3. Modules over noetherian rings.

(3.1) Proposition. Let $A$ be a noetherian ring and let $M \neq 0$ be an $A$-module. Then $M$ has associated prime ideals.

When $M$ is finitely generated there is a chain

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$$

of submodules of $M$ such that each quotient $M_{i-1}/M_i$ is isomorphic to an $A$-module of the form $A/p_i$, where $p_i$ is a prime associated to $M$.

Proof. Let $I$ be the collection of ideals in $A$ that are annihilators of elements in $M$. Then $I$ is not empty because it contains the zero ideal. Since $A$ is noetherian there is a maximal element $p = \text{Ann}(x)$ of $I$. We shall prove that $p$ is a prime ideal and thus associated to $M$. Let $f, g$ be elements in $A$ such that $fg \in p$ and $f \notin p$. Then $fx \neq 0$ and $\text{Ann}_A(fx) \supseteq Ag + p$. Since $p$ is maximal we must have that $p = \text{Ann}_A(fx)$ and thus that $g \in p$.

To prove the second part we let $\mathcal{L}$ be the collection of submodules of $M$ for which the Proposition holds. Then $\mathcal{L}$ is not empty because it contains the zero module. Since $A$ is noetherian and $M$ is finitely generated it follows from Proposition (2.5) that $M$ is noetherian. Thus there is a maximal element $L$ in $\mathcal{L}$. We shall show that $L = M$. Assume to the contrary that $L \neq M$. Then there is an associated prime ideal $p$ of $M/L$. Let $p = \text{Ann}(y)$ for some $y \in M/L$, and denote by $x$ an element of $M$ whose class in $M/L$ is $y$. We have an isomorphism $A/p \to Ay = (Ax + L)/L$. Since the Proposition holds for $L$ it will consequently hold for $Ax + L$. Hence $Ax + L$ is in $\mathcal{L}$, which is impossible since $L$ is maximal in $\mathcal{L}$. This contradicts the assumption that $L \neq M$. Hence we we must have that $L = M$, and the Proposition holds for $M$.

(3.2) Proposition. Let $A$ be a noetherian ring and let $M$ be an $A$-module. An element $f \in A$ is contained in an associated prime ideal if and only if there is an element $x \neq 0$ in $M$ such that $fx = 0$.

Proof. Let $\text{Ann}(x)$ be an associated prime ideal in $A$. If $f \in \text{Ann}(x)$ we have that $x \neq 0$ and $fx = 0$.

Conversely, assume that $fx = 0$ for some $x \neq 0$. It follows from Proposition (3.1) that $Ax$ has an associated prime ideal $p$. Then $p = \text{Ann}_A(gx)$ for some $g \in A$, and consequently $p$ is associated to $M$ and $f \in p$. Thus $f$ is contained in an associated ideal.

(3.3) Proposition. Let $A$ be a noetherian ring and let $M$ be an $A$-module.

1. The support of $M$ consists of the prime ideals in $A$ that contain an associated prime.

2. The intersection $\bigcap_{p \in \text{Supp}(M)} p$, which is thus the intersection of all associated ideals of $M$, consists of all elements $f \in A$ such that $f_M : M \to M$ is locally nilpotent.
Proof. (1) Assume that \( p \) is in the support of \( M \), that is, we have \( M_p \neq 0 \). Then there is an element \( x \in M \) such that \( (Ax)_p \neq 0 \). It follows from Proposition (3.1) that there is a prime ideal \( q \) that is associated to the \( A \)-module \( (Ax)_p \). Then there is an element \( f \in A \) and an element \( s \in A \setminus p \) such that \( q = \text{Ann}_A((fx)/s) \). We have that \( p \supseteq q \) because, if \( t \in q \setminus p \), then \((tfx)/s = 0\) and \( 0 = (1/t)((tfx)/s) = (fx)/s \) in \((Ax)_p\), contradicting that \((fx)/s \neq 0\) in \((Ax)_p\).

