4. Irreducible sets.

(4.1) Definition. A topological space X is *irreducible* if X is non-empty, and if any two non-empty open subsets of X intersect. Equivalently X is irreducible if $X \neq \emptyset$ and X is *not* the union of two closed subsets different from X. A subset Y of X is irreducible if it is an irreducible topological space with the induced topology.

(4.2) Proposition. *Let X be a topological space.*

(1) *A subset Y of X is irreducible if and only if the closure $\overline{Y}$ is irreducible.*
(2) *Every irreducible subset Y of X is contained in a maximal irreducible subset.*
(3) *The maxmial irreducible subsets of X are closed, and they cover X.*

Proof. (i) The first claim follows easily from the observation that every open subset that intersects $\overline{Y}$ also intersects Y.

n (ii) Let Y be an irreducible subset of X, and let !!$\mathcal{I}$ be the family consisting of
n all irreducible subsets of X that contain Y. For every chain !!$\mathcal{J} = \{Z_\alpha\}_{\alpha \in I}$ in $\mathcal{I}$ we have that $Z = \cup_{\alpha \in I} Z_\alpha$ is irreducible. This is because, when U and V are open sets that intersect Z there are α and β in $\mathcal{I}$ such that $U \cap Z_\alpha$ and $V \cap Z_\beta$ are non-empty. Since $\mathcal{J}$ is a chain we have that either the sets $U \cap Z_\alpha$ and $V \cap Z_\alpha$, or the sets $U \cap Z_\beta$ and $V \cap Z_\beta$, are non-empty. In particular $(U \cap Z) \cap (V \cap Z)$ is non-empty. Since all chains have maximal elements it follows from Zorns Lemma that $\mathcal{I}$ has maximal elements.

→ (iii) The third claim is an immediate consequence of assertions (1) and (2).

(4.3) Definition. The maximal irreducible subsets of X are called the *irreducible components* of X.

(4.4) Example. The irreducible components of the topological space with the trivial topology is X itself.

(4.5) Example. The irreducible components of the topological space X with the discrete topology are the points of X.

(4.6) Example. The topological space X with the finite complement topology is irreducible exactly when X consists of infinitely many points, or consists of one point.

(4.7) Example. Let x be a point of the topological space X. Then the closure
n !!$\overline{\{x\}}$ is irreducible.

(4.8) Definition. Let X be an irreducible topological space. If there is a point x in X such that $X = \overline{\{x\}}$ we call x a *generic point* of X.

(4.9) Definition. A topological space X is *compact* if every open covering $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover, that is, there is a *finite* subset J of I such that $X = \cup_{\beta \in J} U_\beta$.

(4.10) Example. The topological space X with the trivial topology is compact.

(4.11) Example. The topological space X with the discrete topology is compact if and only if the set X is finite.

(4.12) Example. The topological space X with the finite complement topology is compact.

(4.13) Definition. The *combinatorial dimension*, or simply the *dimension*, of a topological space X is the supremum of the length n of all chains

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

of irreducible closed subsets X_i of X. We denote the dimension of X by $\dim(X)$.

Let Y be a closed irreducible subset of X. The *combinatorial codimension*, or simply the *codimension*, of Y in X is the supremum of the length n of all chains

$$Y = X_0 \subset X_1 \subset \cdots \subset X_n$$

of irreducible closed subsets X_i of X. We denote the codimension of Y in X by $\operatorname{codim}(Y, X)$.

(4.14) Example. The topological space X with the trivial topology has dimension 0.

(4.15) Example. The topological space with the discrete topology has dimension 0.

(4.16) Example. Let $X = \{x_0, x_1\}$ be the topological space consisting of two points and with open sets $\{\emptyset, X, \{x_0\}\}$. Then X has dimension 1.

(4.17) Remark. Let X be a topological space and $\{X_\alpha\}_{\alpha \in I}$ its irreducible components. Then $\dim(X) = \sup_{\alpha \in I} \dim(X_\alpha)$.

(4.18) Remark. For every subset Y of X with the induced topology we have that $\dim(Y) \leq \dim(X)$. This is because when Z is closed and irreducible in Y, then the
→ closure $\overline{Z}$ of Z in X is irreducible by Proposition (4.2), and since Z is closed in Y we obtain that $\overline{Z} \cap Y = Z$.

(4.19) Remark. A topological space X is *noetherian* if the open subsets of X satisfy the maximum condition. That is, every chain of open subsets of X has a maximal element. Equivalently the space X is noetherian if the closed subsets of X satisfy the minimum condition. That is, every chain of closed subsets have a minimal element. A space is *locally noetherian* if every point $x \in X$ has a neighbourhood that is noetherian.

