2. Hilbert polynomials.

(2.1) Definition. Let $A$ be a ring. An additive function $\lambda = \lambda_A$ on finitely generated $A$-modules associates to every finitely generated $A$-modules $M$ an integer $\lambda(M)$ and satisfies the property:

For every exact sequence of finitely generated $A$-modules

$$0 \to M' \to M \to M'' \to 0$$

we have

$$\lambda(M) = \lambda(M') + \lambda(M'').$$

(2.2) Remark. Let $M' = (0)$ and thus $M = M''$. We see that $\lambda((0)) = 0$.

(2.3) Example. It follows from Proposition (CHAINS 1.15) that when $A$ is an artinian ring the *length* is an additive function on finitely generated $A$-modules. In particular the vectors space dimension is an additive function on finite dimensional vector spaces.

(2.4) Remark. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded ring that is finitely generated as an $A_0$-algebra, and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated $A$-module. Then each $M_n$ is a finitely generated $A_0$-module. In fact when we replace, if necessary, a set of generators for the $A_0$-algebra $A$, and a set of generators for the $A$-module $M$ by their homogeneous components, we see that the $A_0$-algebra $A$ can be generated by a finite set $f_1, f_2, \ldots, f_p$ of homogeneous elements of $A$, respectively that the $A$-module $M$ can be generated by a finite set $x_1, x_2, \ldots, x_q$ of homogeneous elements of $M$. If $f_i \in A_{m_i}$ for $i = 1, 2, \ldots, p$ and $x_i \in M_{n_i}$ for $i = 1, 2, \ldots, q$ we clearly have that $M_n$ is generated, as an $A_0$-module, by the elements $f_i_1 f_i_2 \cdots f_i_r x_j$ for all collections of integers $i_1, i_2, \ldots, i_r$ between 1 and $p$ and and $j$ between 1 and $q$, and with $m_{i_1} + m_{i_2} + \cdots + m_{i_r} + n_j = n$.

In particular, it follows from Proposition (1.6) that for any noetherian graded ring $A$ and finitely generated graded module $M$, the homogeneous part $M_n$ is finitely generated for all $n$.

(2.5) Definition. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded ring that is finitely generated as an $A_0$-algebra, and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated $A$-module. Moreover let $\lambda$ be an additive function on finitely generated $A_0$-modules. The *Poincaré series* of the $A$-module $M$ is the power series!!

$$P_\lambda(M, t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n$$

in the variable $t$ with coefficients in $\mathbb{Z}$.
(2.6) Example. Let \( A = K[t_1, t_2, \ldots, t_n] \) be the polynomial ring in the variables \( t_1, t_2, \ldots, t_n \) over a field \( K \). Then \( A = \oplus_{i=0}^{\infty} A_i \) where \( A_i \) is the vector space of all homogeneous polynomials of degree \( n \). Let \( \lambda(M) = \dim_K(M) \) for all finite dimensional vector spaces \( M \) over \( K \). Then \( P_\lambda(A, t) = \sum_{i=0}^{\infty} (i^{n+1}) t^i = 1/(1 - t)^n \).

(2.7) Example. Let \( K[t_1, t_2] \) be the polynomial ring in the variables \( t_1, t_2 \) over a field \( K \). Moreover let \( A = K[t_1, t_2]/(t_1^2, t_1 t_2) \) be the residue ring of the polynomial ring \( K[t_1, t_2] \) modulo the ideal \( (t_1^2, t_1 t_2) \), and let \( u \) and \( v \) be the residue classes of \( t_1 \), respective \( t_2 \) in \( A \). Then \( u^2 = 0 = uv \) and we have that \( A = K \oplus ( Ku + K v ) \oplus K v^2 \oplus K v^3 \oplus \cdots \). Hence \( P_\lambda(A, t) = 1 + 2t + t^2 + t^3 + \cdots = (1 + t - t^2)/(1 - t) \).

(2.8) Example. Let \( K[t_1, t_2] \) be the polynomial ring in the variables \( t_1, t_2 \) over a field \( K \). Moreover let \( A = K[t_1, t_2]/(t_1^2 + t_2^2) \) be the residue ring of the polynomial ring \( K[t_1, t_2] \) modulo the ideal \( (t_1^2 + t_2^2) \), and let \( u \) and \( v \) be the residue classes of \( t_1 \) respectively \( t_2 \) in \( A \). Then \( u^2 + v^2 = 0 \) and \( A = K \oplus ( Ku + K v ) \oplus ( Kw + K v^2 ) \oplus ( K v + K v^3 ) \oplus \cdots \). Hence \( P_\lambda(A, t) = 1 + 2t + 2t^2 + \cdots = (1 + t - t^2)/(1 - t) \).

