2. Artinian and noetherian rings.

(2.1) Definition. A ring A is *noetherian*, respectively *artinian*, if it is noetherian, respectively artinian, considered as an A-module. In other words, the ring A is noetherian, respectively artinian, if every chain $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ of ideal $\mathfrak{a}_i$ in A is stable, respectively if every chain $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ of ideals $\mathfrak{a}_i$ in A is stable.

(2.2) Example. Let $K[t]$ be the polynomial ring in the variable t with coefficients in a field K. Then the residue ring $K[t]/(t^n)$ is artinian and noetherian for all positive integers n. This is because $K[t]/(t^n)$ is a finite dimensional vector space of dimension n.

(2.3) Example. The ring $\mathbf{Z}$ is noetherian, but not artinian. All rings with a finite number of ideals, like $\mathbf{Z}/n\mathbf{Z}$ for $n \in \mathbf{Z}$, and fields are artinian and noetherian.

(2.4) Example. The polynomial ring $A[t_1, t_2, \dots]$ in the variables $t_1, t_2, \dots$ over a ring A is not noetherian since it contains the infinite chain $(t_1) \subset (t_1, t_2) \subset \cdots$ of ideals. It is not artinian either since it contains the infinite chain $(t_1) \supset (t_1^2) \supset (t_1^3) \supset \cdots$.

(2.5) Proposition. *Let A be a ring and let M be a finitely generated A-module.*

(1) *If A is a noetherian ring then M is a noetherian A-module.*
(2) *If A is an artinian ring then M is an artinian A-module.*

→ *Proof.* (1) It follows from Proposition (MODULES 1.20) that we have a surjective map $\varphi : A^{\oplus n} \to M$ from the sum of the ring A with itself n times to M. Hence it
→ follows from Proposition (1.7) that M is noetherian.

(2) The proof of the second part is analogous to the proof of the first part.

(2.6) Corollary. *Let $\varphi : A \to B$ be a surjective map from the ring A to a ring B.*

(1) *If the ring A is noetherian then the ring B is noetherian.*
(2) *If the ring A is artinian then the ring B is artinian.*

Proof. (1) Since φ is surjective B is a finitely generated A-module with generator 1. It follows from the Proposition that B is noetherian as an A-module. Then B is clearly noetherian as a B-modules.

(2) The proof of the second part is analogous to the proof of the first part.

(2.7) Proposition. *Let S be a multiplicatively closed subset of a ring A.*

(1) *If A is noetherian then $S^{-1}A$ is noetherian.*
(2) *If A is artinian then $S^{-1}A$ is artinian.*

→ *Proof.* (1) It follows from Remark (MODULES 3.13) that every ideal $\mathfrak{b}$ in the localization $S^{-1}A$ satisfies $\varphi_{S^{-1}A}(\mathfrak{b})S^{-1}A = \mathfrak{b}$. Every chain $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$ of ideals in $S^{-1}A$ therefore gives a chain $\varphi_{S^{-1}A}^{-1}(\mathfrak{b}_1) \subseteq \varphi_{S^{-1}A}^{-1}(\mathfrak{b}_2) \subseteq \cdots$ of ideals in A. Since A is noetherian there is a positive integer n such that $\varphi_{S^{-1}A}^{-1}(\mathfrak{b}_n) = \varphi_{S^{-1}A}^{-1}(\mathfrak{b}_{n+1}) = \cdots$. Consequently we have that $\mathfrak{b}_n = \mathfrak{b}_{n+1} = \cdots$. Hence $S^{-1}A$ is noetherian.

(2) The proof of the second part is analogous to the proof of the first part.
chains2

(2.8) Remark. A noetherian ring has only a finite number of minimal prime ideals. This is because Spec(A) is a noetherian topological space since the descending chains of closed subsets of Spec(A) correspond to ascending chains of ideals in A by Remark (RINGS 5.2). By Proposition (TOPOLOGY 4.25) Spec(A) has only a finite number of irreducible components. However, it follows from Proposition (TOPOLOGY 5.13) that the irreducible components of Spec(A) correspond bijectively to the minimal prime ideals in A.

(2.9) Remark. The radical $\mathfrak{r}(A)$ of a noetherian ring A is nilpotent, that is, we have $\mathfrak{r}(A)^n = 0$ for some integer n. This follows from Remark (RINGS 4.8) because $\mathfrak{r}(A)$ is finitely generated ideal.

