\, \widetilde{\mathrm{E}}_t\big[e^{-\sigma\int_t^T (T-s)\,dW(s)}\big] \\
&= \tfrac{Z(T)}{Z(t)} e^{-\frac{\sigma^2}{2}(T-t)Tt}\, e^{-\sigma(T-t)W(t)}
\end{aligned}$$

We are now in a position to write down a formula for the price of a European option on a zero coupon bond maturing at T (the bond) maturing at time t (the option). Let $\Phi\big(Z(t,T)\big)$ be the payoff of the option at maturity t. The price of that option today is then

$$p = \widetilde{\mathrm{E}}\left[\Phi\big(Z(t,T)\big)e^{-\int_0^t r(s)\,ds}\right] = Z(t)\widetilde{\mathrm{E}}\left[\Phi\big(Z(t,T)\big)e^{-\frac{\sigma^2}{6}t^3 - \sigma\int_0^t (t-s)\,dW(s)}\right]$$

The problem is to find an analytical expression for this expectation. At the end of Lecture Note 7 we found the Black-Scholes-Merton pricing formula for a European option on a stock. In that case, we were able to find an integral expression for the relevant expected value, since only one integral of dW appeared in the expression, namely $W(t) = \int_0^t dW(s)$. It was then an easy task to write down an integral formula, and clean it up by making a change of variables in the integral. This change of variables was actually a derivation of a special case of Girsanov's theorem.

The current situation is somewhat more complicated, for *two* different integrals of dW appear: first $W(t) = \int_0^t dW(s)$ appears in the expression for $Z(t,T)$, and the integral $\int_0^t (t-s)\,dW(s)$ appears in the exponent stemming from the discounting. The remedy is again a change of variables, but infinitely many such, and that process is the content of Girsanov's theorem.

Changing measure and employing Girsanov's theorem

We will change measure to the one having the zero coupon bond as a numéraire. This is the reverse of what we did in note 3 – we will go from the money market account as a numéraire to the zero coupon bond as numéraire. We denote expectations operator with respect to the risk adjusted probability measure for contracts maturing at time t with zero coupon bond as a numeraire by $\mathrm{E}^{(t)}$ (in note 3 we used the notation E^*), and the relation between the two measures is expressed by

$$\begin{aligned} Z(t)\mathrm{E}^{(t)}\left[Y\right] &= \widetilde{\mathrm{E}}\left[Ye^{-\int_0^t r(s)\,ds}\right] \\ &= Z(t)\widetilde{\mathrm{E}}\left[Ye^{-\frac{\sigma^2}{6}t^3-\sigma\int_0^t (t-s)\,dW(s)}\right] \end{aligned}$$

for any stochastic variabke Y realized at t. Hence

$$\mathrm{E}^{(t)}\left[Y\right] = \widetilde{\mathrm{E}}\left[Ye^{-\frac{\sigma^2}{6}t^3-\sigma\int_0^t (t-s)\,dW(s)}\right]$$

Note that the factor appearing in the expectation above is

$$e^{-\frac{\sigma^2}{6}t^3-\sigma\int_0^t (t-s)\,dW(s)} = e^{-\frac{\sigma^2}{2}\int_0^t (t-s)^2\,ds-\sigma\int_0^t (t-s)\,dW(s)}$$

Girsanov's theorem now says: *under the new probability measure* $dP^{(t)} \overset{\text{def}}{=} e^{-\frac{\sigma^2}{2}\int_0^t (t-s)^2\,ds-\sigma\int_0^t (t-s)\,dW(s)}\,d\widetilde{P}$*, the Wiener process* $W(s)$ *is equal to* $W^{(t)}(s)-\sigma\int_0^s (t-u)\,du = W^{(t)}(s)-\sigma(ts-\frac{1}{2}s^2)$ *where* $W^{(t)}(s)$ *is a Wiener process under the new* $P^{(t)}$*-measure.*

Thus, under the $P^{(t)}$-measure, the price at time t of a zero coupon bond maturing at T is

$$\begin{aligned} Z(t,T) &= \frac{Z(T)}{Z(t)}e^{-\sigma^2(T-t)Tt/2}\,e^{-\sigma(T-t)(W^{(t)}(t)-\sigma t^2/2)} \\ &= \frac{Z(T)}{Z(t)}e^{-\sigma^2(T-t)^2t/2-\sigma(T-t)W^{(t)}(t)} \end{aligned}$$

Hence, we can write:

$$\begin{aligned} p &= Z(t)\mathrm{E}^{(t)}\left[\Phi\left(\frac{Z(T)}{Z(t)}e^{-\sigma^2(T-t)^2t/2-\sigma(T-t)W^{(t)}(t)}\right)\right] \\ &= \frac{Z(t)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi\left(G_0e^{-\hat{\sigma}^2t/2+\hat{\sigma}\sqrt{t}\,z}\right)e^{-z^2/2}\,dz \end{aligned}$$

where $G_0 = \frac{Z(T)}{Z(t)}$ is the *forward price* of $Z(t,T)$ and $\widehat{\sigma} \overset{\text{def}}{=} \sigma(T-t)$.