(2.9) Lemma. Let \( A = \oplus_{n=0}^{\infty} A_n \) be a noetherian graded ring and let \( M = \oplus_{n=0}^{\infty} M_n \) be a finitely generated \( A \)-module. Moreover let \( \lambda \) be an additive function on finitely generated \( A_0 \)-modules. For every homogeneous element \( f \in A_m \) with \( m > 0 \) we have an exact sequence of \( A \)-modules

\[
0 \rightarrow L \rightarrow M \xrightarrow{f_M} M \rightarrow N \rightarrow 0
\]

where \( L \) and \( N \) are finitely generated \((A/fA)\)-modules, and

\[
(1 - t^m) P_\lambda(M, t) = P_\lambda(N, t) - t^m P_\lambda(L, t).
\]  

(2.9.1)

Proof. For each integer \( n \geq -m \) we have an exact sequence

\[
0 \rightarrow L_n \rightarrow M_n \xrightarrow{f_M} M_{m+n} \rightarrow N_{m+n} \rightarrow 0
\]  

(2.9.2)

where \( L_n \) and \( N_{m+n} \) are defined as the kernel, respectively the cokernel of the map \( f_M \). Let \( L = \oplus_{n=0}^{\infty} L_n \) and \( N = \oplus_{n=0}^{\infty} N_n \). Then \( L \) and \( N \) are \( A \)-modules, and we have an exact sequence (2.6.1). Since \( M \) is noetherian by Lemma (CHAINS 1.6) it follows from Proposition (1.7) that \( L \) and \( N \) are noetherian \( A \)-modules. In particular it follows from Remark (2.4) that \( L_n \) and \( N_n \) are finitely generated \( A_0 \)-modules for all \( n \). It follows from (2.9.3) that we have equations

\[
\lambda(M_{m+n}) - \lambda(M_n) = \lambda(N_{m+n}) - \lambda(L_n) \quad \text{for } n = -m, -m + 1, \ldots.
\]  

(2.9.4)

Multiply both sides of (2.9.4) by \( t^{n+m} \) for \( n = -m, -m + 1, \ldots \), and sum the right and left hand sides of the resulting equations. We obtain equation (2.9.2) of the Lemma.

Finally we note that \( fL = 0 \) and \( fN = 0 \). Hence \( L \) and \( N \) are in fact \( A/fA \)-modules.
(2.10) Theorem. (Hilbert-Serre) Let \( A \) be a noetherian graded ring, generated as an \( A_0 \)-module by \( m \) homogeneous elements of degrees \( p_1, p_2, \ldots, p_m \). Moreover let \( M \) be a finitely generated graded \( A \)-module, and \( \lambda \) an additive function on finitely generated \( A_0 \)-modules. Then

\[
P_\lambda(M, t) = f(t) / \prod_{i=1}^{m} (1 - t^{p_i})
\]

in the ring \( \mathbb{Z}[[t]] \) of power series in the variable \( t \) over the integers, where \( f(t) \) is a polynomial in \( \mathbb{Z}[t] \) and \( 1/(1 - t^{p_i}) = 1 + t^{p_i} + t^{2p_i} + \cdots \).

Proof. We prove the Theorem by induction on \( m \). When \( m = 0 \) we have that \( A = A_0 \), and since \( M \) is finitely generated \( M_n = 0 \) for all sufficiently large \( n \). Consequently \( P_\lambda(M, t) \) is a polynomial when \( m = 0 \).

Assume that \( m > 0 \) and that the Theorem holds for \( m - 1 \). Let \( f_1, f_2, \ldots, f_m \) be homogeneous elements of positive degrees \( p_1, p_2, \ldots, p_m \) respectively that generate \( A \) as an \( A_0 \)-algebra. It follows from Lemma (2.9) with \( f = f_m \) that

\[
(1 - t^{p_m}) P_\lambda(M, t) = P_\lambda(N, t) - t^{p_m} P_\lambda(L, t)
\]

where \( L \) and \( N \) are \( (A/f_m A) \)-modules. We have that the \( A_0 \)-algebra \( A/f_m A = A_0[f_1, f_2, \ldots, f_m]/f_m A \) is generated by the residue classes of \( f_1, f_2, \ldots, f_{m-1} \). It follows from the induction hypothesis that \( P_\lambda(N, t) = g(t) / \prod_{i=1}^{m-1} (1 - t^{p_i}) \) and \( P_\lambda(L, t) = h(t) / \prod_{i=1}^{m-1} (1 - t^{p_i}) \), where \( g(t) \) and \( h(t) \) are polynomials in \( \mathbb{Z}[t] \). Equation (2.10.1) consequently follows from equation (2.10.2).

