2. Artinian and noetherian rings.

(2.1) Definition. A ring $A$ is noetherian, respectively artinian, if it is noetherian, respectively artinian, considered as an $A$-module. In other words, the ring $A$ is noetherian, respectively artinian, if every chain $a_1 \subseteq a_2 \subseteq \cdots$ of ideal $a_i$ in $A$ is stable, respectively if every chain $a_1 \supseteq a_2 \supseteq \cdots$ of ideals $a_i$ in $A$ is stable.

(2.2) Example. Let $K[t]$ be the polynomial ring in the variable $t$ with coefficients in a field $K$. Then the residue ring $K[t]/(t^n)$ is artinian and noetherian for all positive integers $n$. This is because $K[t]/(t^n)$ is a finite dimensional vector space of dimension $n$.

(2.3) Example. The ring $\mathbb{Z}$ is noetherian, but not artinian. All rings with a finite number of ideals, like $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{Z}$, and fields are artinian and noetherian.

(2.4) Example. The polynomial ring $A[t_1, t_2, \ldots]$ in the variables $t_1, t_2, \ldots$ over a ring $A$ is not noetherian since it contains the infinite chain $(t_1) \subset (t_1, t_2) \subset \cdots$ of ideals. It is not artinian either since it contains the infinite chain $(t_1) \supset (t_1^2) \supset (t_1^3) \supset \cdots$.

(2.5) Proposition. Let $A$ be a ring and let $M$ be a finitely generated $A$-module.

1. If $A$ is a noetherian ring then $M$ is a noetherian $A$-module.
2. If $A$ is an artinian ring then $M$ is an artinian $A$-module.

→ Proof. (1) It follows from Proposition (MODULES 1.20) that we have a surjective map $\varphi : A^\oplus n \to M$ from the sum of the ring $A$ with itself $n$ times to $M$. Hence it follows from Proposition (1.7) that $M$ is noetherian.
→ (2) The proof of the second part is analogous to the proof of the first part.

(2.6) Corollary. Let $\varphi : A \to B$ be a surjective map from the ring $A$ to a ring $B$.

1. If the ring $A$ is noetherian then the ring $B$ is noetherian.
2. If the ring $A$ is artinian then the ring $B$ is artinian.

Proof. (1) Since $\varphi$ is surjective $B$ is a finitely generated $A$-module with generator $1$. It follows from the Proposition that $B$ is noetherian as an $A$-module. Then $B$ is clearly noetherian as a $B$-modules.
→ (2) The proof of the second part is analogous to the proof of the first part.

(2.7) Proposition. Let $S$ be a multiplicatively closed subset of a ring $A$.

1. If $A$ is noetherian then $S^{-1}A$ is noetherian.
2. If $A$ is artinian then $S^{-1}A$ is artinian.

→ Proof. (1) It follows from Remark (MODULES 3.13) that every ideal $b$ in the localization $S^{-1}A$ satisfies $\varphi_{S^{-1}A}(b)S^{-1}A = b$. Every chain $b_1 \subseteq b_2 \subseteq \cdots$ of ideals in $S^{-1}A$ therefore gives a chain $\varphi_{S^{-1}A}(b_1) \subseteq \varphi_{S^{-1}A}(b_2) \subseteq \cdots$ of ideals in $A$. Since $A$ is noetherian there is a positive integer $n$ such that $\varphi_{S^{-1}A}(b_n) = \varphi_{S^{-1}A}(b_{n+1}) = \cdots$. Consequently we have that $b_n = b_{n+1} = \cdots$. Hence $S^{-1}A$ is noetherian.
→ (2) The proof of the second part is analogous to the proof of the first part.
(2.8) Remark. A noetherian ring has only a finite number of minimal prime ideals. This is because Spec($A$) is a noetherian topological space since the descending chains of closed subsets of Spec($A$) correspond to ascending chains of ideals in $A$ by Remark (RINGS 5.2). By Proposition (TOPOLOGY 4.25) Spec($A$) has only a finite number of irreducible components. However, it follows from Proposition (TOPOLOGY 5.13) that the irreducible components of Spec($A$) correspond bijectively to the minimal prime ideals in $A$.

(2.9) Remark. The radical $\tau(A)$ of a noetherian ring $A$ is nilpotent, that is, we have $\tau(A)^n = 0$ for some integer $n$. This follows from Remark (RINGS 4.8) because $\tau(A)$ is finitely generated ideal.

(2.10) Theorem. (The Hilbert basis theorem) Let $A$ be a noetherian ring and $B$ a finitely generated algebra over $A$. Then $B$ is a noetherian ring.