(4.20) Example. The topological space X with the trivial topology is noetherian.

(4.21) Example. The topological space X with the discrete topology is noetherian exactly when the space consists of a finite number of points.

(4.22) Example. A topological space with the finite complement topology is noetherian.

(4.23) Remark. Let X be a noetherian topological space. Then every subspace Y of X is noetherian. This is because a chain $\{Z_\alpha\}_{\alpha\in I}$ of closed subsets in Y gives a chain $\{\overline{Z}_\alpha\}_{\alpha\in I}$ of closed subsets in X, where $\overline{Z}_\alpha$ is the closure of Z_α in X. We have that $\overline{Z}_\alpha \cap Y = Z_\alpha$ and consequently that when $Z_\alpha \subset Z_\beta$ then $\overline{Z}_\alpha \subset \overline{Z}_\beta$.

(4.24) Remark. A noetherian topological space X is compact. This is because if $\{U_\alpha\}_{\alpha\in I}$ is an open covering of X without a finite subcovering we can find, by induction on n, a sequence of indices $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ in I such that $U_{\alpha_1} \subset U_{\alpha_1} \cup U_{\alpha_2} \subset U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3} \subset \cdots$. Hence X is not noetherian.

Conversely, if every open subset of X is compact, then X is noetherian. This is because if X is not noetherian then we can find an infinite sequence of open subsets $U_1 \subset U_2 \subset \cdots$ of X. Then the union $\cup_{n=1}^{\infty} U_n$ is an open subset of X with a covering $\{U_n\}_{n\in\mathbf{N}}$ that does not have a finite subcovering.

(4.25) Proposition. *A noetherian topological space X has only a finite number of distinct irreducible components $X_1, X_2, \dots, X_n$. Moreover we have that X is not contained in $\cup_{i\neq j} X_j$ for $i = 1, 2, \dots, n$.*

Proof. Let $\mathcal{I}$ be the collection of all closed subsets of the topological space X for which the Lemma does not hold. Assume that $\mathcal{I}$ is not empty. Since X is noetherian the collection $\mathcal{I}$ then has a minimal element Y. Then Y can not be irreducible, so Y is the union $Y = Y' \cup Y''$ of two closed subsets Y', Y'' different from Y. By the minimality of Y the sets Y' and Y'' both have a finite number of irreducible components. Consequently Y can be written as a union of a finite number of closed
→ irreducible subsets. It follows from Proposition (4.2) that Y has only a finite number of irreducible components. This contradicts the assumption that $\mathcal{I}$ is not empty. Hence $\mathcal{I}$ is empty and the Proposition holds.

If i is such that $X_i \subseteq \cup_{i\neq j} X_j$ we have that X_i is covered by the closed subsets $X_i \cap X_j$ for $i \neq j$. Since X_i is irreducible it follows that X_i must be contained in one of the X_j, which contradicts the maximality of X_i.

(4.26) Exercises.

1. Find the generic points of the topological space X with the trivial topology.

Let X with a distinguished element x_0 be the topological space with open subsets consisting of all subsets that contain x_0.

(1) Find the irreducible subsets of X.
(2) Find the generic point of all the irreducible subsets.

2. A topological space X is called a *Kolmogorov space* if there for every pair x, y of distinct points of X is an open set which contains one of the points, but not the other. Show that when X is a Kolmogorov space which is irreducible and has a generic point, then there is only one generic point.

3. A topological space is called a *Hausdorff space* if there for every pair of distinct points x, y of X are two open disjoint subsets of X such that one contains x and the other contains y. Determine the irreducible components of a Hausdorff space.

4. Let X be an irreducible topological space, and $f : X \to Y$ a continuous map to a topological space Y.

(1) Show that the the image $f(X)$ of X is an irreducible subset of Y.
(2) Show that if x is a generic point of X, then $f(x)$ is a generic point of $f(X)$.

5. Let X be an irreducible topological space. Show that all open subsets are irreducible.

6. Let $X = \mathbf{N}$ be the natural numbers and let $\mathcal{U}$ be the collection of sets consisting of X, $\emptyset$ and the subsets $\{0, 1, \ldots, n\}$ for all $n \in \mathbf{N}$.

(1) Show that X with the collection of sets $\mathcal{U}$ is a topological space.
→ (2) Show that the topological space of part (1) is irreducible.
→ (3) Show that the topological space of part (1) has exactly one generic point.
(4) What is the dimension of X?