Splitting algebras, factorization algebras, and residues

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26 Januar 2009

Abstract.

We define and construct splitting and factorization algebras. In order to study these algebras we introduce residues that generalize classical Schur polynomials. In particular we show how residues induce Gysin maps between splitting and factorization algebras.

We base our presentation upon well known and classical results on alternating and symmetric polynomials, that we prove. Throughout we focus on results that are used to describe the cohomology theories for flag and Grassmann manifolds, both in the classical, quantum, equivariant, and quantum-equivariant sense. As a consequence we obtain, for example, an interpretation by factorization algebras of the the astonishing description by L. Gatto of the cohomology of grassmannians via exterior powers.

Contents

1991 *Mathematics Subject Classification.* 14N15, 14M15, 05E05, 13F20.

Key words and phrases. alternating polynomials, determinantal formulas, exterior products, factorization algebras, formal Laurent series, formal power series, Jacobi-Trudi Lemma, residues, Schubert cycles, splitting algebras, symmetric groups, symmetric polynomials,.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Splitting and factorization algebras appear in several branches of mathematics. A well known and illustrative example is polynomial rings considered as algebras over the symmetric polynomials. This is the generic example of splitting algebras (see [B], [EL], [LT1], and [PZ], and further references there).

Another well known example we obtain by adjoining a root to a monic polynomial in one variable with coefficients in a ring. This useful, and frequently used, construction splits off a linear term from the polynomial. Successively adjoining roots we obtain splitting algebras of higher order of the polynomial.

An important class of examples of splitting and factorization algebras, that is our main motivation for writing these notes, is the cohomology rings of flag and Grassmann manifolds (see [G], [GS1], [LT1], [LT2] and [LT3], and further references there). The cohomology ring of the manifold of complete flags in a vector space V of dimension n is, for example, the splitting algebra for T^n. Moreover, the cohomology ring of the grassmannian of d-dimensional subspaces of V is the factorization algebra of T^n in factors of degrees d and $n-d$. For families of flags or grassmannians we get the corresponding splitting, respectively factorization, algebra of the Chern polynomial of the locally free sheaf that defines the family. Quantum, equivariant, and quantum-equivariant cohomology of grassmannians give more examples. These are obtained from the factorization algebras by changing bases or by varying the polynomial we factor (see [GS2], [L1] and [L2], and further references there).

A more unusual example comes from Galois theory, where splitting algebras can be used to give a presentation of the theory that lies close to the original point of view (see [EL], [K], and [T]).

In this article we give a self-contained presentation of those parts of the theory of splitting and factorization algebras that are related to the above examples. In particular we give bases for factorization algebras that correspond to bases consisting of classical Schur polynomials for symmetric polynomials and to Schubert cycles in cohomology. We also construct Gysin maps from splitting algebras to factorization algebras that correspond to the Gysin maps from cohomology rings of flag manifolds to cohomology rings of grassmannians in geometry. Our treatment is different from that in the works mentioned above in that it is based upon the theory of symmetric polynomials, and thus illustrates the connections between splitting and factorization algebras and the theory of symmetric polynomials.

It is worth noticing that as a consequence of our presentation we immediately obtain an interpretation by factorization algebras of the astonishing description of the cohomology of grassmannians via exterior products given by Letterio Gatto (see [G], [GS1], [LT1] and [LT2]). In particular we obtain the fundamental determinantal formulas that correspond to the Giambelli formula and the determinantal formula in Schubert calculus.

We also indicate how the connection between splitting algebras and polynomial algebras, mentioned above, gives somewhat exotic proofs of the different parts of the Main Theorem of Symmetric Polynomials.

The prerequisites for reading this article are knowledge of the definition of rings and ideals, and of the residue ring of a ring by an ideal. In addition, some knowledge is needed of basic results on polynomial rings and determinants.

1. Formal Laurent series and formal power series

In this section we remind the reader of formal Laurent series, and power series over commutative rings with unit. We also recall some basic results on inversion of power series, and in particular, the usual relation between elementary and complete symmetric polynomials.

1.1 Algebras. By a *ring* A we always mean a *commutative ring* with *unity*. An A-*algebra* $\varphi : A \to B$ is a *homomorphism* of rings. We shall throughout denote by $A[T]$ the A-algebra of polynomials in the variable T with coefficients in A.

1.2 Formal Laurent series. A *formal Laurent series* in the *variable* $\frac{1}{T}$ is a *formal expression*

$$g(T) = \cdots + a_{-2}T^2 + a_{-1}T + a_0 + \frac{a_1}{T} + \frac{a_2}{T^2} + \cdots ,$$

where each a_i lies in a ring A. For every A-algebra $\varphi : A \to B$ we write

$${}^{\varphi}g(T) = \cdots + \varphi(a_{-2})T^2 + \varphi(a_{-1})T + \varphi(a_0) + \frac{\varphi(a_1)}{T} + \frac{\varphi(a_2)}{T^2} + \cdots .$$

1.3 Formal power series. When $0 = a_1 = a_2 = \cdots$ we say that the formal Laurent series $g(T)$ is a *formal power series*, and we change the *indexing* to

$$g(T) = b_0 + b_1T + b_2T^2 + \cdots .$$

1.4 Inverting formal power series. Let

$$p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$$

be a *polynomial* in the *variable* T with *coefficients* in the ring A. An easy calculation shows that the *equation*

$$1 = (1 - c_1T + \cdots + (-1)^n c_nT^n)(1 + s_1T + s_2T^2 + \cdots)$$

of formal power series has an *unique solution* with $s_1, s_2, \ldots$ in A, and that each element s_i can be expressed as a polynomial in $c_1, \ldots, c_n$ with *integer* coefficients. Conversely, an equally easy calculation shows that the elements $s_1, s_2, \ldots$ determine $c_1, \ldots, c_n$ uniquely and that each element c_i can be expressed as a polynomial in $s_1, \ldots, s_n$ with integer coefficients.

1.5 Example. For every natural number h we have

$$\begin{aligned} \frac{T^h}{p(T)} = T^{h-n}\frac{1}{1 - \frac{c_1}{T} + \cdots + (-1)^n\frac{c_n}{T^n}} &= T^{h-n}\left(1 + \frac{s_1}{T} + \frac{s_2}{T^2} + \cdots\right) \\ &= \cdots + \frac{s_{h-n+1}}{T} + \frac{s_{h-n+2}}{T^2} + \cdots + \frac{s_h}{T^n} + \cdots . \end{aligned} \tag{1.5.1}$$

1.6 Formal power series and symmetric polynomials. Let $T_1, \dots, T_n$ be *algebraically independent* elements over the ring A and write

$$P(T) = (T - T_1) \cdots (T - T_n) = T^n - C_1 T^{n-1} + \cdots + (-1)^n C_n.$$

Then

$$C_j = \sum_{0 \le i_1 < \cdots < i_j \le n} T_{i_1} \cdots T_{i_j}$$

is the j'th *elementary symmetric polynomial* in the variables $T_1, \dots, T_n$. We obtain

$$\frac{1}{1 - C_1 T + \cdots + (-1)^n C_n T^n} = \frac{1}{(1 - T_1 T) \cdots (1 - T_n T)}$$
$$= \prod_{i=1}^{n} (1 + T_i T + T_i^2 T^2 + \cdots) = 1 + S_1 T + S_2 T^2 + \cdots ,$$

where

$$S_j = \sum_{0 \le i_1, \dots, i_j \le n} T_{i_1} \cdots T_{i_j}$$

is the j'th *complete symmetric polynomial* in the variables $T_1, \dots, T_n$.