Proof. It follows from Proposition (RINGS 3.6) that we have a surjective homomorphism $A[t_1, t_2, \ldots, t_n] \to B$ of $A$-algebras from the polynomial ring $A[t_1, t_2, \ldots, t_n]$ in the variables $t_1, t_2, \ldots, t_n$ over $A$. Hence it follows from Corollary (2.6) that suffices to prove that the polynomial ring $A[t_1, t_2, \ldots, t_n]$ is noetherian. If we can prove that the polynomial ring $C[t]$ in one variable $t$ over a noetherian ring $C$ is noetherian, it clearly follows by induction on $n$ that $A[t_1, t_2, \ldots, t_n]$ is noetherian. Hence it suffices to prove that $A[t]$ is noetherian.

Let $b$ be an ideal in $A[t]$. We shall show that $b$ has a finite number of generators. Let $a$ be the collection of elements $f \in A$ such that there is a polynomial $f_0 + f_1t + \cdots + f_{n-1}t^{n-1} + ft^n$ in $b$. It is clear that $a$ is an ideal in $A$. Since $A$ is noetherian we can find generators $g_1, g_2, \ldots, g_m$ of $a$. For every $i = 1, 2, \ldots, m$ we can find a polynomial $p_i(t) = g_{i,0} + g_{i,1}t + \cdots + g_{i,d_i}t^{d_i} + gi t^{d_i}$ in $b$. Let $d = \max_{i=1}^m (d_i)$.

For each polynomial $f(t) = f_0 + f_1t + \cdots + ft^n$ in $b$ we can find elements $h_1, h_2, \ldots, h_m$ in $A$ such that $f_i = h_1g_1 + h_2g_2 + \cdots + h_m g_m$. If $e \geq d$ the polynomial $f(t) = h_1t^e - d_1p_1(t) - h_2t^e - d_2p_2(t) - \cdots - h_m t^e - d_m p_m(t)$ is of degree strictly less than $e$. It follows by descending induction on $e$ that we can find polynomials $h_1(t), h_2(t), \ldots, h_m(t)$ such that $g(t) = f(t) - \sum_{i=1}^m h_i(t)p_i(t)$ is of degree strictly less than $d$. Since $f(t) \in b$, and all the polynomials $p_i(t)$ are in $b$, we have that $g(t) \in b$. Hence $g(t)$ is in the $A$-module $M = (A + tA + \cdots + t^{d-1}A) \cap b$. It follows from Corollary (1.8) and Proposition (1.7) that $M$ is a noetherian module. Hence we can find a finite number of generators $q_1(t), q_2(t), \ldots, q_n(t)$ of $M$. Then $b$ will be generated by the polynomials $p_1(t), p_2(t), \ldots, p_m(t), q_1(t), q_2(t), \ldots, q_n(t)$. Hence $b$ is finitely generated as we wanted to prove. Since all ideals $b$ of $B$ are finitely generated it follows from Lemma (1.6) that $B$ is noetherian as a module over itself, and hence noetherian.

(2.11) Proposition. In an artinian ring all the prime ideals are maximal.

Proof. Let $p$ be a prime ideal. We must show that for each element $f \in A \setminus p$ we have that $Af + p = A$. Since $A$ is artinian the chain $Af + p \supset Af^2 + p \supset \cdots$ must
have an infinite chain stabilize. Hence there is a positive integer \( n \) such that \( f^n = gf^{n+1} + h \) for some \( g \in A \) and \( h \in p \). Hence \( f^n(1 - gf) \in p \). Since \( p \) is a prime ideal and \( f \notin p \) we have that \( 1 - gf \in p \). Hence there is an \( e \in p \) such that \( 1 - gf = e \). The ideal \( Af + p \) consequently contains the element \( gf - e = 1 \) and thus is equal to \( A \) is we wanted to prove.

(2.12) Proposition. Let \( A \) be a ring and \( m_1, m_2, \ldots \) different maximal ideals in \( A \). Then \( m_1m_2 \cdots m_n \) is a proper submodule of \( m_1m_2 \cdots m_{n-1} \).

Proof. Since the ideals \( m_i \) are maximal we can for each \( i = 1, 2, \ldots, n - 1 \) find an element \( f_i \in m_i \setminus m_{i+1} \). Assume that \( m_1m_2 \cdots m_{n-1} = m_1m_2 \cdots m_n \). Then we have that \( f_1f_2 \cdots f_{n-1} \in m_1m_2 \cdots m_{n-1} = m_1m_2 \cdots m_n \subseteq m_i \), which is impossible since \( m_i \) is a prime ideal and \( f_i \notin m_i \) for \( i = 1, 2, \ldots, n - 1 \). This contradicts the assumption that \( m_1m_2 \cdots m_n = m_1m_2 \cdots m_{n-1} \). Hence \( m_1m_2 \cdots m_n \) is a proper submodule of \( m_1m_2 \cdots m_{n-1} \).

