3. Localization.

(3.1) Construction. Let $S$ be a multiplicatively closed subset of the ring $A$. For every $A$-module $M$ we define a relation $\sim$ on the cartesian product $M \times S$ by $(x, s) \sim (y, t)$ if there is an element $r \in S$ such that $r(tx - sy) = 0$ in $M$. It is clear that the relation $\sim$ is reflexive, that is $x \sim x$, and symmetric, that is $x \sim y$ implies $y \sim x$. It is transitive because if $(x, s) \sim (x', s')$ and $(x', s') \sim (x'', s'')$ there are elements $t, t'$ in $S$ such that $t(s'x - sx') = 0$ and $t'(s''x' - s'x'') = 0$. Then we have that $tt's'(s''x - sx'') = tt's's''x - tt's'sx'' = t's'tsx' - tst's''x' = 0$, and consequently that $(x, s) \sim (x'', s'')$.

Let $S^{-1}M = M \times S/\sim$ be the residue classes of $M \times S$ modulo the equivalence relation $\sim$. The class of the element $(x, s)$ we denote by $x/s$. There is a canonical map

$$i_M^S : M \to S^{-1}M$$

defined by $i_M^S(x) = x/1$.

On the set $S^{-1}M$ there is a unique addition such that $S^{-1}M$ becomes a group and such that the canonical map $i_M^S$ is a group homomorphism. The sum of two elements $x/s$ and $y/t$ in $S^{-1}M$ is defined by $x/s + y/t = (tx + sy)/st$. We have that the addition is independent of the choice of representative $(x, s)$ for the class $x/s$ because if $x/s = x'/s'$ there is an element $r \in S$ such that $r(s'x - sx') = 0$. Consequently we have that $r(s't(tx + sy) - st(tx' + s'y)) = t^2rs'x + rs'tsy - rst^2x' - rst's'y = 0$, and thus $(tx + sy)/st = (tx' + s'y)/s't$. Symmetrically the addition is independent of the choice of representative $(y, t)$ of the class $y/t$. It is easily checked that $S^{-1}M$ with this addition becomes an abelian group with $0 = 0/1$.

We define the product of an element $f/s \in S^{-1}A$ with an element $x/t \in S^{-1}M$ by $(f/s)(x/t) = (fx)/(st)$. A simple calculation shows that the multiplication is independent of the choice of representatives $(f, s)$ and $(x, t)$ of the classes $f/s$, respectively $x/t$. In particular we obtain a multiplication on $S^{-1}A$ and it is easily seen that this multiplication together with the group structure on $S^{-1}A$ makes $S^{-1}A$ into a ring. Moreover, with this ring structure the above operation of $S^{-1}A$ on $S^{-1}M$ makes $S^{-1}M$ into a $(S^{-1}A)$-module.

The canonical map!!

$$i_M^S : A \to S^{-1}A$$

that maps an element $f$ in $A$ to $f/1$ is a ring homomorphism.

(3.2) Definition. We call $S^{-1}M$ the localization of $M$ by the multiplicative set $S$.

(3.3) Proposition. Let $A$ be a ring and $S$ a multiplicatively closed subset. Moreover, let $M$ be an $A$-module and $N$ an $S^{-1}A$-module. For every homomorphism

$$u : M \to N_{i_M^S}$$

modules3
of $A$-modules there is a unique $S^{-1}A$-module homomorphism
\[ v : S^{-1}M \rightarrow N_{[i_M^S]} \]
such that $u = vi_M^S$.

The canonical map $i_A^S : A \rightarrow S^{-1}A$ has the universal property:

For every homomorphism of rings $\varphi : A \rightarrow B$ where $\varphi(s)$ is invertible in $B$ for every element $s \in S$, there is a unique ring homomorphism $\chi : S^{-1}A \rightarrow B$ such that $\varphi = \chi i_A^S$.

Proof. If $v$ exists we have, for all $x \in M$ and $s \in S$, that $v(x/s) = v((1/s)i_M^S(x)) = (1/s)v(i_M^S(x)) = (1/s)u(x)$. Hence $v$ is uniquely determined if it exists.