2. Residues

Residues will play an important role in the remaining part of this article. We here define residues and give their main properties.

2.1 Definition. Let

$$g_i(T) = \cdots + a_{i\,-2} T^2 + a_{i\,-1} T + a_{i\,0} + \frac{a_{i\,1}}{T} + \frac{a_{i\,2}}{T^2} + \cdots$$

for $i = 1, \dots, n$ be formal Laurent series in the variable $\frac{1}{T}$. We write

$$\operatorname{Res}(g_1, \dots, g_n) = \det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

The main properties of *residues* are summarized in the following result:

2.2 Lemma. *Let A be a ring.*

(1) *Res is A-linear in $g_1, \dots, g_n$. That is, for every index i, for every formal Laurent series g_i', and for every pair of elements a, a' in A, we have*

$$\operatorname{Res}(g_1, \dots, a g_i + a' g_i', \dots, g_n) = a \operatorname{Res}(g_1, \dots, g_i, \dots, g_n) + a' \operatorname{Res}(g_1, \dots, g_i', \dots, g_n).$$

(2) Res *is alternating in $g_1, \dots, g_n$. That is, if $g_i = g_j$ for some $i \neq j$ we have*

$$\operatorname{Res}(g_1, \dots, g_n) = 0.$$

(3) Res *is zero on polynomials. That is, if at least one g_i is a polynomial in T we have*

$$\operatorname{Res}(g_1, \dots, g_n) = 0.$$

(4) *Res is functorial. That is, when $\varphi : A \to B$ is an A-algebra and $g_1, \dots, g_n$ have coefficients in A, then*

$$\varphi(\operatorname{Res}(g_1, \dots, g_n)) = \operatorname{Res}({}^{\varphi}g_1, \dots, {}^{\varphi}g_n).$$

Proof. All the properties of Res follow directly from Definition 2.1, and the corresponding properties of determinants.

The following result shows, in particular, that the residue generalizes the *classical Schur polynomials* (see e.g. [La2], [M2] and [Ma]). It will be used repeatedly in the following.

2.3 Proposition. *Let $p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$ be in the algebra $A[T]$ of polynomials in T with coefficients in A. Moreover, let $s_{-i} = 0$ for $i = 1, 2, \dots,$ let $s_0 = 1$, and let $s_1, s_2, \dots$ be determined by the equation*

$$1 = (1 - c_1T + \cdots + (-1)^n c_n T^n)(1 + s_1T + s_2T^2 + \cdots),$$

of formal power series. For all natural numbers $h_1, \dots, h_d$, we have

$$\operatorname{Res}\left(\frac{T^{h_1}}{p}, \dots, \frac{T^{h_d}}{p}\right) = \det\begin{pmatrix} s_{h_1-n+1} & \cdots & s_{h_1-n+d} \\ \vdots & \ddots & \vdots \\ s_{h_d-n+1} & \cdots & s_{h_d-n+d} \end{pmatrix}.$$

In particular, when $0 \le h_i \le n - i$ for $i = 1, \dots, d$, we have

$$\operatorname{Res}\left(\frac{T^{h_1}}{p}, \dots, \frac{T^{h_d}}{p}\right) = \begin{cases} 1 & \text{when} \quad h_i = n - i \quad \text{for} \quad i = 1, \dots, d \\ 0 & \text{when} \quad h_j < n - j \quad \text{for some} \quad j. \end{cases}$$

Proof. The first part of the proposition follows immediately from Definition 2.1 and equation (1.5.1).

From the first part of the proposition it follows that when $0 \le h_i \le n - i$ for $i = 1, \dots, d$ the $d \times d$-matrix (s_{h_i-n+j}) is upper triangular. Moreover, it follows that there are ones on the diagonal when $h_i = n - i$ for $i = 1, \dots, d$, and a zero on the diagonal in position (j, j) when $h_j < n - j$. Thus the last part of the proposition holds.

3. Symmetric and alternating polynomials

Here we recall some terminology concerning symmetric and alternating polynomials.

3.1 Notation, terminology and elementary properties. Let A be a ring and let $T_1, \dots, T_n$ be algebraically independent elements over A. We denote by $A[T_1, \dots, T_n]$ the A-algebra of polynomials in these variables with coefficients in A.

The *symmetric group* $\mathfrak{S}_n$, that is, the group of *permutations* of $1, 2, \dots, n$, *operates* on $A[T_1, \dots, T_n]$ by

$$(\sigma f)(T_1, \dots, T_n) = f(T_{\sigma(1)}, \dots, T_{\sigma(n)})$$

for all $f \in A[T_1, \dots, T_n]$ and $\sigma \in \mathfrak{S}_n$. A polynomial $f \in A[T_1, \dots, T_n]$ is *symmetric* when

$$(\sigma f)(T_1, \dots, T_n) = f(T_1, \dots, T_n) \quad \text{for all} \quad \sigma \in \mathfrak{S}_n.$$

The symmetric polynomials in $A[T_1, \dots, T_n]$ form an A-algebra $A[T_1, \dots, T_n]^{\text{sym}}$.

For every element $f \in A[T_1, \dots, T_n]$ we write

$$\operatorname{alt}(f) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma)(\sigma f),$$

where $\operatorname{sign}(\sigma)$ is the *sign* of the permutation σ. A polynomial of the form $\operatorname{alt}(f)$ for $f \in A[T_1, \dots, T_n]$ is called *alternating*. The alternating polynomials form an A-submodule $A[T_1, \dots, T_n]^{\mathrm{alt}}$ of $A[T_1, \dots, T_n]$.

For all polynomials $f_1, \dots, f_n$ in $A[T]$ we write

$$(f_i(T_j)) = \begin{pmatrix} f_1(T_1) & \dots & f_1(T_n) \\ \vdots & \ddots & \vdots \\ f_n(T_1) & \dots & f_n(T_n) \end{pmatrix}.$$

Then

$$\operatorname{alt}(f_1(T_1) \cdots f_n(T_n)) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) f_1(T_{\sigma(1)}) \cdots f_n(T_{\sigma(n)}) = \det(f_i(T_j)).$$

3.2 Remark. The determinant $\det(T_j^{n-i}) = \operatorname{alt}(T_1^{n-1} \cdots T_n^0)$ is an alternating polynomial and is called the *Vandermonde determinant*. An easy calculation shows that it is equal to $\prod_{0 \le i < j \le n}(T_i - T_j)$. We want however to make the point that this is not needed in the following. All that we need is that the Vandermonde determinant is not a zero divisor in $A[T_1, \dots, T_n]$, or equivalently, that it is not zero. This follows, for example, from the expansion of the determinant $\det(T_j^{n-i})$ that contains a single monomial of the form $T_1^{n-1} \cdots T_n^0$.

3.3 Lemma. *Let $T_1, \dots, T_n$ be algebraically independent elements over A, and let*

$$P(T) = (T - T_1) \cdots (T - T_n) = T^n - C_1 T^{n-1} + \cdots + (-1)^n C_n.$$

For all $f_1, \dots, f_n$ in $A[C_1, \dots, C_n]$ we have that $\operatorname{Res}(\frac{f_1}{P}, \dots, \frac{f_n}{P})$ is in $A[C_1, \dots, C_n]$.

Proof. It follows from 1.4 and 1.6 that $\frac{f_i}{P}$ is contained in $A[C_1, \dots, C_n]$. The assertion of the lemma thus follows from Definition 2.1.

