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Du havde naturligvis ret, Lemma 1.2 for en $A[X]$-modul M var ikke dkket af det jeg skrev. Jeg har tilfjet det herunder, og samtidig rettet et par mindre fejl.

1. The alternator. Let M be an A-module. The symmetric group $\mathfrak{S}_n$ has a natural A-linear action on the tensor power $\bigotimes_A^n M$. We denote by

$$\Big(\bigotimes_A^n M\Big)^{\text{sym}} \qquad \text{and} \qquad \Big(\bigotimes_A^n M\Big)^{\text{a-sym}} \quad \text{respectively,}$$

the A-submodule of symmetric and anti-symmetric elements, that is, elements $x \in \bigotimes_A^n M$ such that, for all σ,

$$\sigma(x) = x, \quad \text{respectively} \quad \sigma(x) = \operatorname{sign}(\sigma)x.$$

The *alternator* is the A-linear map $\operatorname{alt}\colon \bigotimes_A^n M \to \bigotimes_A^n M$ defined by

$$\operatorname{alt}(x) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma)\sigma(x). \tag{1.1}$$

The tensors in the image are called *alternating*, and we denote by $(\bigotimes_A^n M)^{\text{alt}}$ the submodule of alternating tensors. Clearly, $\operatorname{alt}(x)$ is an anti-symmetric tensor. Hence

$$\Big(\bigotimes_A^n M\Big)^{\text{alt}} \subseteq \Big(\bigotimes_A^n M\Big)^{\text{a-sym}}.$$

[*Remark.* Clearly, if x is anti-symmetric, then $\operatorname{alt}(x) = n!\,x$. Hence, over $\mathbb{Q}$, equality holds in the inclusion above. It follows from the observation below, that equality also holds if M is A-free and 2 is a non zero-divisor in A.]

2. The alternator on a free module. Assume that M is free with basis e_i, $i \in I$. Then the tensors $e(i_1, \dots, i_n) := e_{i_1} \otimes \cdots \otimes e_{i_n}$ form a basis for $\bigotimes_A^n M$. For any $x \in \bigotimes_A^n M$, we denote by $x(i_1, \dots, i_n)$ the coefficient to $e(i_1, \dots, i_n)$ when x is expanded in terms of this basis. We may view $x(i_1, \dots, i_n)$ as a function defined on n-tupples of indices. Then:

3. Observation. *In the setup with a free module M:*

(i) *x is symmetric, if and only if $x(i_1, \dots, i_n)$ is symmetric.*
(ii) *x is anti-symmetric, if and only if $x(i_1, \dots, i_n)$ is anti-symmetric.*
(iii) *x is alternating, if and only if $x(i_1, \dots, i_n)$ is alternating, that is, $x(i_1, \dots, i_n)$ is anti-symmetric and vanishes when $i_\nu = i_\mu$ for some $\nu \neq \mu$.*

Moreover, the submodule $(\bigotimes_A^n M)^{\text{alt}}$ is free with basis

$$\operatorname{alt} e(i_1, \dots, i_n),$$

where the $i_1, \dots, i_n$ runs through all striclty decreasing sequences of indices $i_1 > \cdots > i_n$. [Here we assume for simplicity that the index set I is totally ordered.]

Proof. Only the assertion about the alternator requires an argument. Clearly, the image

$$\operatorname{alt} e(i_1,\dots,i_n) = \sum \operatorname{sign}(\sigma) e(i_{\sigma^{-1}1},\dots,i_{\sigma^{-1}n}) \tag{3.1}$$

is alternating in the $i_1,\dots,i_n$. So the special images $\operatorname{alt} e(i_1,\dots,i_n)$ for $i_1 > \cdots > i_n$ generate $(\bigotimes_A^n)^{\mathrm{alt}}$. The special images are linearly independent, since the base elements involved in the sums (3.1) for the special images are different and disjoint. So the special images form a basis for $(\bigotimes_A^n M)^{\mathrm{alt}}$. The characterization (iii) is an obvious consequence.

4. Lemma. *The alternator is alternating, that is, the map* alt *induces an A-linear map,*

$$\bigwedge\nolimits_A^n M \to (\bigotimes\nolimits_A^n M)^{\mathrm{alt}}. \tag{4.1}$$

If M is free, then the map (4.1) is an isomorphism.

Proof. By construction, the kernel of the surjection $\bigotimes_A^n M \to \bigwedge_A^n M$ is the A-submodule generated by tensors of the form,

$$y = y_1 \otimes \cdots \otimes y_n, \quad \text{with } y_\nu = y_\mu \quad \text{for some } \nu < \mu.$$

Since, obviously, the alternator alt vanishes on tensors of this form, the first assertion holds. To prove the second, it suffices to note that, in the setup of 2, the basis for $\bigwedge_A^n M$ given by the n-vectors $e_{i_1} \wedge \cdots \wedge e_{i_n}$ for $i_1 > \cdots > i_n$ is mapped to the basis for $\big(\bigotimes_A^n M\big)^{\mathrm{alt}}$ described in observation 3.