To prove the first part of the Proposition we prove that \( q \) is an associated prime ideal of \( M \). Let \( f_1, f_2, \ldots, f_n \) be generators for \( q \). Since \( q \) is the annihilator of the element \( fx/s \) in the \( A \)-module \( (Ax)_p \) we can find elements \( s_1, s_2, \ldots, s_n \) in \( A \setminus p \) such that \( s_if_1fx = 0 \) in \( M \) for \( i = 1, 2, \ldots, n \). Consequently we have an inclusion \( q \subseteq \text{Ann}_A(s_1s_2\cdots s_nfx) \). We shall prove the opposite inclusion. Take an element \( g \in \text{Ann}_A(s_1s_2\cdots s_nfx) \). Since \( s_1s_2\cdots s_nfx = 0 \) and \( s_1s_2\cdots s_n \notin p \), we have that \((gfx)/s = 0 \) in \((Ax)_p \). However, then we have that \( g \in q \), and we have proved that \( q = \text{Ann}_A(s_1s_2\cdots s_nfx) \). Hence the prime ideal \( q \) is associated to \( M \). We have proved that every ideal in the support contains an associated prime ideal. In Remark (MODULES 4.13) we saw that every associated prime ideal is contained in the support. Hence every prime ideal that contains an associated ideal is in the support.

(2) Assume that \( f \in A \) is not in the intersection of all the prime ideals in the support. Then there is a prime ideal \( p \) of \( A \) with \( M_p \neq 0 \) and \( f \notin p \). Let \( x \in M \) and \( s \notin p \) be such that \( x/s \neq 0 \) in \( M_p \). Since \( f \notin p \) we have that \( f^nx/s \neq 0 \) in \( M_p \), and thus \( f^nx \neq 0 \) in \( M \) for all positive integers \( n \). Consequently \( f \) is not locally nilpotent.

Finally let \( f \in A \) be an element in the intersection of all the ideals in the support of \( M \). We shall show that \( f_M \) is locally nilpotent. Assume to the contrary that \( f \) is not locally nilpotent. Then there is an \( x \in M \) such that \( f^n \notin \text{Ann}(x) \) for all positive integers \( n \). It follows from Proposition (RINGS 4.16) that we can find a prime ideal \( p \) that contains \( \text{Ann}(x) \) but does not contain \( f \). Then we have that \( (Ax)_p \neq 0 \), and thus \( p \) is contained in the support of \( M \). This contradicts the assumption that \( f \) is in the intersection of all ideals in the support. Hence we have proved that \( f_M \) is locally nilpotent.

\( \) \( (3.4) \) Remark. Let \( A \) be a noetherian ring and \( M \) a finitely generated \( A \)-module. It follows from Remark (MODULES 3.8) that the locally nilpotent elements are the elements of \( \text{r}(\text{Ann}(M)) \) and hence from Proposition (3.3) that the radical \( \text{r}(\text{Ann}(M)) \) of \( M \) is equal to the intersection of the prime ideals of the support of \( M \), or equivalently, to the intersection of the associated ideals of \( M \).

In particular we have that \( \text{Supp}(M) = V(\text{Ann}(M)) \).

\( \) \( (3.5) \) Proposition. Let \( A \) be a noetherian ring and let \( M \) be an \( A \)-module. The following assertions are equivalent:

\( 1 \) The module \( M \) has exactly one associated prime ideal.

\( 2 \) We have that \( M \neq 0 \) and for every element \( f \) in \( A \) either \( f_M \) is injective or
locally nilpotent.

When the assertions hold the associated ideal of $M$ consists of the locally nilpotent elements.

Proof. $(1) \Rightarrow (2)$ If there is only one associated prime ideal $p$ it follows from Proposition (3.3) that when $f \in p$ the map $f_M$ is locally nilpotent. Moreover it follows from Proposition (3.2) that for $f \notin p$ the map $f_M$ is injective.