4. Residues and symmetric polynomials

In this section we first show how we can use residues to give a natural proof of a general version of the Jacobi-Trudi Lemma. This Lemma is then used to give the well known bases for the alternating polynomials as an A-module, and as a module over the symmetric polynomials. As a consequence we obtain the well known bases of the symmetric polynomials in terms of Schur polynomials. We also observe that the residue can be used to define a Gysin type map from a polynomial algebra to the corresponding algebra of symmetric polynomials. The results of this section will later be generalized to splitting algebras.

4.1 The Jacobi-Trudi Lemma. *Let $T_1, \dots, T_n$ be algebraically independent elements over the ring A and let*

$$P(T) = (T - T_1) \cdots (T - T_n).$$

For all polynomials $f_1, \dots, f_n$ in $A[T]$ we obtain

$$\det(f_i(T_j)) = \operatorname{Res}\left(\frac{f_1}{P}, \dots, \frac{f_n}{P}\right) \det(T_j^{n-i}).$$

Proof. We use the division algorithm to the polynomial $f_i(T)$ modulo $P(T)$ over the algebra $A[T_1, \dots, T_n]$ and obtain

$$f_i(T) = q_i(T)P(T) + r_i(T)$$

where $q_i(T)$ and $r_i(T)$ have coefficients in $A[T_1, \dots, T_n]$, and $r_i(T)$ is of degree less that n in T. Since Res is linear in $f_1, \dots, f_n$ and zero on polynomials by Lemma 2.2 we obtain

$$\operatorname{Res}\left(\frac{f_1}{P}, \dots, \frac{f_n}{P}\right) = \operatorname{Res}\left(q_1 + \frac{r_1}{P}, \dots, q_n + \frac{r_n}{P}\right) = \operatorname{Res}\left(\frac{r_1}{P}, \dots, \frac{r_n}{P}\right).$$

Moreover, since $P(T_j) = 0$ for $j = 1, \dots, n$, we obtain the equations $\det(f_i(T_j)) = \det(q_i(T_j)P(T_j) + r(T_j)) = \det(r_i(T_j))$. Since both Res and det are linear in $r_1, \dots, r_n$ it suffices to prove the lemma when $r_i = T^{h_i}$ with $0 \leq h_i < n$ for $i = 1, \dots, n$. Moreover, since Res is alternating in $r_1, \dots, r_n$ by Lemma 2.2, and the same is true for det, we can assume that $n > h_1 > \dots > h_n \geq 0$, that is, we can assume that $h_i = n - i$ for $i = 1, \dots, n$. However, then the lemma follows from the equality $\operatorname{Res}(\frac{T^{n-1}}{P}, \dots, \frac{T^0}{P}) = 1$ of Proposition 2.3.

The next result gives the bases of $A[T_1, \dots, T_n]^{\mathrm{alt}}$ alluded to above. As a corollary we obtain a basis for $A[T_1, \dots, T_n]^{\mathrm{sym}}$ as an A-module in terms of Schur polynomials.

4.2 Theorem. *Let $T_1, \dots, T_n$ be algebraically independent elements over the ring A.*

(1) *The A-module $A[T_1, \dots, T_n]^{\mathrm{alt}}$ is an $A[T_1, \dots, T_n]^{\mathrm{sym}}$-submodule of the algebra $A[T_1, \dots, T_n]$, and the homomorphism $A[T_1, \dots, T_n] \to A[T_1, \dots, T_n]^{\mathrm{alt}}$ that maps f to $\operatorname{alt}(f)$ is $A[T_1, \dots, T_n]^{\mathrm{sym}}$-linear.*

(2) *As an $A[T_1, \dots, T_n]^{\mathrm{sym}}$-module $A[T_1, \dots, T_n]^{\mathrm{alt}}$ is free of rank 1 with basis the Vandermonde determinant $\det(T_j^{n-i})$.*

(3) *As an A-module $A[T_1, \dots, T_n]^{\mathrm{alt}}$ is free with basis $\det(T_j^{h_i})$ for $h_1 > \dots > h_n \geq 0$.*

Proof. (1) For $g \in A[T_1, \dots, T_n]^{\mathrm{sym}}$ and $f \in A[T_1, \dots, T_n]^{\mathrm{alt}}$ we have

$$\operatorname{alt}(gf) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma)\sigma(gf) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) g\sigma(f) = g \operatorname{alt}(f),$$

that proves assertion (1).

(2) It follows from assertion (1) that we have an inclusion of $A[T_1,\dots,T_n]^{\text{sym}}$-modules $A[T_1,\dots,T_n]^{\text{sym}}\det(T_j^{n-i})\subseteq A[T_1,\dots,T_n]^{\text{alt}}$. Since $\det(T_j^{n-i})$ is not a zero divisor in $A[T_1,\dots,T_n]$, as observed in 3.2, it suffices to show the converse inclusion.

From the definition of $A[T_1,\dots,T_n]^{\text{alt}}$ it is clear that this A-module is generated by the elements $\text{alt}(f_1(T_1)\cdots f_n(T_n))=\det(f_i(T_j))$ for all $f_1,\dots f_n$ i $A[T]$. Thus it follows from the Jacobi-Trudi Lemma 4.1 that the A-module $A[T_1,\dots,T_n]^{\text{alt}}$ is generated by the elements $\text{Res}(\frac{f_1}{P},\dots,\frac{f_n}{P})\det(T_j^{n-i})$, where $P(T)=(T-T_1)\cdots(T-T_n)$. Since $\text{Res}(\frac{f_1}{P},\dots,\frac{f_n}{P})$ is in $A[T_1,\dots,T_n]^{\text{sym}}$ for all $f_1,\dots,f_n$ in $A[T]$, as we noted in Lemma 3.3, it follows that $A[T_1,\dots,T_n]^{\text{alt}}\subseteq A[T_1,\dots,T_n]^{\text{sym}}\det(T_j^{n-i})$, as we wanted to show.

(3) As we just saw the A-module $A[T_1,\dots,T_n]^{\text{alt}}$ is generated by the elements $\text{Res}(\frac{f_1}{P},\dots,\frac{f_n}{P})\det(T_j^{n-i})$ for all $f_1,\dots,f_n$ in $A[T]$. Since Res is linear in $f_1,\dots,f_n$ by Lemma 2.2 it is generated by the elements with $f_i=T^{h_i}$ for $i=1,\dots,n$, and since Res is alternating by the same lemma we can assume that $h_1>\dots>h_n\geq 0$. However, the elements $\det(T_j^{h_i})=\text{Res}(\frac{T^{h_1}}{P},\dots,\frac{T^{h_n}}{P})\det(T_j^{n-i})$ for $h_1>\dots>h_n\geq 0$ are linearly independent over A because $\det(T_j^{h_i})$ is the only one of these polynomials in $T_1,\dots,T_n$ that contains the monomial $T_1^{h_1}\cdots T_n^{h_n}$.

4.3 Corollary. *Let* $P(T)=(T-T_1)\cdots(T-T_n)=T^n-C_1T^{n-1}+\dots+(-1)^nC_n$.

(1) *The homomorphism*

$$\partial(P):A[T_1,\dots,T_n]\to A[T_1,\dots,T_n]^{\text{sym}}$$

defined by $\partial(P)(f_1(T_1)\cdots f_n(T_n))=\text{Res}(\frac{f_1}{P},\dots,\frac{f_n}{P})$ *is linear as a homomorphism of* $A[T_1,\dots,T_n]^{\text{sym}}$*-modules.*

(2) *The* A*-module* $A[T_1,\dots,T_n]^{\text{sym}}$ *is free with basis* $\text{Res}(\frac{T^{h_1}}{P},\dots,\frac{T^{h_n}}{P})$ *for* $h_1>\dots>h_n\geq 0$.