(2.11) Corollary. Let \( A \) be a noetherian graded ring that is finitely generated as an \( A_0 \)-algebra by \( m \) elements of degree 1. Moreover let \( M \) be a finitely generated graded \( A \)-module. Write

\[
P_\lambda(M, t) = f(t) / (1 - t)^m = g(t) / (1 - t)^p
\]

where \( 0 \leq p \leq m \) and \( g(t) \) is a polynomial in \( \mathbb{Z}[t] \) with \( g(1) \neq 0 \). Then there is a polynomial \( h(t) \) in \( \mathbb{Q}[t] \) of degree \( p - 1 \) such that \( \lambda(M_n) = h(n) \) for all sufficiently large \( n \). Here we define the degree of the zero polynomial as \(-1\).

Proof. When \( p = 0 \), that is, when \( (1 - t)^m \) divides \( f(t) \) we have that \( P_\lambda(M, t) \) is a polynomial. Consequently we have that \( \lambda(M_n) = 0 \) when \( n \) is larger than the degree of \( P_\lambda(M, t) \). Hence the Corollary holds when \( (1 - t)^m \) divides \( f(t) \).

Assume that \( 0 < p \leq m \). Write \( g(t) = \sum_{n=0}^{q} g_n t^n \) with \( g_n \in \mathbb{Z} \). Since \( 1/(1 - t)^p = \sum_{n=0}^{\infty} \frac{(n+p-1)}{p-1} t^n \) in \( \mathbb{Z}[[t]] \), we have that

\[
g(t) / (1 - t)^p = \sum_{n=0}^{\infty} \sum_{i+j=n} g_i \binom{j+p-1}{p-1} t^n.
\]
Consequently $\lambda(M_n) = \sum_{i+j=n} g_i (t^{i+p-1}) = \sum_{i=0}^{q} g_i (t^{n-i+p-1})$. We write $(t) = (1/n!)t(t-1)\cdots(t-n+1)$ in $\mathbb{Q}[t]$, and we let $h(t) = \sum_{i=0}^{q} g_i (t^{i+p-1})$. Then $h(t)$ is a polynomial of degree $p - 1$ because the coefficient of $t^{p-1}$ is $(1/(p-1)!) \sum_{i=0}^{q} g_i = (1/(p-1)!)(1) \neq 0$. Moreover we have that $\lambda(M_n) = h(n)$ when $n \geq q$, and we have proved the Corollary.

(2.12) Definition. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded noetherian ring which is generated as an $A_0$-module by elements of degree 1. Moreover let $M$ be a finitely generated $A$-module, and let $\lambda$ be an additive function on finitely generated $A_0$-modules. The polynomial $!h(t)!$ in $\mathbb{Q}[t]$ such that $h(n) = \lambda(M_n)$ for all sufficiently large $n$ is called the Hilbert polynomial of $M$ with respect to $\lambda$. We denote by $!!d_\lambda(M)$ the degree of the Hilbert polynomial. Here we define the degree of the zero polynomial as $-1$.

(2.13) Example. Let $K[t_1, t_2, \ldots, t_n]$ be the polynomial ring in the variables $t_1, t_2, \ldots, t_n$ over a field $K$. We saw in Example (2.6) that the Hilbert polynomial $h(t)$ is $(t^{n+1})/(n+1) = (1/(n-1)!(t + n - 1)(t + n - 2)\cdots(t + 1)$.

(2.14) Example. Let $A = K[u, v]$ with $u^2 = 0 = uv$ be the ring of Example (2.7). Then the Hilbert polynomial $h(t)$ is equal to 1.

(2.15) Example. Let $A = K[u, v]$ with $u^2 + v^2 = 0$ be the ring of Example (2.8). Then the Hilbert polynomial $h(t)$ is equal to 2.

(2.16) Lemma. Let $A$ be a noetherian graded ring that is generated as an $A_0$-module by elements of degree 1. Moreover let $M$ be a finitely generated $A$-module, and let $\lambda$ be an additive function on finitely generated $A_0$-modules. For every homogeneous element $f \in A$ of positive degree which is $M$-regular we have that

$$d_\lambda(M) = d_\lambda(M/fM) + 1.$$  

Proof. Let $f$ be homogeneous of degree $m > 0$. Since $f$ is $M$-regular the map $f_M : M \to M$ is injective. Hence it follows from the exact sequence (2.9.1) that $L = 0$, and we obtain from equation (2.9.2) that

$$(1 - t^m)P_\lambda(M, t) = P_\lambda(M/fM, t).$$

Write $P_\lambda(M, t) = g(t)/(1 - t)^p$ and $P_\lambda(M/fM, t) = h(t)/(1 - t)^q$ where $g(t)$ and $h(t)$ are polynomials in $\mathbb{Z}[t]$ with $g(1) \neq 0$ respectively $h(1) \neq 0$. Then $(1 - t^m)(1 - t)^qg(t) = (1 - t)^ph(t)$. Since $1 - t^m = (1 - t)(1 + t + \cdots + t^{m-1})$ and $(1 + t + \cdots + t^{m-1})(1) = m \neq 0$ we have that $p = q + 1$. That is, we have $d_\lambda(M) = d_\lambda(M/fM) + 1$, and we have proved the Lemma.