(2.13) Corollary. An artinian ring has a finite number of maximal ideals.

Proof. If it had an infinite number of maximal ideals we could find an infinite sequence \( m_1, m_2, \ldots \) of maximal ideals. Then it follows from the Proposition that we have an infinite chain \( m_1 \supset m_1m_2 \supset \cdots \) of ideals in \( A \). This contradicts that \( A \) is artinian. Thus \( A \) has only a finite number of maximal ideals.

(2.14) Proposition. In an artinian ring the radical is nilpotent.

Proof. Since \( A \) is artinian the sequence of ideals \( r(A) \supset r(A)^2 \supset \cdots \) is stable. Thus there is a positive integer \( n \) such that \( a := r(A)^n = r(A)^{n+1} = \cdots \). We shall prove that \( a = 0 \). Assume to the contrary that \( a \neq 0 \). Consider the collection \( \mathcal{B} \) of ideals \( b \) in \( A \) such that \( ab \neq 0 \). Then \( \mathcal{B} \) is not empty since \( a \) is in \( \mathcal{B} \). Since \( A \) is artinian we have that \( \mathcal{B} \) contains a minimal element \( c \). Then there is an \( f \in c \) such that \( af \. Since \( c \) is minimal in \( \mathcal{B} \) and \( (f) \subseteq c \) we must have that \( c = (f) \). We have that \( (fa)a = fa^2 = fa \neq 0 \) and \( (fa) \subseteq (f) = c \). By the minimality of \( c \) we obtain that \( (fa) = (f) \). Hence there is an element \( g \in a \) such that \( fg = f \). Hence \( f = fg = fg^2 = \cdots \). However, since \( g \in a \subseteq r(A) \), we have that \( g^n = 0 \) for some positive integer \( n \). Thus \( f = 0 \) which is impossible since \( af = ac \neq 0 \). This contradicts the assumption that \( a \neq 0 \). Hence \( a = 0 \) as we wanted to prove.

(2.15) Lemma. Let \( A \) be a ring and let \( m_1, m_2, \ldots, m_n \) be, not necessarily different, maximal ideals in \( A \) such that \( m_1m_2 \cdots m_n = 0 \). Then \( A \) is artinian if and only if \( A \) is noetherian.

Proof. We have a chain \( A = m_0 \supset m_1 \supset m_1m_2 \supset m_1m_2m_3 \supset \cdots \supset m_1m_2 \cdots m_n = 0 \) of ideals in \( A \). Let \( M_i = m_1m_2 \cdots m_{i-1}/m_1m_2 \cdots m_i \) for \( i = 1, 2, \ldots, n \). Then each \( M_i \) is an \( A/m_i \)-module, that is, a vector space over \( A/m_i \). Hence \( M_i \) is artinian if and only if it is noetherian. For \( i = 1, 2, \ldots, n \) we have an exact sequence

\[ 0 \rightarrow m_1m_2 \cdots m_i \rightarrow m_1m_2 \cdots m_{i-1} \rightarrow M_i \rightarrow 0. \]
It follows from Proposition (1.7) that $m_1m_2\cdots m_i$ and $M_i$ are artinian, respectively noetherian, if and only if $m_1m_2\cdots m_{i-1}$ is artinian, respectively noetherian. By descending induction on $i$ starting with $M_n = m_1m_2\cdots m_{n-1}$ we obtain that the module $m_1m_2\cdots m_i$ is artinian if and only if it is noetherian. For $i = 0$ we obtain that $A$ is artinian if and only if it is noetherian.

(2.16) Remark. Let $A$ be a local noetherian ring with maximal ideal $m$, and let $q$ be an $m$-primary ideal. Then $A/q$ is an artinian ring. To show this we first note that $m = r(q)$. Since $A$ is noetherian $m$ is finitely generated, and thus it follows from Remark (RINGS 4.8) that a power of the maximal ideal in the noetherian local ring $A/q$ is zero. Hence it follows from Lemma (2.15) that $A/q$ is artinian.

(2.17) Theorem. A ring is artinian if and only if it is noetherian and has dimension 0.