(3) *We have* $A[T_1,\dots,T_n]^{\text{sym}}=A[C_1,\dots,C_n]$.

Proof. It follows from assertion (2) of the theorem that multiplication by $\det(T_j^{n-i})$ gives an isomorphism $\mu: A[T_1,\dots,T_n]^{\text{sym}}\to A[T_1,\dots,T_n]^{\text{alt}}$ of $A[T_1,\dots,T_n]^{\text{sym}}$-modules. Moreover, it follows from the Jacobi-Trudi Lemma 4.1 that $\mu\partial(P)=\text{alt}$. Thus assertions (1) and (2) follow from assertions (1) and (3) of the theorem.

(3) It is clear that we have an inclusion $A[C_1,\dots,C_n]\subseteq A[T_1,\dots,T_n]^{\text{sym}}$. The converse inclusion follows from assertion (2) since Lemma 3.3 shows that the element $\text{Res}(\frac{T^{h_1}}{P},\dots,\frac{T^{h_n}}{P})$ lies in $A[C_1,\dots,C_n]$.

5. Splitting and factorization algebras

Here we first define splitting and factorization algebras and give the two most common constructions of splitting algebras. We obtain a basis for splitting algebras as a module, and also a proof of the existence of factorization algebras.

Both splitting and factorization algebras can be constructed in many alternative ways (see [B], [EL], [LT1] and [PZ]). Each construction provides a different perspective of the field. Properties that are obvious in one construction may be complicated in another. The connections between the different constructions are thus of separate interest.

5.1 Definition. Let

$$p(T) = T_n - c_1T^{n-1} + \cdots + (-1)^n c_n$$

be in the algebra $A[T]$ of polynomials in the variable T over the ring A. Moreover, let $m_1, \dots, m_r$ be positive integers such that

$$m_1 + \cdots + m_r = n.$$

A *factorization* of $p(T)$ in *factors of degrees* $m_1, \dots, m_r$ over an A-algebra $\psi : A \to B$ is an *ordered* set of polynomials $q_1, \dots, q_r$ of degrees $m_1, \dots, m_r$ in $B[T]$ such that

$${}^{\psi}p(T) = T^n - \psi(c_1)T^{n-1} + \cdots + (-1)^n\psi(c_n) = q_1(T)\cdots q_r(T).$$

We say that an A-algebra $\varphi : A \to \mathrm{Fact}_A^{m_1,\dots,m_r}(p)$ is a *factorization algebra* for $p(T)$ over A in *factors of degrees* $m_1, \dots, m_r$ when we have a factorization

$${}^{\varphi}p(T) = p_1(T)\cdots p_r(T)$$

over $\mathrm{Fact}_A^{m_1,\dots,m_r}(p)$ in *factors of degrees* $m_1, \dots, m_r$, and when φ satisfies the following *universal property*:

For every A-algebra $\psi : A \to B$ such that we have a factorization

$${}^{\psi}p(T) = q_1(T)\cdots q_r(T)$$

of $p(T)$ over B in factors of degrees $m_1, \dots, m_r$, there is a *unique* A-algebra homomorphism

$$\chi : \mathrm{Fact}_A^{m_1,\dots,m_r}(p) \to B$$

such that

$${}^{\chi\varphi}p_i(T) = q_i(T) \quad \text{for} \quad i = 1, \dots, r.$$

We call ${}^{\varphi}p(T) = p_1(T)\cdots p_r(T)$ the *universal factorization* of $p(T)$ over A.

When $r = d + 1$ and $1 = m_1 = \cdots = m_d$ we write

$$\mathrm{Fact}_A^{1,\dots,1,m_r}(p) = \mathrm{Split}_A^d(p)$$

and call $\mathrm{Split}_A^d(p)$ the d'th *splitting algebra* for $p(T)$ over A. The universal factorization

$${}^{\varphi}p(T) = (T - \xi_1)\cdots(T - \xi_d)p_{d+1}(T)$$

we call the *universal splitting* and we call $\xi_1, \dots, \xi_d$ the *universal roots*. Moreover, we let

$$\mathrm{Split}_A(p) = \mathrm{Split}_A^n(p)$$

and call $\mathrm{Split}_A(p)$ the *splitting algebra* for $p(T)$ over A. For every integer d such that $0 \le d \le n$ we write

$$\mathrm{Fact}_A^{d,n-d}(P) = \mathrm{Fact}_A^d(p).$$

5.2 Example. As mentioned in the introduction we obtain a first splitting algebra for a monic polynomial p of degree n in $A[T]$ by adjunction of a root of $p(T)$ to A. More precisely, we have that $A[T]/(p)$ is a first splitting algebra $\mathrm{Split}_A^1(p)$ of $p(T)$ over A, or a factorization algebra $\mathrm{Fact}_A^1(p)$ of $p(T)$ over A in factors of degrees 1 and $n-1$. The universal splitting is ${}^{\varphi}p(T) = (T-\xi)p_2(T)$, where ξ is the class of T modulo $p(T)$. Similarly $A[T]/(p)$ is a factorization algebra $\mathrm{Fact}_A^{n-1}(p)$ of $p(T)$ over A in factors of degrees $n-1$ and 1 with universal factorization $p_1(T)(T-\xi)$.

5.3 Remark and convention. From Example 5.2 and the well known properties of the A-algebra $A[T]/(p)$ it follows that $\mathrm{Split}_A^1(p)$ is a free A-module with a basis $1, \xi, \dots, \xi^{n-1}$. Since 1 is part of this basis the map $A \to \mathrm{Split}_A^1(p)$ is injective. When it can cause no confusion we shall identify A with its image in $\mathrm{Split}_A^1(p)$ via this map.

5.4 Remark. Two factorization algebras B_1, B_2 for $p(T)$ over A in factors of degrees $m_1, \dots, m_r$ are canonically isomorphic as A-algebras. This is because it follows from the universal properties of B_1 and B_2 that we have unique A-algebra homomorphisms $\psi_2 : B_2 \to B_1$, respectively $\psi_1 : B_1 \to B_2$, and, again by the uniqueness, we have that $\psi_2\psi_1$ and $\psi_1\psi_2$ are the identity maps of B_1, respectively of B_2.

We now prove the existence of splitting algebras.

5.5 Theorem (Construction 1). *Let $p(T) = T^n - c_1T^{n-1} + \dots + (-1)^n c_n$ be in $A[T]$ and let*

$$P(T) = (T-T_1)\cdots(T-T_n) = T^n - C_1T^{n-1} + \dots + (-1)^n C_n,$$

where $T_1, \dots, T_n$ are algebraically independent elements over A. Then the residue algebra

$$A[T_1, \dots, T_n]/(C_1 - c_1, \dots, C_n - c_n)$$

of the polynomial algebra $A[T_1, \dots, T_n]$ modulo the ideal generated by the elements $C_1 - c_1, \dots, C_n - c_n$ is a splitting algebra for $p(T)$ over A. The residue classes $\xi_1, \dots, \xi_n$ of $T_1, \dots, T_n$ are the universal roots.

Proof. Let

$$\varphi : A \to A[\xi_1, \dots, \xi_n]$$

be the A-algebra. The residue class of C_i by the canonical map

$$\chi : A[T_1, \dots, T_n] \to A[\xi_1, \dots, \xi_n]$$

is then $\varphi(c_i)$, and we have

$$\begin{aligned}{}^{\varphi}p(T) &= T^n - \varphi(c_1)T^{n-1} + \dots + (-1)^n\varphi(c_n)\\ &= T^n - \chi(C_1)T^{n-1} + \dots + (-1)^n\chi(C_n) = {}^{\chi}P(T) = (T-\xi_1)\cdots(T-\xi_n).\end{aligned}$$

Thus $p(T)$ splits completely over $A[\xi_1, \dots, \xi_n]$.