Note that this expression for the price of the option coincides with that of Black's pricing formula (see Lecture note 4) if the volatility of the underlying asset $Z(t,T)$ is set to $\widehat{\sigma} = \sigma(T-t)$.

Note 10: Looking Back and some PDE:s

In Lecture note 7 we introduced a new measure $\widehat{P}$. We saw that the discounted price $\widetilde{p}_t$ at time t of a contract is a martingale under this new measure. The measure is thus the *Equivalent Martingale Measure (EMM)* when the money market account is the numéraire. Elsewhere we have denoted this measure by a $\widetilde{}$ rather than $\widehat{}$, but in order to not confuse with the notation for discounting, we continue to use the $\widehat{}$ symbol here. The pricing formula $\widetilde{p}_t = \widehat{\mathrm{E}}_t[\widetilde{X}]$ can also be written as

$$p_t = \widehat{\mathrm{E}}_t[X e^{-\int_t^T r_s\,ds}]$$

in more conformity with the notation used elsewhere in these notes.

The true probability measure P and the *EMM* are related by the Radon-Nikodym derivative $u_T = e^{-\frac{1}{2}\int_0^T \lambda_s^2\,ds - \int_0^T \lambda_s\,dW_s}$. By Girsanov's theorem, this implies that the underlying Wiener process W_t can be written as

$$dW_t = d\widehat{W}_t - \lambda\,ds$$

where $\widehat{W}_t$ is a Wiener process under the *EMM*-measure. In particular, it means that

$$\begin{aligned} dS &= \mu S\,dt + \sigma S\,dW = \mu S\,dt + \sigma S(d\widehat{W} - \lambda\,ds) \\ &= rS\,dt + \sigma S\,d\widehat{W} \end{aligned}$$

This shows that the drift of the underlying asset S is equal to the risk free rate (which we already knew) and that the volatility of the asset inder the *EMM*-measure is the same as under the true measure, in accordance with what we claimed earlier (note 4.) It now follows easily that the expected rate of return under the *EMM* measure of the derivative is also r. Indeed, with the notation from note 7,

$$\begin{aligned} dp_t = dV_t &= \phi_t\,dB_t + \psi_t\,dS_t = \phi_t r_t B_t\,dt + \psi_t(r_t S_t\,dt + \sigma_t S_t\,d\widehat{W}_t) \\ &= r_t V_t\,dt + \psi_t \sigma_t S_t\,d\widehat{W}_t = r_t p_t\,dt + \psi_t \sigma_t S_t\,d\widehat{W}_t \end{aligned}$$

This fact is true not only under the Black-Scholes model, but for any contract. Indeed, under the *EMM* measure, the discounted price $\widetilde{p}_t$ of any contract is a martingale. Hence, the actual price p_t satisfies

$$dp_t = d(e^{\int_0^t r_s\,ds}\,\widetilde{p}_t) = r_t\,e^{\int_0^t r_s\,ds}\,\widetilde{p}_t\,dt + e^{\int_0^t r_s\,ds}\,d\widetilde{p}_t$$

Since $\displaystyle\int_0^t r_s\,ds$ is realized at t and $\widehat{\mathrm{E}}_t[d\widetilde{p}_t] = 0$ (since $\widetilde{p}_t$ is a martingale), we get, with somewhat sloppy notation,

$$\widehat{\mathrm{E}}_t[dp_t] = r_t\,e^{\int_0^t r_s\,ds}\,\widetilde{p}_t\,dt = r_t\,p_t\,dt$$

The PDE:s

Let $f(S_t, t)$ be the price of a derivative of the underlying asset S_t at time t. We assume that the price of the derivative only depends of the current value of the underlying asset. At time dt later, the price is

$$f(S_t + dS_t, t + dt) = f(S_t, t) + f_t(S_t, t)\, dt + f_S(S_t, t)\, dS_t + \tfrac{1}{2} f_{SS}(S_t, t)\, dS_t^2$$

Taking expectation w.r.t. the *EMM*-measure gives, since the price trend of the derivative f also must be that of the risk free rate:

$$f + rf\, dt = f + f_t\, dt + rSf_S\, dt + \tfrac{\sigma^2}{2} S_t^2 f_{SS}\, dt$$

since the expected value of dS^2 is $\sigma^2 S^2\, dt$. The calculation is a somewhat informal way of using Itô's formula. Hence we have the *Black-Scholes-Merton differential equation:*

$$rf = f_t + rSf_S + \tfrac{\sigma^2}{2} S_t^2 f_{SS}$$

By the very same token, if $f(r, t)$ is the price of a interest rate derivarive, depending only on the current short rate r, we get

$$rf = f_t + \mu f_r + \tfrac{\sigma^2}{2} f_{rr}$$

where σ is the volatility of the short rate; μ is the trend of the interest rate *under the* EMM-*measure!* This equation is called *"the Term Structure Equation"*. Someimes one defines $\lambda = \dfrac{\nu - \mu}{\sigma}$ where ν is tha true trend of the short rate, and thus have $\mu = \nu - \lambda\sigma$, and calls λ "the market price of risk". However, interest rate models typically model the dynamics of the interest w.r.t. the *EMM*-measure directly, so they doesn't specify what the "market price of risk" is.