To show that $v$ exists we let $v(x/s) = (1/s)u(x)$ for all $x \in M$ and $s \in S$. This definition is independent of the choice of representative $(x, s)$ for the class $x/s$ because if $x/s = y/t$ with $y \in M$ and $t \in S$ there is an $r \in S$ such that $u(r(tx - sy)) = ru(tu(x) - su(y)) = 0$. Hence we have that $r(tu(x) - su(y)) = 0$ in $N$ and consequently that $u(x)/s = u(y)/t$ in the $S^{-1}A$-module $N$. It is clear that $v$ is an $S^{-1}A$-module homomorphism and that $u = vi_M^S$.

Finally when $\varphi : A \rightarrow B$ is a ring homomorphism such that $\varphi(s)$ is invertible in $B$ for all $s \in S$ we have that $B$ is an $S^{-1}A$-module by the multiplication $(f/s)g = \varphi(f)\varphi(s)^{-1}g$ for all $f/s \in S^{-1}A$ and $g \in B$. It is easily checked that the definition is independent of the representative $(f, s)$ of the element $f/s$ and that $B$ becomes an $S^{-1}A$-module. Hence it follows from the first part of the Proposition that we have a map $\chi : S^{-1}A \rightarrow B$ of $S^{-1}A$-modules, and it is clear that $\chi$ is a ring homomorphism.

(3.4) Remark. The universal property characterizes $i_A^S : A \rightarrow S^{-1}A$ up to an isomorphism of rings. In fact let $\psi : A \rightarrow T$ be a homomorphism of rings with the same universal property as $i_A^S$. That is, for each homomorphism of rings $\varphi : A \rightarrow B$ with $\varphi(s)$ invertible in $B$ for all $s \in S$ there is a unique homomorphism $\tau : T \rightarrow B$ such that $\varphi = \tau \psi$. Then the universal properties give unique ring homomorphisms $\omega : S^{-1}A \rightarrow T$ and $\tau : T \rightarrow S^{-1}A$ such that $\omega i_A^S = \psi$ and $\tau \psi = i_A^S$. Hence we have that $i_A^S = \tau \omega i_A^S$ and $\psi = \omega \tau \psi$. By unicity, we obtain that $\tau$ and $\omega$ are inverse maps.

(3.5) Example. Let $S = \mathbb{Z} \setminus \{0\}$. Then $S^{-1}\mathbb{Z}$ are the rational numbers $\mathbb{Q}$.

(3.6) Example. Let $A$ be a ring and $S$ a multiplicatively closed subset of $A$ containing 0. Then $S^{-1}A = 0$.

(3.7) Example. Let $A = \mathbb{Z}/6\mathbb{Z}$ and let $S = \{1, 2, 2^2, \ldots \}$. Then we have that $S^{-1}A = \mathbb{Z}/3\mathbb{Z}$. The map $i_A^S : \mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$ coincides with the canonical residue map of $\mathbb{Z}/6\mathbb{Z}$ modulo the ideal $3\mathbb{Z}/6\mathbb{Z}$. In particular we have that $i_A^S$ is not injective.

(3.8) Remark. Let $A$ be a ring and let $S$ be a multiplicatively closed subset that consist of non-zero divisors different from 0. Then the map $i_A^S : A \rightarrow S^{-1}A$ is injective. We often identify $A$ with its image by $i_A^S$. When $A$ is an integral domain and $S = A \setminus \{0\}$ we have that $S^{-1}A$ is a field.
(3.9) Definition. The total quotient ring, or total fraction ring of a ring \( A \) is the localization \( S^{-1}A \) of \( A \) in the multiplicative set \( S \) consisting of all non-zero divisors different from 0. When \( A \) is an integral domain we call the field \( S^{-1}A \) the quotient field, or field of fractions of \( A \).

(3.10) Notation. Let \( f \) be an element of \( A \). The set \( S = \{ 1, f, f^2, \ldots \} \) is a multiplicatively closed subset of \( A \). We write \( \mathcal{M}^{-1}M = M_f \). Let \( p \) be a prime ideal of \( A \). Then the set \( T = A \setminus p \) is a multiplicatively closed subset of \( A \). We write \( T^{-1}M = M_p \). The \( A_p \)-module \( M_p \) is called the localization of \( M \) at \( p \). Moreover we write \( i^S_M = i^f_M \) and \( i^T_M = i^p_p \).