Let $\psi : A \to B$ be an A-algebra such that

$${}^{\psi}p(T) = (T-b_1)\cdots(T-b_n)$$

in $B[T]$. We define an A-algebra homomorphism

$$\chi' : A[T_1, \dots, T_n] \to B$$

by $\chi'(T_i) = b_i$ for $i = 1, \dots, n$. Then ${}^{\chi'}P(T) = (T-b_1)\cdots(T-b_n)$, and thus ${}^{\psi}p(T) = {}^{\chi'}P(T)$, that is, we have $\psi(c_i) = \chi'(C_i)$ for $i = 1, \dots, n$. Thus $\chi'(\varphi(c_i)) = \chi'(C_i)$ for $i = 1, \dots, n$. Consequently χ' factors via an A-algebra homomorphism

$$\chi : A[\xi_1, \dots, \xi_n] \to B$$

such that $\chi(\xi_i) = b_i$ for $i = 1, \dots, n$. The A-algebra homomorphism χ is uniquely determined by the equalities $\chi(\xi_i) = b_i$ for $i = 1, \dots, n$. Hence we have proved that $A[\xi_1, \dots, \xi_n]$ is a splitting algebra for $p(T)$ over A with universal roots $\xi_1, \dots, \xi_n$.

5.6 Example. With notation as in Theorem 5.5 we obtain that $A[T_1, \dots, T_n]$ is a splitting algebra for $P(T)$ over $A[C_1, \dots, C_n]$ with universal roots $T_1, \dots, T_n$.

In Example 5.6 we saw a property of splitting algebras that is an immediate consequence of the construction of 5.5. With this construction it is however hard to give the standard basis for a splitting algebra as an A-module. To illustrate that different constructions of splitting algebras can give easy proofs of some properties but make it hard to prove other results, we give a second construction in which the assertion of Example 5.6 is slightly hard to verify, but where we immediately obtain an A-module basis. This basis is fundamental to the theory of splitting algebras as we shall see.

5.7 Theorem (Construction 2). *Let $p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$ be in $A[T]$. We construct, by induction on d, a series of algebras*

$$A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_d$$

and polynomials $p(T) = q_1(T), \dots, q_{d+1}(T)$ with q_i in $A_{i-1}[T]$ by,

$$A_i = A_{i-1}[T]/(q_i) \quad \textit{and} \quad q_{i+1}(T) = \frac{q_i(T)}{T - \xi_i},$$

where ξ_i is the class of T in A_i. Then A_d is a d'th splitting algebra for $p(T)$ over A with universal roots $\xi_1, \dots, \xi_d$.

In particular we have that A_d is a free A-module with basis $\xi_1^{h_1} \cdots \xi_d^{h_d}$ with $0 \leq h_i \leq n - i$ for $i = 1, \dots, d$.

Proof. We first note that, by definition, $q_i(T)$ is of degree $n-i-1$, and by Remark 5.3 A_i is a free A_{i-1}-module with basis $1, \xi_i, \dots, \xi_i^{n-i}$. Moreover, by the same Remark, we consider A_{i-1} as a subset of A_i. This justifies the inclusions in the theorem, and also proves the last part of the theorem.

We show the first part of the theorem by induction on d. It holds for $d = 0$ because $A_0 = A$ is a zeroth splitting algebra for $p(T)$ over A. Assume that the theorem holds for $d - 1$. Let $\varphi : A \to B$ be an algebra such that

$${}^{\varphi}p(T) = (T - b_1)\cdots(T - b_d)q(T)$$

in $B[T]$. By the induction assumption we have an A-algebra homomorphism

$$\chi : A_{d-1} \to B$$

uniquely determined by $\chi(\xi_i) = b_i$ for $i = 1, \ldots, d-1$. We thus obtain that ${}^{\chi}q_d(T) = \frac{{}^{\chi}p(T)}{(T-b_1)\cdots(T-b_{d-1})} = (T - b_d)q(T)$. It follows from Example 5.2 that A_d is a first splitting algebra for $q_d(T)$ over A_{d-1} with universal root ξ_d. Consequently we have a unique A_{d-1}-algebra homomorphism

$$\psi : A_d \to B$$

such that $\psi(\xi_d) = b_d$. Since the restriction of ψ to A_{d-1} is χ it follows that the equations $\psi(\xi_i) = b_i$ for $i = 1, \ldots, d$ uniquely determine ψ. We have thus proved that A_d is a d'th splitting algebra for $p(T)$ over A with universal roots $\xi_1, \ldots, \xi_d$.

5.8 Remark and convention. Since 1 is part of the basis given in Theorem 5.7 it follows that the algebra homomorphism $\varphi : A \to \mathrm{Split}_A(p)$ is injective. When it can cause no confusion we shall identify A with its image in $\mathrm{Split}_A(p)$ by φ. In particular we write ${}^{\varphi}p(T) = p(T)$.

The following result is one of many similar results that follow from the universal properties of splitting and factorization algebras. We have chosen this particular version because it is simple and illustrates well the underlying principles, and because we shall use it later.

5.9 Lemma. *Let $p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$ be in $A[T]$ and let $A_1 = \mathrm{Fact}^d_A(p)$ be a factorization algebra for $p(T)$ over A with universal factorization $p(T) = p_1(T)p_2(T)$ where $p_1(T)$ is of degree d. Then a splitting algebra $\mathrm{Split}_{A_1}(p_1)$ for $p_1(T)$ over A_1 is a d'th splitting algebra $\mathrm{Split}^d_A(p)$ for $p(T)$ over A, and the universal roots in $\mathrm{Split}_{A_1}(p_1)$ are the universal roots in $\mathrm{Split}^d_A(p)$.*

Proof. Let $\varphi : A \to B$ be an A-algebra homomorphism such that

$${}^{\varphi}p(T) = (T - b_1)\cdots(T - b_d)q(T)$$

in $B[T]$. From the universal property of $A_1 = \mathrm{Fact}^d_A(p)$ we obtain a unique A-algebra homomorphism

$$\chi : \mathrm{Fact}^d_A(p) \to B$$

such that

$${}^{\chi}p_1(T) = (T - b_1)\cdots(T - b_d).$$

Let $\omega : A \to A_1 = \mathrm{Fact}^d_A(p)$ and $\omega_1 : A_1 = \mathrm{Fact}^d_A(p) \to \mathrm{Split}_{A_1}(p_1)$ denote the algebra homomorphisms. Then

$${}^{\omega_1\omega}p(T) = (T - \pi_1)\cdots(T - \pi_d){}^{\omega_1}p_2(T)$$

where $\pi_1, \ldots, \pi_d$ are the universal roots of $\mathrm{Split}_{A_1}(p_1)$. Consequently it follows from the universal property of $\mathrm{Split}_{A_1}(p_1)$, used to the A_1-algebra $\chi : A_1 \to B$ and the polynomial $p_1(T)$, that we have a unique A_1-algebra homomorphism

$$\psi : \mathrm{Split}_{A_1}(p_1) \to B$$

such that $\psi(\pi_i) = b_i$ for $i = 1, \dots, d$.