(2.17) Exercises.
1. Let $K[u, v]$ be the ring of polynomials in the independent variables $u, v$ with coefficients in a field $K$. Moreover, let $S = K[u, v]/(u^2, uv^m)$ be the residue ring of $K[u, v]$ modulo the ideal $(u^2, uv^m)$.

(1) Determine the polynomial $g(t)$ in $\mathbb{Z}[t]$ and the non-negative integer $p$ such that

$$P_\lambda(S, t) = g(t)/(1 - t)^p$$

and $g(1) \neq 0$, when $\lambda = \dim_K$.

(2) Determine the Hilbert polynomial of $S$ with respect to $\dim_K$.

2. Let $K[u, v]$ be the ring of polynomials in the independent variables $u, v$ with coefficients in a field $K$. Let $S = K[u, v]/(u^2, v^m)$ be the residue ring of $K[u, v]$ modulo the ideal $(u^2, v^m)$.

(1) Determine the polynomial $g(t)$ in $\mathbb{Z}[t]$ and the non-negative integer $p$ such that

$$P_\lambda(S, t) = g(t)/(1 - t)^p$$

and $g(1) \neq 0$, when $\lambda = \dim_K$.

(2) Determine the Hilbert polynomial of $S$ with respect to $\dim_K$.

3. Let $K[t_1, t_2, \ldots, t_n]$ be the ring of polynomials in the independent variables $t_1, t_2, \ldots, t_n$ over a field $K$. Moreover, let $f(t_1, t_2, \ldots, t_n)$ be a polynomial of degree $d > 0$, and let $S = K[t_1, t_2, \ldots, t_n]/(f(t_1, t_2, \ldots, t_n)$ be the residue ring of $K[t_1, t_2, \ldots, t_n]$ modulo the ideal $(f(t_1, t_2, \ldots, t_n)$ generated by $f(t_1, t_2, \ldots, t_n)$.

(1) Determine the polynomial $g(t)$ in $\mathbb{Z}[t]$ and the non-negative integer $p$ such that

$$P_\lambda(S, t) = g(t)/(1 - t)^p$$

and $g(1) \neq 0$, when $\lambda = \dim_K$.

(2) Determine the Hilbert polynomial of $S$ with respect to $\dim_K$.

4. Let $K[t_0, t_1, \ldots, t_n]$ be the ring of polynomials in the independent variables $t_0, t_1, \ldots, t_n$ with coefficients in a field $K$ with infinitely many elements. For every point $b = (b_0, b_1, \ldots, b_n$ in the cartesian product $K^{n+1}$ of the field $K$ with itself $n+1$ times, and for every element $\kappa$ in $K$ we write $\kappa b = (\kappa b_0, \kappa b_1, \ldots, \kappa b_n$. Moreover for every collection of points $a_1, a_2, \ldots, a_m$ in $K^{n+1}$ we write

$$\mathcal{I}(a_1, a_2, \ldots, a_m) = \{ f \in K[t_0, t_1, \ldots, t_n]: f(\kappa a_i) = 0 \text{ for } i = 1, 2, \ldots, m \text{ and all }\kappa \in K \}.$$ 

(1) Show that $\mathcal{I}(a_1, a_2, \ldots, a_m)$ is a homogeneous ideal in $K[t_0, t_1, \ldots, t_n]$.

(2) Let $S = K[t_0, t_1, \ldots, t_n]/\mathcal{I}(a_1, a_2, \ldots, a_m)$. Show that

$$\dim_K(\mathcal{I}(a_1, a_2, \ldots, a_m)) \geq \max(0, \left(\binom{n+d}{d} - m\right)).$$
(3) Show that for every non-empty collection \( \mathcal{P} \) of homogeneous polynomials in \( K[t_0, t_1, \ldots, t_n] \) of positive degree the subset

\[
V(\mathcal{P}) = \{ b \in K^{n+1} : g(b) = 0 \text{ for all } f \in \mathcal{P} \}
\]

of \( K^{n+1} \) is different from \( K^{n+1} \).

(4) Show that we can find points \( a_1, a_2, \ldots, a_m \) in \( K^{n+1} \) such that

\[
dim_K(\mathcal{I}(a_1, a_2, \ldots, a_m)) = \max(0, \binom{n + d}{d} - m).
\]

(5) Determine the polynomial \( g(t) \) in \( \mathbb{Z}[t] \) and the non-zero integer \( p \) such that

\[
P_\lambda(S, t) = g(t)/(1 - t)^p
\]

and \( g(1) \neq 0 \), when \( \lambda = \dim_K \).