Proof. When $A$ is artinian it follows from Proposition (2.11) that $\dim(A) = 0$. It follows from Corollary (2.13) that the ring $A$ has a finite number of maximal ideals $m_1, m_2, \ldots, m_n$. We have that $m_1m_2\cdots m_n \subseteq m_1 \cap m_2 \cap \cdots \cap m_n \subseteq r(A)$. Since $r(A)$ is nilpotent by Proposition (2.14) it follows from Lemma (2.15) that $A$ is noetherian.

Conversely assume that $A$ is noetherian of dimension 0. Then every prime ideal is maximal, and from Remark (2.8) it follows that $A$ has finitely many maximal ideals $m_1, m_2, \ldots, m_n$. Again $m_1m_2\cdots m_n \subseteq r(A)$. If follows from Remark (RINGS 4.8) that $r(A)$ is nilpotent. Hence it follows from Lemma (2.15) that $A$ is artinian.

(2.18) Proposition. An artinian ring is isomorphic to the direct product of a finite number of local artin rings.

More precisely, when $A$ is an artinian ring the canonical map $A \to \prod_{x \in \text{Spec}(A)} A_{j_x}$ obtained from the localization maps $A \to A_{j_x}$ is an isomorphism.

Proof. By Corollary (2.13) we have that $\text{Spec}(A)$ consists of a finite number of points, and by Proposition (2.11) the points are closed. Hence $\text{Spec}(A)$ is a discrete topological space. Since $O_{\text{Spec}(A)}$ is a sheaf there is an injective map $A = \Gamma(\text{Spec}(A), O_{\text{Spec}(A)}) \to \prod_{x \in \text{Spec}(A)} A_{j_x} = \prod_{x \in \text{Spec}(A)} O_{\text{Spec}(A), x}$. However each point $x$ is open in $\text{Spec}(A)$. Hence $A_{j_x} = \Gamma(\{x\}, O_{\text{Spec}(A)})$, and $\{x\} \cap \{y\} = \emptyset$ when $x \neq y$. It follows that we can glue any collection of sections $s_x \in \Gamma(\{x\}, O_{\text{Spec}(A)})$ for $x \in \text{Spec}(A)$ to a section $s \in \Gamma(\text{Spec}(A), O_{\text{Spec}(A)})$. Hence the map $A \to \prod_{x \in \text{Spec}(A)} \prod A_{j_x}$ is also surjective.

(2.19) Exercises.

1. Show that if $S$ is a multiplicatively closed subset of a ring $A$ such that $S^{-1}A$ is noetherian. Then $A$ is not necessarily noetherian.

2. Let $K[t_1, t_2, \ldots]$ be the polynomial ring in the infinitely many variables $t_1, t_2, \ldots$ over a field $K$. Moreover let $K(t_1, t_2, \ldots)$ be the localization of $K[t_1, t_2, \ldots]$ in the multiplicatively closed subset of $K[t_1, t_2, \ldots]$ consisting of all non-zero elements.

   (1) Show that $K(t_1, t_2, \ldots)$ is noetherian.
(2) Show that $K(t_1, t_2, \ldots) \otimes_K K(t_1, t_2, \ldots)$ is not Noetherian.

3. Let $A$ be a ring. Give an example of a ring $A$ that is not noetherian, but is such that $\text{Spec}(A)$ is noetherian.

4. Let $M$ be a noetherian $A$-module. Show that the ring $A / \text{Ann}_A(M)$ is noetherian.

5. Prove that there is only a finite number of minimal primes in a noetherian ring $A$ without using properties of the topological space $\text{Spec}(A)$.

6. Let $A$ be a ring. We say that two ideals $a$ and $b$ in $A$ are coprime if $a + b = A$. Let $a_1, a_2, \ldots, a_n$ be ideals of $A$ that are pairwise comprime. We define a map

$$\varphi : A \to \prod_{i=1}^n A / a_i$$

by $\varphi(f) = (\varphi_{A/a_1}(f), \varphi_{A/a_2}(f), \ldots, \varphi_{A/a_n}(f))$ for all $f \in A$.

1. Show that if $a$ and $b$ are coprime, then $a^m$ and $b^n$ are coprime for all positive integers $m$ and $n$.

2. Show that for all $i$ the ideals $a_i$ and $\cap_{i \neq j} a_j$ are coprime.

3. Show that the homomorphism $\varphi$ is a ring homomorphism with kernel $\cap_{i=1}^n a_i$.

4. Show that the homomorphism $\varphi$ is surjective.

5. Use parts (1), (2), (3), and (4) to prove that an artin ring is the direct product of a finite number of artinian rings.