The equations $\psi(\pi_i) = b_i$ for $i = 1, \dots, n$ determine ψ uniquely as an A_1-algebra homomorphism, and since the equation ${}^{\chi}p_1(T) = (T - b_1) \cdots (T - b_d)$ determines χ we obtain that ψ is also determined by the equations $\psi(\pi_i) = b_i$ for $i = 1, \dots, d$ as an A-algebra homomorphism. We have thus proved that $\mathrm{Split}_{A_1}(p_1)$ is a d'th splitting algebra for $p(T)$ over A with universal roots $\pi_1, \dots, \pi_d$.

We can now show that factorization algebras exist and simultaneously give an explicit description of such algebras.

5.10 Theorem. *Let $\mathrm{Split}_A(p)$ be a splitting algebra for the monic polynomial p of degree n in $A[T]$ with universal roots $\xi_1, \dots, \xi_n$. Moreover, let $m_1, \dots, m_r$ be positive integers such that $m_1 + \cdots + m_r = n$. For $j = 1, \dots, r$ and $i = 1, \dots, m_j$ we let*

$$\xi_i(j) = \xi_{i+m_1+\cdots+m_{j-1}} \quad \textit{and} \quad p_j(T) = (T - \xi_1(j)) \cdots (T - \xi_{m_j}(j))$$

with $m_0 = 0$. Then the A-algebra in $\mathrm{Split}_A(p) = A[\xi_1, \dots, \xi_n]$ generated by the coefficients of the polynomials $p_1(T), \dots, p_r(T)$ is a factorization algebra for $p(T)$ in factors of degrees $m_1, \dots, m_r$ and

$$p(T) = p_1(T) \cdots p_r(T)$$

is the universal splitting.

Proof. Let $\varphi : A \to B$ be an A-algebra such that we have a factorization

$${}^{\varphi}p(T) = q_1(T) \cdots q_r(T)$$

in $B[T]$ in factors of degrees $m_1, \dots, m_r$. Using the convention of 5.8 we construct a sequence of algebras $B = B_0 \subset B_1 \subset \cdots \subset B_r$ with $B_j = \mathrm{Split}_{B_{j-1}}(q_j)$ for $j = 1, \dots, r$. Let $\pi_1(j), \dots, \pi_{m_j}(j)$ be the universal roots in $B_j = \mathrm{Split}_{B_{j-1}}(q_j)$. Then we have a splitting

$${}^{\varphi}p(T) = \prod_{j=1}^{r} (T - \pi_1(j)) \cdots (T - \pi_{m_j}(j)) = q_1(T) \cdots q_r(T)$$

over $B_r[T]$. It consequently follows from the universal property of $\mathrm{Split}_A(p)$ that we have an A-algebra homomorphism

$$\chi : \mathrm{Split}_A(p) \to B_r$$

such that $\chi(\xi_i(j)) = \pi_i(j)$ for $j = 1, \dots, r$ and $i = 1, \dots, m_j$. Then ${}^{\chi}p_j(T) = q_j(T)$ for $j = 1, \dots, r$. In particular we have that the homomorphism χ induces an A-algebra homomorphism $\psi : C \to B$ from the A-algebra C in $A[\xi_1, \dots, \xi_d]$ generated by the coefficients of $p_1(T), \dots, p_r(T)$ such that ${}^{\psi}p_j(T) = q_j(T)$ for $j = 1, \dots, r$. Moreover, ψ is uniquely determined by its values on these coefficients. Consequently C is a factorization algebra for $p(T)$ in factors of degrees $m_1, \dots, m_r$, and with universal splitting $p(T) = p_1(T) \cdots p_r(T)$, as we wanted to prove.

5.11 Corollary. *Let d be an integer such that $0 \le d \le n$. We have, with the notation of the theorem, that $A[\xi_1, \dots, \xi_d]$ is a d'th splitting algebra $\mathrm{Split}_A^d(p)$ of $p(T)$ over A with universal roots $\xi_1, \dots, \xi_d$.*

The A-algebra in $A[\xi_1, \dots, \xi_d]$ that is generated by the coefficients of the polynomial $p_1(T) = (T - \xi_1) \cdots (T - \xi_d)$ is a factorization algebra $\mathrm{Fact}_A^d(p)$ for $p(T)$ over A with universal splitting $p(T) = p_1(T)p_2(T)$, where $p_2(T) = (T - \xi_{d+1}) \cdots (T - \xi_n)$.

Proof. The first assertion of the corollary follows immediately from the second construction 5.7.

To prove the second assertion we take $r = 2$ in the theorem. From the universal factorization $p(T) = p_1(T)p_2(T)$ we obtain $p_2(T) = \frac{p(T)}{p_1(T)}$. Thus it follows from 1.4 that the coefficients of $p_2(T)$ lie in the A-algebra generated by the coefficients of $p_1(T)$.

6. Applications to symmetric polynomials

Splitting algebras provide a slightly different point of view of symmetric polynomials than the usual one. We shall illustrate this by explaining how to prove the Main Theorem of Symmetric Polynomials via splitting algebras.

There are many analogies between polynomial algebras in a finite number of variables and splitting algebras. We mention, without proofs, how the symmetric group operates on splitting algebras and some results on the algebra of invariant elements under this action.

6.1 Lemma. *Let $P(T) = (T - T_1) \cdots (T - T_n) = T^n - C_1 T^{n-1} + \cdots + (-1)^n C_n$, where the elements $T_1, \dots, T_n$ are algebraically independent over the ring A. Then the elementary symmetric polynomials $C_1, \dots, C_n$ are algebraically independent over A.*

Proof. Let $U_1, \dots, U_n$ be algebraically independent elements over A and denote by $A[U_1, \dots, U_n][\pi_1, \dots, \pi_n]$ a splitting algebra for $Q(T) = T^n - U_1 T^{n-1} + \cdots + (-1)^n U_n$ over $A[U_1, \dots, U_n]$ with universal roots $\pi_1, \dots, \pi_n$. Since the coefficients of a polynomial can be expressed as elementary symmetric polynomials in its roots we have an equality $A[U_1, \dots, U_n][\pi_1, \dots, \pi_n] = A[\pi_1, \dots, \pi_n]$.

Let

$$\varphi : A[U_1, \dots, U_n] \to A[C_1, \dots, C_n]$$

be the A-algebra homomorphism defined by $\varphi(U_i) = C_i$ for $i = 1, \dots, n$. Then

$${}^{\varphi}Q(T) = T^n - C_1 T^{n-1} + \cdots + (-1)^n C_n = (T - T_1) \cdots (T - T_n)$$

over $A[T_1, \dots, T_n]$. It follows from the universal property for the splitting algebra $A[\pi_1, \dots, \pi_n]$ that there is a unique homomorphism of $A[U_1, \dots, U_n]$-algebras

$$\chi : A[\pi_1, \dots, \pi_n] \to A[T_1, \dots, T_n]$$

such that $\chi(\pi_i) = T_i$ for $i = 1, \dots, n$, where $A[T_1, \dots, T_n]$ is an $A[U_1, \dots, U_n]$-algebra via φ and the inclusion of $A[C_1, \dots, C_n]$ in $A[T_1, \dots, T_n]$. Since $T_1, \dots, T_n$ are algebraically independent over A we must have that $\pi_1, \dots, \pi_n$ are algebraically independent over A. Thus χ is an isomorphism of A-algebras. The restriction of χ to

$A[U_1, \dots, U_n]$ clearly induces the homomorphism φ. Consequently φ is an isomorphism. In particular, we have that $C_1, \dots, C_n$ are algebraically independent over A.

We have in previous sections proved two of the parts of the Main Theorem of Symmetric Polynomials. The above lemma gives the final piece of this result.