(2.10) Theorem. (The Hilbert basis theorem) *Let A be a noetherian ring and B a finitely generated algebra over A. Then B is a noetherian ring.*

Proof. It follows from Proposition (RINGS 3.6) that we have a surjective homomorphism $A[t_1, t_2, \dots, t_n] \to B$ of A-algebras from the polynomial ring $A[t_1, t_2, \dots, t_n]$ in the variables $t_1, t_2, \dots, t_n$ over A. Hence it follows from Corollary (2.6) that is suffices to prove that the polynomial ring $A[t_1, t_2, \dots, t_n]$ is noetherian. If we can prove that the polynomial ring $C[t]$ in one variable t over a noetherian ring C is noetherian, it clearly follows by induction on n that $A[t_1, t_2, \dots, t_n]$ is noetherian. Hence it suffices to prove that $A[t]$ is noetherian.

Let $\mathfrak{b}$ be an ideal in $A[t]$. We shall show that $\mathfrak{b}$ has a finite number of generators. Let $\mathfrak{a}$ be the collection of elements $f \in A$ such that there is a polynomial $f_0 + f_1 t + \cdots + f_{n-1}t^{n-1} + ft^n$ in $\mathfrak{b}$. It is clear that $\mathfrak{a}$ is an ideal in A. Since A is noetherian we can find generators $g_1, g_2, \dots, g_m$ of $\mathfrak{a}$. For every $i = 1, 2, \dots, m$ we can find a polynomial $p_i(t) = g_{i,0} + g_{i,1}t + \cdots + g_{i,d_i-1}t^{d_i-1} + g_i t^{d_i}$ in $\mathfrak{b}$. Let $d = \max_{i=1}^m(d_i)$.

For each polynomial $f(t) = f_0 + f_1 t + \cdots + f_e t^e$ in $\mathfrak{b}$ we can find elements $h_1, h_2, \dots, h_m$ in A such that $f_e = h_1 g_1 + h_2 g_2 + \cdots + h_m g_m$. If $e \geq d$ the polynomial $f(t) = h_1 t^{e-d_1} p_1(t) - h_2 t^{e-d_2} p_2(t) - \cdots - h_m t^{d-d_m} p_m(t)$ is of degree stricly less than e. It follows by descending induction on e that we can find polynomials $h_1(t), h_2(t), \dots, h_m(t)$ such that $g(t) = f(t) - \sum_{i=1}^m h_i(t)p_i(t)$ is of degree strictly less than d. Since $f(t) \in \mathfrak{b}$, and all the polynomials $p_i(t)$ are in $\mathfrak{b}$, we have that $g(t) \in \mathfrak{b}$. Hence $g(t)$ is in the A-module $M = (A + tA + \cdots + t^{d-1}A) \cap \mathfrak{b}$. It follows from Corollary (1.8) and Proposition (1.7) that M is a noetherian module. Hence we can find a finite number of generators $q_1(t), q_2(t), \dots, q_n(t)$ of M. Then $\mathfrak{b}$ will be generated by the polynomials $p_1(t), p_2(t), \dots, p_m(t), q_1(t), q_2(t), \dots, q_n(t)$. Hence $\mathfrak{b}$ is finitely generated as we wanted to prove. Since all ideals $\mathfrak{b}$ of B are finitely generated it follows from Lemma (1.6) that B is noetherian as a module over itself, and hence noetherian.

(2.11) Proposition. *In an artinian ring all the prime ideals are maximal.*

Proof. Let $\mathfrak{p}$ be a prime ideal. We must show that for each element $f \in A \setminus \mathfrak{p}$ we have that $Af + \mathfrak{p} = A$. Since A is artinian the chain $Af + \mathfrak{p} \supseteq Af^2 + \mathfrak{p} \supset \cdots$ must stabilize. Hence there is a positive integer n such that $f^n = gf^{n+1} + h$ for some $g \in A$ and $h \in \mathfrak{p}$. Hence $f^n(1 - gf) \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal and $f \notin \mathfrak{p}$ we have that $1 - gf \in \mathfrak{p}$. Hence there is an $e \in \mathfrak{p}$ such that $1 - gf = e$. The ideal $Af + \mathfrak{p}$ consequently contains the element $gf - e = 1$ and thus is equal to A is we wanted to prove.

(2.12) Proposition. *Let A be a ring and $\mathfrak{m}_1, \mathfrak{m}_2, \cdots$ different maximal ideals in A. Then $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$ is a proper submodule of $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}$.*

Proof. Since the ideals $\mathfrak{m}_i$ are maximal we can for each $i = 1, 2, \ldots, n-1$ find an element $f_i \in \mathfrak{m}_i \setminus \mathfrak{m}_n$. Assume that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1} = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$. Then we have that $f_1 f_2 \cdots f_{n-1} \in \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1} = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n \subseteq \mathfrak{m}_n$, which is impossible since $\mathfrak{m}_n$ is a prime ideal and $f_i \notin \mathfrak{m}_n$ for $i = 1, 2, \ldots, n-1$. This contradicts the assumption that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}$. Hence $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n$ is a proper submodule of $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{n-1}$.