(3.11) Proposition. Let \( A \) be a ring and \( S \) a multiplicatively closed subset. For every prime ideal \( p \) in \( A \) that does not intersect \( S \) we have that \( \mathcal{M}^{-1}A = \{ f/s \in S^{-1}A : f \in \mathcal{M} \} \) is a prime ideal in \( S^{-1}A \). The correspondence that maps \( p \) to \( \mathcal{M}^{-1}A \) is a bijection between the prime ideals in \( A \) that do not intersect \( S \) and the prime ideals of \( S^{-1}A \). The inverse correspondence associates to a prime ideal \( q \) in \( S^{-1}A \) the ideal \( (i^S_A)^{-1}(q) \) in \( A \).

Proof. Let \( q \) be a prime ideal in \( S^{-1}A \). It is clear that \( (i^S_A)^{-1}(q) \) is a prime ideal in \( A \) that does not intersect \( S \).

Let \( p \) be a prime ideal in \( A \) that does not intersect \( S \). If \( (f/s)(g/t) \in \mathcal{M}^{-1}A \) there is an \( r \in S \) such that \( rfg \in \mathcal{M} \). Since \( r \notin p \) we have that \( f \) or \( g \) are in \( p \), and thus that \( f/s \) or \( g/t \) is in \( \mathcal{M}^{-1}A \). Moreover we have that \( (i^S_A)^{-1}(\mathcal{M}^{-1}A) = p \) since, if \( i^S_A(f) = g/t \) with \( g \notin p \), then there is an \( r \notin p \) such that \( r(tf - g) = 0 \) in \( A \). We obtain that \( rtf \notin p \), and thus that \( f \notin p \).

It remains to prove that if \( p = (i^S_A)^{-1}(q) \) then \( \mathcal{M}^{-1}A = q \). However it is clear that \( \mathcal{M}^{-1}A \subseteq q \). Conversely if \( f/s \in q \) we must have that \( f \in p \).

(3.12) Corollary. Let \( p \) be a prime ideal in the ring \( A \). Then the localization \( A_p \) of \( A \) at \( p \) is a local ring with maximal ideal \( pA_p \).

Proof. In this case \( S = A \setminus p \) so \( p \) is maximal among the ideals in \( A \) that do not intersect \( S \).

(3.13) Remark. Let \( b \) be an ideal in \( S^{-1}A \) and let \( a = (i^S_A)^{-1}(b) \). Then, \( b = \mathcal{M}^{-1}A = \{ f/s \in S^{-1}A : f \in \mathcal{M} \} \). It is clear that \( \mathcal{M}^{-1}A \subseteq b \). Conversely, when \( f/s \in b \) we have that \( f/1 \in b \) and consequently \( f \in a \). Hence \( f/s = (f/1)(1/s) \in aS^{-1}A \).

(3.14) Proposition. There is a canonical isomorphism of \( S^{-1}A \)-modules

\[
M \otimes_A S^{-1}A \rightarrow S^{-1}M
\]  
(3.14.1)

that is uniquely determined by mapping \( x \otimes_A (f/s) \) to \( (fx)/s \) for all \( f \in A \), \( s \in S \) and \( x \in M \).

Proof. It follows from the explicit description of the map (3.14.1) that it is a map of \( S^{-1}A \)-modules if it exists.
To prove the existence we consider the map \( M \times S^{-1}A \to S^{-1}M \) that maps \((x, f/s)\) to \((fx)/s\). It is clear that this map is \( A \)-bilinear. Consequently we obtain an \( A \)-linear map \( M \otimes_A S^{-1}A \to S^{-1}M \) that maps \( x \otimes_A f/s \) to \((fx)/s\). It is clear that this map is an \((S^{-1}A)\)-homomorphism.

In order to show that the map is an isomorphism we construct an inverse \( S^{-1}M \to M \otimes_A S^{-1}A \) by mapping \( x/s \) to \( x \otimes_A 1/s \). The latter map is independent of the choice of representative \((x, s)\) of the class \(x/s\). In fact if \( x/s = y/t \) there is an \( r \in S \) such that \( r(tx - sy) = 0 \) in \( A \). We obtain that \( x \otimes_A (1/s) = x \otimes_A ((rt)/(rst)) = rtx \otimes_A (1/(rst)) = rsy \otimes_A (1/(rst)) = y \otimes_A ((rs)/(rst)) = y \otimes_A (1/t) \).

It is clear that the two maps are inverses of each other.