6.2 The Main Theorem of Symmetric Polynomials. *Let $C_1, \dots, C_n$ be the elementary symmetric polynomials in the algebraically independent elements $T_1, \dots, T_n$ over A, and let $A[T_1, \dots, T_n]^{\mathrm{sym}}$ be the polynomials that are invariant under the action of the symmetric group $\mathfrak{S}_n$.*

(1) $A[T_1, \dots, T_n]^{\mathrm{sym}} = A[C_1, \dots, C_n]$.
(2) *The elements $C_1, \dots, C_n$ are algebraically independent over A.*
(3) *$A[T_1, \dots, T_n]$ is a free $A[T_1, \dots, T_n]^{\mathrm{sym}}$-module with basis $T_1^{h_1} \cdots T_n^{h_n}$, where $0 \leq h_i \leq n - i$ for $i = 1, \dots, n$.*

Proof. (1) The first assertion is Corollary 4.3 (3).

(2) The second assertion is Lemma 6.1.

(3) It follows from Example 5.6 that $A[T_1, \dots, T_n]$ is the splitting algebra for the polynomial $P(T) = T^n - C_1 T^{n-1} + \cdots + (-1)^n C_n$ over $A[C_1, \dots, C_n]$, and consequently over $A[T_1, \dots, T_n]^{\mathrm{sym}}$ by the first assertion. Assertion (3) thus follows from Theorem 5.7.

6.3 The action of the symmetric group. Let $\mathrm{Split}_A(p)$ be a splitting algebra of the monic polynomial p of degree n in $A[T]$ with universal roots $\xi_1, \dots, \xi_n$. It follows from the universal property of $\mathrm{Split}_A(p)$ that for every permutation σ in the symmetric group $\mathfrak{S}_n$ we have a unique A-algebra homomorphism

$$\varphi_\sigma : \mathrm{Split}_A(p) \to \mathrm{Split}_A(p)$$

such that $\varphi_\sigma(\xi_i) = \xi_{\sigma(i)}$ for $i = 1, \dots, n$. This clearly defines an *action* of $\mathfrak{S}_n$ on $\mathrm{Split}_A(p) = A[\xi_1, \dots, \xi_n]$. It is an interesting problem to determine the invariants of the splitting algebra under this action. In [EL] we prove the following partial result:

The invariants of $\mathrm{Split}_A(p)$ *under the action of $\mathfrak{S}_n$ are A when at least one of the following two conditions are fulfilled:*

(1) *The element $\prod_{i \neq j}(\xi_i - \xi_j)$ is regular in A.*
(2) *The element 2 is regular in A.*

A more complete and refined result is given by Anders Thorup in [T].

7. Gysin maps and module bases for factorization algebras

Gysin maps are important in geometry. Here we shall show how the Gysin map for polynomial algebras in 4.3 (1) can be used to obtain Gysin maps on splitting algebras. We also show how the Gysin maps can be used to find module bases for factorization algebras.

We define Gysin maps in terms of the residue. In [M1], [F], [La1], [La2], and [Ma] (see also further references there) it is defined by divided difference operators. The connection between the two approaches is explained in [LT1] (where one can also find further references).

7.1 Theorem. *Let* $p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^n c_n$ *be in* $A[T]$ *and let* $\operatorname{Split}_A(p) = A[\xi_1, \dots, \xi_n]$ *be a splitting algebra for* $p(T)$ *over* A *with universal roots* $\xi_1, \dots, \xi_n$. *Then we have a surjective* A*-module homomorphism*

$$\partial(p) : A[\xi_1, \dots, \xi_n] \to A$$

determined by $\partial(p)(f_1(\xi_1) \cdots f_n(\xi_n)) = \operatorname{Res}(\frac{f_1}{p}, \dots, \frac{f_n}{p})$ *for all polynomials* $f_1, \dots, f_n$ *in* $A[T]$.

Proof. Let $T_1, \dots, T_n$ be algebraically independent elements over A and let

$$P(T) = (T - T_1) \cdots (T - T_n) = T^n - C_1T^{n-1} + \cdots + (-1)^n C_n.$$

It follows from Corollary 4.3 (1) that we have an $A[T_1, \dots, T_n]^{\mathrm{sym}}$-linear homomorphism

$$\partial(P) : A[T_1, \dots, T_n] \to A[T_1, \dots, T_n]^{\mathrm{sym}}$$

determined by $\partial(P)(f_1(T_1) \cdots f_n(T_n)) = \operatorname{Res}(\frac{f_1}{P}, \dots, \frac{f_n}{P})$. Since $C_1, \dots, C_n$ are algebraically independent over A by Lemma 6.1, and $A[T_1, \dots, T_n]^{\mathrm{sym}} = A[C_1, \dots, C_n]$ by Corollary 4.3 (3) we have an A-algebra homomorphism

$$\varphi : A[T_1, \dots, T_n]^{\mathrm{sym}} \to A$$

defined by $\varphi(C_i) = c_i$ for $i = 1, \dots, n$. In particular, we have ${}^{\varphi}P(T) = p(T)$. From Lemma 2.2 (4) it follows that $\varphi\partial(P)(f_1(T_1) \cdots f_n(T_n)) = \varphi \operatorname{Res}(\frac{f_1}{P}, \dots, \frac{f_n}{P}) = \operatorname{Res}(\frac{f_1}{p}, \dots, \frac{f_n}{P})$ for all $f_1, \dots, f_n$ in $A[T]$. In order to show the existence of $\partial(p)$ it thus follows from Theorem 5.5 that we must show that $\varphi\partial(P)$ vanishes on the ideal in $A[T_1, \dots, T_n]$ generated by the elements $C_1 - c_1, \dots, C_n - c_n$. However, we observed that $\partial(P)$ is $A[T_1, \dots, T_n]^{\mathrm{sym}}$-linear, and thus

$$\begin{aligned} &\varphi\partial(P)((C_i - c_i)f_1(T_1) \cdots f_n(T_n)) \\ &\qquad = \varphi(C_i)\partial(P)(f_1(T_1) \cdots f_n(T_n)) - \varphi(c_i)\partial(P)(f_1(T_1) \cdots f_n(T_n)) = 0. \end{aligned}$$

Consequently we have shown the existence of $\partial(p)$.

We have that $\partial(p)$ is surjective since it is A-linear and $\partial(p)(\xi_1^{n-1} \cdots \xi_n^0) = 1$ by Proposition 2.3.

7.2 Corollary. *Let* $\operatorname{Split}_A^d(p)$ *be a* d*'th splitting algebra for* $p(T)$ *over* A *with universal roots* $\xi_1, \dots, \xi_d$, *and let* $\operatorname{Fact}_A^d(p)$ *be a factorization algebra for* $p(T)$ *over* A *with universal splitting* $p(T) = p_1(T)p_2(T)$, *where* $p_1(T)$ *is of degree* d.

(1) *We have a surjective homomorphism of* $\operatorname{Fact}_A^d(p)$*-modules*

$$\partial(p_1) : \operatorname{Split}_A^d(p) \to \operatorname{Fact}_A^d(p)$$

such that $\partial(p_1)(f_1(\xi_1) \cdots f_d(\xi_d)) = \operatorname{Res}(\frac{f_1}{p_1}, \cdots, \frac{f_d}{p_1})$ *for all* $f_1, \dots, f_d$ *in* $A[T]$.