5. The symmetric structure. Let B be a commutative A-algebra. Then the tensor power $\bigotimes_A^n B$ is an A-algebra, and the submodule $S := (\bigotimes_A^n B)^{\mathrm{sym}}$ of symmetric tensors is a subalgebra. If M is a B-module, then the tensor power $\bigotimes_A^n M$ is a module over $\bigotimes_A^n B$, and in particular an S-module. Moreover, the A-submodules,

$$(\bigotimes\nolimits_A^n M)^{\mathrm{sym}}, \qquad (\bigotimes\nolimits_A^n M)^{\text{a-sym}}, \qquad \text{and} \quad (\bigotimes\nolimits_A^n M)^{\mathrm{alt}},$$

are S-submodules, and the alternator is S-linear

$$\operatorname{alt}\colon \bigotimes\nolimits_A^n M \to (\bigotimes\nolimits_A^n M)^{\mathrm{alt}}.$$

Lemma. If M is A-free, or if B is A-free, or if 2 *is invertible in M, then there is a unique structure of $\bigwedge_A^n M$ as an S-module such that the surjection $\bigotimes_A^n M \to \bigwedge_A^n M$ is S-linear.*

Proof. The kernel of the surjection is described in Lemma 4. It suffices to prove, under the assumptions of the Lemma, that if y is an element of the form (4.1) and $f \in \bigotimes_A^n B$ is invariant under the transposition τ interchanging ν and μ, then fy belongs to the kernel. We consider the case $n = 2$; the proof in the general case is similar, but the notation is more involved. Note that the kernel contains all tensors of the form $y' \otimes y'' + y'' \otimes y'$.

Assume first that 2 is invertible in M. Then $f = (f + \tau(f))/2$, and, consequenctly, f is a sum of tensors of the form $b' \otimes b + b \otimes b'$. So it suffices to treat the case when $f = b' \otimes b'' + b'' \otimes b'$. Then, as just noted, $fy = b'y \otimes b''y + b''y \otimes b'y$.

Assume next the B is A-free with basis e_i, $i \in I$. Then, obviously, the tensors of the form $e_i \otimes e_i$ and $e_i \otimes e_j + e_j \otimes e_i$ for $i < j$ form a basis of $(B \otimes_A B)^{\mathrm{sym}}$. So we may assume that f is of one of these forms. If f is of the first form, the assertion is obvious, and if f is of the second form, the assertion was proved in the previos paragraph.

Finally, assume that M is A free. Then, by Lemma 1.4, the kernel is the same as the kernel of the alternator. As the alternator is S-linear, the kernel is en S-submodule, as asserted.

6. Alternating polynomials. For the polynomial ring $A[X]$, the tensor power $\bigotimes_A^n A[X]$ may be identified with ring of polynomials $A[X_1, \dots, X_n]$ in n variables, and the subalgebra $S := A[X_1, \dots, X_n]^{\mathrm{sym}}$ is the algebra of symmetric polynomials.

7. Lemma. *Let $f \in A[X_1, \dots, X_n]$ be an anti-symmetric polynomial. Then f is alternating, iff f vanishes after the substitution $X_j = X_i$ for every $j > i$, iff f is divisible by $\Delta = \prod_{i<j}(X_j - X_i)$. Moreover, $\Delta = \mathrm{alt}(X_1^{n-1} \cdots X_n^0)$.*

Proof. Classical.

8. Corollary. *The exterior power $\bigwedge_A^n A[X]$ is a free S-module of rank 1 generated by $X^{n-1} \wedge \cdots X^1 \wedge X^0$.*

Proof. The structure on $\bigwedge_A^n A[X]$ is described in 5. To prove the claim, it suffices to note that under the S-linear isomorphism $\bigwedge_A^n A[X] \to A[X_1, \dots, X_n]^{\mathrm{alt}}$ described in Lemma 4, the n-vector $X^{n-1} \wedge \cdots X^0$ is mapped to Δ, and Δ is a free S-generator for the module of alternating polynomials by Lemma 7.

9. Noter. En stor del af ovenstående er naturligvis indeholdt i "A determinantal formula ... ", men jeg synes generaliteten ovenfor giver lidt mere indsigt. Et par ting forvirrer mig:

(1). Jeg prøvede et stykke tid at definere alternerende tensorer bare i tilfældet hvor M var fri, og her ved karakteriseringen i Observation 3(iii). Men så løb jeg ind i problemet med om definitionen var uafhængig af valg af basis. At den *er* uafhængig følger naturligvis af at definitionen på alternerende givet her er helt basis-uafhængig. Men er denne uafhngighed for en fri modul a priori oplagt?

(2). I fremstillingen ovenfor kommer strukturen af $\bigwedge_A^n A[X]$ som modul over ringen S af symmetriske polynomier via isomorfien i Observation 3. Det er ikke så forskelligt fra konstruktionen i "A determinantal formula ... ". På den anden side ser det ud som om det foregående viser følgende: Antag, at B er en A-algebra, og at M er en B-modul, fri over A. Da er den ydre potens $\bigwedge_A^n M$ på naturlig måde en modul over $(\bigwedge_A^n B)^{\mathrm{sym}}$. Kan det virkelig passe? Det er da ikke klart for mig, at kernen for

$\bigotimes_A^n M \to \bigwedge_A^n M$ er invariant under $(\bigotimes_A^n B)^{\mathrm{sym}}$. [Dette er behandlet i 5; men jeg synes stadig, at der er noget mystisk i tilfldet hvor M er fri.]

(3). En af grundene til at jeg gjorde ovensstående så generelt var at jeg håbede, at det kunne give lidt indsigt i den fundamentale S-isomorfi ($S = A[X_1, \ldots, X_n]^{\mathrm{sym}}$):

$$\bigwedge\nolimits_A^n A[X] \to \bigwedge\nolimits_S^n S[X_1].$$

Venstresiden er en fri rang-1 modul over S ifølge korollar 8, og det er højresiden også (fordi $S[X_1]$ er en rang-n modul over S). Pointen er, at den naturlige naturlige afbildning er S-lineær, eller om du vil – og det er det du giver et meget smukt bevis for – at de to strukturer af højresiden som S-modul: den naturlige via at $S[X_1]$ er en S-modul, og den symmetriske via det alternerende ovenfor – er den samme struktur.

Men det er vist ikke nemmere via isomorfien fra Lemma 4, anvendt på $M = S[X_1]$ og $A = S$?