$(2) \Rightarrow (1)$ If $f_M$ is nilpotent it follows from Proposition (3.2) that the elements $f \in A$ is contained in some associated prime ideal. On the other hand, if $f_M$ is injective, it follows from Proposition (3.2) that $f$ is not contained in some associated ideal. Hence it follows from Proposition (?) that the union of the associated prime ideals will be equal to their intersection. Hence there can be only one associated prime ideal for $M$.

We saw in the proofs of both $(1) \Rightarrow (2)$ and $(2) \Rightarrow (1)$ that when the assertions of the Proposition holds then the associated ideal consists of the locally nilpotent elements.

(3.6) Corollary. Let $A$ be a noetherian ring and $M$ a finitely generated $A$-module. Moreover let $L$ be a submodule of $M$. The following conditions are equivalent:

1. The module $M/L$ has only one associated ideal.
2. The module $L$ is primary.

When the conditions hold the associated prime ideal of $M/L$ is the prime ideal belonging to $L$.

Proof. $(1) \Rightarrow (2)$ Let $p$ be the associated prime ideal of $M/L$. By the Proposition and Remark (4.8) we have that $M \neq L$ and that $L$ is primary and $p$ is the ideal belonging to $L$.

$(2) \Rightarrow (1)$ If $M \neq L$ and $L$ is primary it follows from the Proposition that $M/L$ has only one associated ideal.

(3.7) Proposition. Let $A$ be a noetherian ring and let $M$ be a finitely generated $A$-module. Then every submodule $L$ of $M$ can be written as an intersection $L = L_1 \cap L_2 \cap \cdots \cap L_n$ of submodules $L_i$ of $M$ such that each module $L_i$ is primary.

Proof. Consider the set $\mathcal{L}$ of submodules $L$ of $M$ that can not be written as $L = L_1 \cap L_2 \cap \cdots \cap L_n$ with all $L_i$ primary. We shall show that $\mathcal{L}$ is empty. Assume to the contrary that it not empty. Since $M$ is noetherian $\mathcal{L}$ then has a maximal element $L$. In particular $L$ is not primary. Thus there is an element $f \in A$ such that the homomorphism $f_{M/L} : M/L \to M/L$ is neither injective nor nilpotent. We therefore obtain a sequence

$$\text{Ker}(f_{M/L}) \subseteq \text{Ker}(f_{M/L}^2) \subseteq \cdots$$

of non-zero proper submodules of $M$. Since $M$ is noetherian this sequence must stop. Assume that $\text{Ker}(f_{M/L}^r) = \text{Ker}(f_{M/L}^{r+1}) = \cdots$ and let $u = f_{M/L}^r$. We have that $\text{Ker}(u)$ is a proper submodule of $M$ and that $\text{Ker}(u) = \text{Ker}(u^2)$. Consequently
Ker(u) \cap \text{Im}(u) = (0). In particular \text{Im}(u) is different from M/L. Let M_1 and M_2 be the inverse images of Ker(u) respectively \text{Im}(u) by the canonical map u_{M/L} : M \to M/L. Then M_1 and M_2 contain L and are different from L, and L = M_1 \cap M_2. By the maximality of L we have that the Proposition holds for M_1 and M_2. Consequently the Proposition holds for L which is impossible since L is in \mathcal{L}. This contradicts the assumption that \mathcal{L} is non-empty, and we have proved the Proposition.

(3.8) Proposition. Let A be a noetherian ring and M a finitely generated A-module. Write 0 = L_1 \cap L_2 \cap \cdots \cap L_n with L_i primary for i = 1, 2, \ldots, n, and assume that for each i we have L_i \not\supseteq \cap_{i \neq j} L_j. Then the associated primes of M coincide with the primes belonging to the primary modules L_i.