**Proposition.** **(3.15) Homomorphisms.** Let \( S \) be a multiplicatively closed subset of \( A \), and let \( u : M \to N \) be a homomorphism of \( A \)-modules. There is a canonical map of \( S^{-1}A \)-modules:!! \[
S^{-1}u : S^{-1}M \to S^{-1}N
\]
that maps \( x/s \) to \( u(x)/s \) for all \( s \in S \) and \( x \in M \). The map is independent of the choice of representative \((x, s)\) of the class \(x/s\) because if \( x/s = y/t \) there is an \( r \in S \) such that \( r(tx - sy) = 0 \), and thus \( u(x)/s = u(y)/t \). It follows from the explicit form of \( S^{-1}u \) that it is an \( S^{-1}A \)-module homomorphism.

**Remark.** **(3.16) When \( v : N \to P \) is a homomorphism of \( A \)-modules we have that \( S^{-1}(vu) = S^{-1}vS^{-1}u \), and \( S^{-1}\text{id}_M = \text{id}_{S^{-1}M} \). In other words, the correspondence that maps an \( A \)-module \( M \) to the \((S^{-1}A)\)-module \( S^{-1}M \) is a covariant functor from \( A \)-modules to \((S^{-1}A)\)-modules.

**Notation.** **(3.17) Let \( f \) be an element of \( A \) and let \( S = \{1, f, f^2, \ldots, \} \). Moreover let \( p \) be a prime ideal of \( A \) and let \( T = A \setminus p \). For every homomorphism \( u : M \to N \) we write \(!!u_f = S^{-1}u \) and \(!!u_p = T^{-1}u \). Moreover we write the canonical maps \( !!!S_A = i_A^f \) and \(!!!T_A = i_A^p \).

**Proposition.** **(3.18) Let \( A \) be a ring and \( S \) a multiplicatively closed subset. Moreover let \( 0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0 \) be an exact sequence of \( A \)-modules. Then the sequence \[
0 \to S^{-1}M' \xrightarrow{S^{-1}u} S^{-1}M \xrightarrow{S^{-1}v} S^{-1}M'' \to 0
\]
is an exact sequence of \( S^{-1}A \)-modules.

**Proof.** We first show that \( S^{-1}u \) is injective. If \( x' \in M' \) and \( s \in S \), and \( S^{-1}u(x'/s) = u(x')/s = 0 \) there is a \( t \in S \) such that \( u(tx') = tu(x') = 0 \). Since \( u \) is injective we have that \( tx' = 0 \) and consequently \( x'/s = 0 \).

It is clear that we have an inclusion \( \text{Im}(S^{-1}u) \subseteq \text{Ker}(S^{-1}v) \). We will show that the opposite inclusion holds. Let \( x/s \in \text{Ker}(S^{-1}v) \), that is \( v(x)/s = 0 \). Then there
is a \( t \in S \) such that \( v(tx) = tv(x) = 0 \). Since \( \text{Im}(u) = \ker(v) \) there is an \( x' \in M' \) such that \( u(x') = tx \). Consequently \( S^{-1}u(x'/(st)) = u(x')/(st) = (tx)/(st) = x/s \), and thus \( x/s \in \text{Im}(S^{-1}u) \).

Finally it is obvious that \( S^{-1}v \) is surjective.

\( \rightarrow \) (3.19) Remark. We paraphrase Proposition (3.17) by saying that the functor \( S^{-1} \) is exact.

(3.20) Proposition. Let \( A \) be a ring and let \( S \) be a multiplicatively closed subset. Moreover let \( \{M_\alpha\}_{\alpha \in I} \) be a collection of \( A \)-modules. Then there is a canonical isomorphism of \( A \)-modules

\[
\bigoplus_{\alpha \in I} S^{-1}M_\alpha \sim S^{-1}\left(\bigoplus_{\alpha \in I} M_\alpha\right)
\]  

(3.20.1)

\( \rightarrow \) such that the composite of the map (3.20.1) with the canonical map \( v_\beta : S^{-1}M_\beta \rightarrow \bigoplus_{\alpha \in I} S^{-1}M_\alpha \) to factor \( \beta \) is the localization \( S^{-1}u_\beta : S^{-1}M_\beta \rightarrow S^{-1}\left(\bigoplus_{\alpha \in I} M_\alpha\right) \) of the canonical map \( u_\beta : M_\beta \rightarrow \bigoplus_{\alpha \in I} M_\alpha \) to factor \( \beta \).