(2.13) Corollary. *An artinian ring has a finite number of maximal ideals.*

Proof. If it had an infinte number of maximal ideals we could find an infinite sequence $\mathfrak{m}_1, \mathfrak{m}_2, \cdots$ of different maximal ideals. Then it follows from the Proposition that we have an infinite chain $\mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots$ of ideals in A. This contradicts that A is artinian. Thus A has only a finite number of maximal ideals.

(2.14) Proposition. *In an artinian ring the radical is nilpotent.*

Proof. Since A is artinian the sequence of ideals $\mathfrak{r}(A) \supseteq \mathfrak{r}(A)^2 \supseteq \cdots$ is stable. Thus there is a positive integer n such that $\mathfrak{a} := \mathfrak{r}(A)^n = \mathfrak{r}(A)^{n+1} = \cdots$. We shall prove
n that $\mathfrak{a} = 0$. Assume to the contrary that $\mathfrak{a} \neq 0$. Consider the collection !!$\mathcal{B}$ of ideals $\mathfrak{b}$ in A such that $\mathfrak{a}\mathfrak{b} \neq 0$. Then $\mathcal{B}$ is not empty since $\mathfrak{a}$ is in $\mathcal{B}$. Since A is artinian we have that $\mathcal{B}$ contains a minimal element $\mathfrak{c}$. Then there is an $f \in \mathfrak{c}$ such that $\mathfrak{a}f \neq 0$. Since $\mathfrak{c}$ is minimal in $\mathcal{B}$ and $(f) \subseteq \mathfrak{c}$ we must have that $\mathfrak{c} = (f)$. We have that $(f\mathfrak{a})\mathfrak{a} = f\mathfrak{a}^2 = f\mathfrak{a} \neq 0$ and $(f\mathfrak{a}) \subseteq (f) = \mathfrak{c}$. By the minimality of $\mathfrak{c}$ we obtain that $(f\mathfrak{a}) = (f)$. Hence there is an element $g \in \mathfrak{a}$ such that $fg = f$. Hence $f = fg = fg^2 = \cdots$. However, since $g \in \mathfrak{a} \subseteq \mathfrak{r}(A)$, we have that $g^n = 0$ for some positive integer n. Thus $f = 0$ which is impossible since $\mathfrak{a}f = \mathfrak{a}\mathfrak{c} \neq 0$. This contradicts the assumption that $\mathfrak{a} \neq 0$. Hence $\mathfrak{a} = 0$ as we wanted to prove.

(2.15) Lemma. *Let A be a ring and let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, m_n$ be, not necessarily different, maximal ideals in A such that $\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$. Then A is artinian if and only if A is noetherian.*

Proof. We have a chain $A = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \supseteq \cdots \supseteq \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_n = 0$ of ideals in A. Let $M_i = \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{i-1}/\mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_i$ for $i = 1, 2, \ldots, n$. Then each M_i is an $A/\mathfrak{m}_i$-module, that is, a vector space over $A/\mathfrak{m}_i$. Hence M_i is artinian if and only if it is noetherian. For $i = 1, 2, \ldots, n$ we have an exact sequence

$$0 \to \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_i \to \mathfrak{m}_1\mathfrak{m}_2 \cdots \mathfrak{m}_{i-1} \to M_i \to 0.$$

It follows from Proposition (1.7) that $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_i$ and M_i are artinian, respectively noetherian, if and only if $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{i-1}$ is artinian, respectively noetherian. By descending induction on i starting with $M_n = \mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_{n-1}$ we obtain that the module $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_i$ is artinian if and only if it is noetherian. For $i = 0$ we obtain that A is artinian if and only if it is noetherian.

(2.16) Remark. Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$, and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. Then $A/\mathfrak{q}$ is an artinian ring. To show this we first note that $\mathfrak{m} = \mathfrak{r}(\mathfrak{q})$. Since A is noetherian $\mathfrak{m}$ is finitely generated, and thus it follows from Remark (RINGS 4.8) that a power of the maximal ideal in the noetherian local ring $A/\mathfrak{q}$ is zero. Hence it follows from Lemma (2.15) that $A/\mathfrak{q}$ is artinian.