Proof. We have an injection
\[ M \to M/L_1 \oplus M/L_2 \oplus \cdots M/L_n \]
which sends x \in N to (u_{M/L_1}(x), u_{M/L_2}(x), \ldots, u_{M/L_n}(x)). It follows from Proposition (MODULES 4.25) that the associated prime ideals of M can be found among the associated primes of M/L_1, M/L_2, \ldots, M/L_n. We shall show that the prime ideal p_i belonging to L_i is associated to M for i = 1, 2, \ldots, n.

We have that L = L_2 \cap L_3 \cap \cdots \cap L_{i-1} \cap L_{i+1} \cdots \cap L_n \neq (0) by assumption. Since L = L/L \cap L_i it follows from Lemma (MODULES 1.13) that we have an injective A-module homomorphism L \to M/L_i. It follows from Proposition (3.1) that L has an associated ideal, and from Corollary (3.6) that this ideal must be p_i. From Proposition (MODULES 4.25) it follows that p_i is also associated to M.

(3.9) Proposition. Let A be a noetherian ring. If A is reduced the associated primes are the minimal prime ideals.

Proof. It follows from Proposition (3.3) that every prime ideal contains an associated prime. Hence every minimal prime ideal is associated.

Conversely let p = \text{Ann}(f) be an associated prime ideal of A. In Remark (2.8) we observed that A has only a finite number of minimal primes p_1, p_2, \ldots, p_n. Assume that p is not minimal. The it follows from Proposition (RINGS 4.22) that we can find an element t \in p \setminus p_1 \cup \cdots \cup p_n. Then tf = 0, and consequently tf = 0. Thus f \in p_1 \cap \cdots \cap p_n. However the intersection of the minimal prime ideals is the radical of A and thus f^n = 0 for some integer n. Since A is reduced f = 0, which is impossible since f \neq 0. This contradicts the assumption that p is not minimal, and we have proved that the associated prime ideals are minimal.

(3.10) Exercises.
1. Find the associated prime ideals of the \mathbf{Z}-module \mathbf{Z}/12\mathbf{Z}, and write (0) in \mathbf{Z}/12\mathbf{Z} as an intersection of primary modules.

2. Let K be a field and let K[u, v] be the polynomial ring over K in the independent variables u, v.

(1) Find the associated prime ideals of the K[u, v]-module M = K[u, v]/(u^2, uv).
(2) Write \((0) \in M\) as an intersection of primary modules.

3. Let \(A\) be a noetherian ring and let \(q\) be a \(p\)-primary ideal. Show that \(p^n \subseteq q\) for some positive integer \(n\).

4. Let \(A\) be a ring. An ideal \(a\) of \(A\) is irreducible if \(a = b \cap c\) implies that \(a = b\) or that \(a = c\).

   (1) Show that \(a\) is irreducible in \(A\) if and only if \((0)\) is irreducible in the residue ring \(A/a\).

   (2) Show that \(a\) is primary in \(A\) if and only if \((0)\) is primary in the residue ring \(A/a\).

   (3) Show that when \(A\) is a noetherian ring then every ideal in \(A\) is the intersection of irreducible ideals.

   (4) Assume that \(A\) is noetherian the ideal \((0)\) in \(A\) is irreducible. Let \(fg = 0\) with \(g \neq 0\) in \(A\). Let \(n\) be such that \(\text{Ann}(f^n) = \text{Ann}(f^{n+1} = \cdots\). Show that \((f^n) \cap (g) = 0\).

   (5) Show that when \(A\) is noetherian then every irreducible ideal is primary.

5. Let \(A\) be a noetherian ring. Moreover let \(m\) be a maximal ideal and \(q\) an ideal contained in \(m\). Show that the following assertions are equivalent:

   (1) The ideal \(q\) is \(m\)-primary.

   (2) \(r(q) = m\).

   (3) There is a positive integer \(n\) such that \(m^n \subseteq q \subseteq m\).