Proof. The canonical map \( u_\beta : M_\beta \rightarrow \bigoplus_{\alpha \in I} M_\alpha \) gives a map \( S^{-1}u_\beta : S^{-1}M_\beta \rightarrow S^{-1}\left(\bigoplus_{\alpha \in I} M_\alpha\right) \) and by the universal property of direct products we obtain the map

\[ \bigoplus_{\alpha \in I} (S^{-1}M_\alpha) \rightarrow S^{-1}\left(\bigoplus_{\alpha \in I} M_\alpha\right) \] of (3.20.1).

To show that the map is an isomorphism we construct the inverse. The canonical maps \( M_\alpha \rightarrow S^{-1}M_\alpha \) for \( \alpha \in I \) define a homomorphism \( \bigoplus_{\alpha \in I} M_\alpha \rightarrow \bigoplus_{\alpha \in I} S^{-1}M_\alpha \).

Consequently it follows from Proposition (3.3) that we have a canonical homomorphism \( S^{-1}\left(\bigoplus_{\alpha \in I} M_\alpha\right) \rightarrow \bigoplus_{\alpha \in I} S^{-1}M_\alpha \) and it is clear that this map is the inverse of the map (3.20.1).

(3.21) Proposition. Let \( A \) be a ring and let \( M \) be an \( A \)-module. The following conditions are equivalent:

1. \( M = \{0\} \).
2. \( M_p = \{0\} \) for all prime ideals \( p \) of \( A \).
3. \( M_m = \{0\} \) for all maximal ideals \( m \) of \( A \).

Proof. (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are clear.

(3) \( \Rightarrow \) (1) Let \( x \in M \) and let \( a_x = \{f \in A : fx = 0\} \). It is clear that \( a_x \) is an ideal in \( A \). We shall show that \( a_x = A \), and hence in particular that \( 1x = x = 0 \). Assume to the contrary that \( a_x \not\subseteq A \). Then there is a maximal ideal \( m \) of \( A \) that contains \( a_x \). Since \( M_m = 0 \) we can find an element \( s \in A \) such that \( sx = 0 \). Then \( s \in a_x \), which is impossible since \( a_x \subseteq m \). This contradicts the assumption that \( a_x \not\subseteq A \). Hence we have proved that \( a_x = A \) for all \( x \in M \) and consequently that \( M = 0 \).

(3.22) Proposition. Let \( A \) be a ring and \( u : M \rightarrow N \) an \( A \)-linear homomorphism. The following conditions are equivalent:

1. \( u \) is injective, respectively surjective.
2. \( u_p \) is injective, respectively surjective, for all prime ideal \( p \) of \( A \).
3. \( u_m \) is injective, respectively surjective, for all maximal ideals \( m \) of \( A \).
Proof. We prove the equivalence of the conditions for injective maps:

(1) ⇒ (2) It follows from Proposition (?) that condition (2) follows from condiiton
→
(1).

(2) ⇒ (3) This implication is clear.

(3) ⇒ (1) Let \( L = \text{Ker}(u) \). When \( u_m \) is injective it follows from Proposition (?)
that \( L_m = 0 \) for all maximal primes \( m \) of \( A \). Hence it follows from Proposition (?)
that \( L = 0 \) and thus that \( u \) is injective.

Similar arguments show the equivalence of the assertions for surjective maps.

(3.23) Corollary. Let \( f \neq 0 \) be an element of \( A \). We have:

1. If \( f \) is not a zero divisor in \( A \) then \( f/1 \) is not a zero divisor in the localization \( A_p \) of \( A \) in \( p \) for all prime ideals \( p \) of \( A \).
2. If \( f/1 \) is not a zero divisor in \( A_\mathfrak{m} \) for all maximal ideals \( \mathfrak{m} \) of \( A \) then \( f \) is not
   a zero divisor in \( A \).

Proof. We have that \( f \) is not a zero divisor in \( A \) if and only if the multiplication
map \( f_A : A \to A \) is injective, and \( f/1 \) is not a zero divisor in \( A_p \) if and only if the
multiplication map \( (f/1)_A : A_p \to A_p \) is not injective. Hence the Corollary follows
from the Proposition.