(2.17) Theorem. *A ring is artinian if and only if it noetherian and has dimension 0.*

Proof. When A is artinian it follows from Proposition (2.11) that $\dim(A) = 0$. It follows from Corollary (2.13) that the ring A has a finite number of maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$. We have that $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n \subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n \subseteq \mathfrak{r}(A)$. Since $\mathfrak{r}(A)$ is nilpotent by Proposition (2.14) it follows from Lemma (2.15) that A is noetherian.

Conversely assume that A is noetherian of dimension 0. Then every prime ideal is maximal, and from Remark (2.8) it follows that A has finitely many maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n$. Again $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n \subseteq \mathfrak{r}(A)$. If follows from Remark (RINGS 4.8) that $\mathfrak{r}(A)$ is nilpotent. Hence it follows from Lemma (2.15) that A is artinian.

(2.18) Proposition. *An artinian ring is isomorphic to the direct product of a finite number of local artin rings.*

More precicely, when A is an artinian ring the canonical map $A \to \prod_{x\in\operatorname{Spec}(A)} A_{\mathfrak{j}_x}$ obtained from the localization maps $A \to A_{\mathfrak{j}_x}$ is an isomorphism.

Proof. By Corollary (2.13) we have that $\operatorname{Spec}(A)$ consists of a finite number of points, and by Proposition (2.11) the points are closed. Hence $\operatorname{Spec}(A)$ is a discrete topological space. Since $\mathcal{O}_{\operatorname{Spec}(A)}$ is a sheaf there is an injective map $A = \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) \to \prod_{x\in\operatorname{Spec}(A)} A_{\mathfrak{j}_x} = \prod_{x\in\operatorname{Spec}(A)} \mathcal{O}_{\operatorname{Spec}(A),x}$. However each point x is open in $\operatorname{Spec}(A)$. Hence $A_{\mathfrak{j}_x} = \Gamma(\{x\}, \mathcal{O}_{\operatorname{Spec}(A)})$, and $\{x\} \cap \{y\} = \emptyset$ when $x \neq y$. It follows that we can glue any collection of sections $s_x \in \Gamma(\{x\}, \mathcal{O}_{\operatorname{Spec}(A)})$ for $x \in \operatorname{Spec}(A)$ to a section $s \in \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$. Hence the map $A \to \prod_{x\in\operatorname{Spec}(A)} \prod A_{\mathfrak{j}_x}$ is also surjective.

(2.19) Exercises.

1. Show that if S is a multiplicatively closed subset of a ring A such that $S^{-1}A$ is noetherian. Then A is not necessarily noetherian.

2. Let $K[t_1, t_2, \dots]$ be the polynomial ring in the infinitely many variables $t_1, t_2, \dots$ over a field K. Morever let $K(t_1, t_2, \dots)$ be the localization of $K[t_1, t_2, \dots]$ in the multiplicatively closed subset of $K[t_1, t_2, \dots]$ consisting of all non-zero elements.

(1) Show that $K(t_1, t_2, \dots)$ is noetherian.

(2) Show that $K(t_1, t_2, \dots) \otimes_K K(t_1, t_2, \dots)$ is not Noetherian.

3. Let A a ring. Give an example of a ring A that is not noetherian, but is such that $\operatorname{Spec}(A)$ is noetherian.

4. Let M be a noetherian A-module. Show that the ring $A/\operatorname{Ann}_A(M)$ is noetherian.

5. Prove that there is only a finite number of minimal primes in a noetherian ring A without using properties of the topological space $\operatorname{Spec}(A)$.

6. Let A be a ring. We say that two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A are *coprime* if $\mathfrak{a} + \mathfrak{b} = A$. Let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n$ be ideals of A that are pairwise comprime. We define a map

$$\varphi : A \to \prod_{i=1}^{n} A/\mathfrak{a}_i$$

by $\varphi(f) = (\varphi_{A/\mathfrak{a}_1}(f), \varphi_{A/\mathfrak{a}_2}(f), \dots, \varphi_{A/\mathfrak{a}_n}(f))$ for all $f \in A$.

(1) Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are coprime, then $\mathfrak{a}^m$ and $\mathfrak{b}^n$ are coprime for all positive integers m and n.
(2) Show that for all i the ideals $\mathfrak{a}_i$ and $\cap_{i \neq j} \mathfrak{a}_j$ are coprime.
(3) Show that the homomorphism φ is a ring homomorphism with kernel $\cap_{i=1}^{n} \mathfrak{a}_i$.
(4) Show that the homomorphism φ is surjective.
(5) Use parts (1), (2), (3), and (4) to prove that an artin ring is the direct product of a finite number of artinian rings.