(3.24) Exercises.
1. Let \( K \) be a field and let \( K[u, v] \) be the polynomial ring in the variables \( u, v \) with
coefficients in \( K \). Moreover let \( A = K[u, v]/(uv) \).
   (1) Show that the ideal \( \mathfrak{p} = (u)/(uv) \) is a prime ideal in \( A \).
   (2) Describe the localization \( A_\mathfrak{p} \).
2. Let \( M \) and \( N \) be \( A \)-modules and let \( S \) be a multiplicatively closed subset of \( A \).
   Show that the \( S^{-1}A \)-modules \( S^{-1}(M \otimes_A N) \) and \( S^{-1}M \otimes_{S^{-1}A} S^{-1}N \) are canonically
   isomorphic.
3. Let \( f \) be a nilpotent element in \( A \), and \( M \) an \( A \)-module. Determine \( M_f \).
4. For every \( f \in A \) and every prime ideal \( \mathfrak{p} \) of \( A \) we let \( f(\mathfrak{p}) \) be the image of \( f \) by the
   composite map \( A \xrightarrow{\mathfrak{p}} A_\mathfrak{p} \xrightarrow{\varphi_{A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}}} A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \). Show that \( f(\mathfrak{p}) = 0 \) for all prime ideals \( \mathfrak{p} \)
   if and only if \( f \) is contained in the radical \( \mathfrak{r}(A) \) of \( A \).
5. Let \( \varphi : A \to B \) be a homomorphism of rings, and let \( S \) be a multiplicatively
   closed subset in \( A \).
   (1) Show that \( T = \varphi(S) \) is a multiplicatively closed subset of \( B \).
   (2) Show that there is a canonical isomorphism between the \( S^{-1}A \)-modules \( T^{-1}B \)
       and \( S^{-1}B = B \otimes_A S^{-1}A \).
6. Let \( A \) be a ring and \( \mathfrak{p} \) a prime ideal.
   (1) Show that if the local ring \( A_\mathfrak{p} \) has no nilpotent elements different from zero
       for all prime ideals \( \mathfrak{p} \) in \( A \) then \( A \) has no nilpotent elements different from zero.
(2) Is it true that if $A$ has no nilpotent elements different from zero then $A_p$ has no nilpotent elements different from zero for all prime ideals $p$ of $A$?

7. Let $M$ be a finitely generated $A$ module, and let $S$ be a multiplicatively closed subset of $M$. Show that $S^{-1}M = 0$ if and only if there is an element $s \in S$ such that $sM = 0$.

8. Let $!!\mathcal{P}$ be the set of all prime number in $\mathbb{Z}$

1. Show that the map $\mathbb{Z} \to \prod_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}$ that sends an integer $n$ to $(n, n, \ldots)$ is injective.
2. Show that for all injective maps $u : G \to H$ of groups the map $u \otimes_{\mathbb{Z}} \text{id}_\mathbb{Q} : G \otimes_{\mathbb{Z}} \mathbb{Q} \to H \otimes_{\text{id}_\mathbb{Q}} \mathbb{Q}$ is injective.
3. Show that $\big((\prod_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$ is not zero
4. Show that $\big((\prod_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$ is not isomorphic to $\prod_{p \in \mathcal{P}} (\mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

9. Let $A$ be a ring and $S$ a multiplicatively closed subset. Moreover let $M$ be an $A$-module. Describe the kernel of the $A$-module homomorphism $M \to M \otimes_A S^{-1}A$ that maps $x \in M$ to $x \otimes_A 1$.

10. Let $A \neq 0$ be a ring and $u : A^m \to A^n$ an $A$-linear map. Moreover let $p$ be a minimal prime ideal in $A$.

1. Let $f_1, f_2, \ldots, f_m$ be elements in $p$. Show that the ideal $b$ in $A_p$ generated by the elements $f_1/1, f_2/1, \ldots, f_n/1$ is nilpotent, that is, we have $b^m = (0)$ for some positive integer $m$.
2. Let $p$ be the integer such that $b^p \neq (0)$ and $b^{p+1} = (0)$ in $A_p$. Show that for all elements $f \in b^p$ we have that $fs \neq 0$ for all $s \in A \setminus p$, and that $f_i ft = 0$ for some $t \in A \setminus p$ for $i = 1, 2, \ldots, m$.
3. Show that if $u$ is injective then the map

$$u_p : (A/p)^m \to (A/p)^n$$

is injective, where the $A/p$-module homomorphism $u_p$ is defined by

$$u_p((u_{A/p}(f_1), u_{A/p}(f_2), \ldots, u_{A/p}(f_n)) = (u_{A/p}(u_1(x)), u_{A/p}(u_2(x)), \ldots, u_{A/p}(u_n(x)))$$

for all $x = (f_1, f_2, \ldots, f_m)$ in $A^m$ and where $u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ in $A^n$.

4. Show that when $u$ is injective then $m \leq n$. 