(2) $\operatorname{Fact}_A^d(p)$ *is a free* A*-module of rank* $\binom{n}{d}$ *with basis*

$$\partial(p_1)(\pi_1^{h_1} \cdots \pi_d^{h_d}) = \operatorname{Res}\left(\frac{T^{h_1}}{p_1}, \dots, \frac{T^{h_d}}{p_1}\right) \quad \text{for} \quad n > h_1 > \cdots > h_d \geq 0.$$

(3) *We have a surjective homomorphism of A-modules*

$$\partial_{\mathrm{S}}(p) : \mathrm{Split}_A^d(p) \to A$$

such that $\partial_{\mathrm{S}}(p)(f_1(\xi_1)\cdots f_d(\xi_d)) = \mathrm{Res}(\frac{f_1}{p},\dots,\frac{f_d}{p})$ *for all* $f_1,\dots,f_d$ *in* $A[T]$.

(4) *We have a unique homomorphism of A-modules*

$$\partial_{\mathrm{F}} : \mathrm{Fact}_A^d(p) \to A$$

such that $\partial_{\mathrm{S}}(p) = \partial_{\mathrm{F}}(p)\partial(p_1)$. *This homomorphism is determined by the equality* $\partial_{\mathrm{F}}(p)(\mathrm{Res}(\frac{f_1}{p_1},\dots,\frac{f_d}{p_1})) = \mathrm{Res}(\frac{f_1}{p},\dots,\frac{f_d}{p})$ *for all* $f_1,\dots,f_d$ *in* $A[T]$.

Proof. (1) It follows from Lemma 5.9 that $\mathrm{Split}_A^d(p)$ is a splitting algebra $\mathrm{Split}_{A_1}(p_1)$ for $p_1(T)$ over $A_1 = \mathrm{Fact}_A^d(p)$ with universal roots $\xi_1,\dots,\xi_d$. The existence of $\partial(p_1)$ thus follows from the theorem.

(2) Since Res is linear in $f_1,\dots,f_d$ by Lemma 2.2 it follows, in particular, that $\mathrm{Fact}_A(p)$ is generated as an A-module by the elements $\partial(p_1)(\pi_1^{h_1}\cdots\pi_d^{h_d})$ where $0 \le h_i \le n-i$ for $i = 1,\dots,d$. Since $\partial(p_1)$ is alternating in $f_1,\dots,f_d$ by Lemma 2.2 we have that $\mathrm{Fact}_A^d(p)$ is generated as an A-module by the elements $\partial(p_1)(\pi_1^{h_1}\cdots\pi_d^{h_d})$ with $n > h_1 > \cdots > h_d \ge 0$.

It remains to prove that these elements are linearly independent over A. To show this we use the natural inclusions of algebras $A \subseteq \mathrm{Fact}_A^d(p) \subseteq \mathrm{Split}_A^d(p)$. Here $\mathrm{Split}_A^d(p)$ is a free A-module of rank $n(n-1)\cdots(n-d+1)$, and is a free $\mathrm{Fact}_A^d(p)$-module of rank $d(d-1)\cdots 1$. Since $\mathrm{Fact}_A^d(p)$ is generated, as an A-module by the $\binom{n}{d} = \frac{n(n-1)\cdots(n-d+1)}{d(d-1)\cdots 1}$ elements $\partial(p_1)(\pi_1^{h_1}\cdots\pi_d^{h_d})$ for $n > h_1 > \cdots > h_d \ge 0$, it follows that these elements form an A-module basis for $\mathrm{Fact}_A^d(p)$.

(3) It follows from Definition 2.1 and Proposition 2.3 that $\mathrm{Res}(\frac{f_1}{p},\dots,\frac{f_d}{p}) = \mathrm{Res}(\frac{f_1}{p},\dots,\frac{f_d}{p},\frac{T^{n-d-1}}{p},\dots,\frac{T^0}{p})$ for all $f_1,\dots,f_d$ in $A[T]$. Thus we obtain $\partial_{\mathrm{S}}(p)$ as the restriction of $\partial(p)$ of the theorem to the A-submodule in $A[\xi_1,\dots,\xi_d]$ generated by the elements $f_1(\xi_1)\cdots f_d(\xi_d)\xi_{d+1}^{n-d-1}\cdots\xi_n^0$.

(4) It follows from assertion (2) that we can define an A-module homomorphism $\partial_{\mathrm{F}}(p)$ by $\partial_{\mathrm{F}}(p)(\mathrm{Res}(\frac{f_1}{p_1},\dots,\frac{f_d}{p_1})) = \mathrm{Res}(\frac{f_1}{p},\dots,\frac{f_d}{p})$ for all $f_1,\dots,f_d$ in $A[T]$. It is clear that we have $\partial_{\mathrm{S}}(p) = \partial_{\mathrm{F}}(p)\partial(p_1)$, and since $\partial(p_1)$ is surjective this equality determines $\partial_{\mathrm{F}}(p)$ uniquely.

8. Exterior products and factorization algebras

We finally show how exterior products introduced by Gatto ([G], [GS1]) to describe the cohomology of grassmannians (see also [LT1] and [LT2]) fit into the theory we have described in the previous chapters.

The first of the two below results is the main result of [LT1], and the second is the main result of [LT2].

8.1 Theorem. *Let $T_1,\dots,T_n$ be algebraically independent elements over the ring A, and let*

$$P(T) = (T - T_1)\cdots(T - T_n).$$

Then there is an isomorphism of A-modules

$$\bigwedge^n A[T] \to A[T_1, \dots, T_n]^{\mathrm{sym}} \tag{8.1.1}$$

that maps $f_1 \wedge \cdots \wedge f_n$ to $\operatorname{Res}(\frac{f_1}{P}, \dots, \frac{f_n}{P})$.

The isomorphism (8.1.1) induces a structure on $\bigwedge^n A[T]$ as an $A[T_1, \dots, T_n]^{\mathrm{sym}}$-module that makes the canonical homomorphism $A[T_1, \dots, T_n] \to \bigwedge^n A[T]$ that takes the element $f_1(T_1) \cdots f_n(T_n)$ to $f_1 \wedge \cdots \wedge f_n$ to an $A[T_1, \dots, T_n]^{\mathrm{sym}}$-module homomorphism.

With this $A[T_1, \dots, T_n]^{\mathrm{sym}}$-module structure on $\bigwedge^n A[T]$ we have the determinantal formula

$$f_1 \wedge \cdots \wedge f_n = \operatorname{Res}\left(\frac{f_1}{P}, \dots, \frac{f_n}{P}\right) T^{n-1} \wedge \cdots \wedge T^0$$

for all $f_1, \dots, f_n$ in $A[T]$.

Proof. The map

$$\partial(P) : A[T_1, \dots, T_n] \to A[T_1, \dots, T_n]^{\mathrm{sym}}$$

of Lemma 4.3 that is determined by $\partial(P)(f_1(T_1) \cdots f_n(T_n)) = \operatorname{Res}(\frac{f_1}{P}, \dots, \frac{f_n}{P})$ is multilinear and alternating in $f_1, \dots, f_n$ by Lemma 2.2. Consequently it factors via the canonical homomorphism $A[T_1, \dots, T_n] \to \bigwedge^n A[T]$ in an A-module homomorphism (8.1.1). This homomorphism maps the A-module basis $T^{h_1} \wedge \cdots \wedge T^{h_n}$ with $h_1 > \cdots > h_n \geq 0$ of $\bigwedge^n A[T]$ to the elements $\operatorname{Res}(\frac{T^{h_1}}{P}, \dots, \frac{T^{h_n}}{P})$ with $h_1 > \cdots > h_n \geq 0$. The latter elements form an A-module basis for $A[T_1, \dots, T_n]^{\mathrm{sym}}$ by Corollary 4.3 (2). Consequently (8.1.1) is an A-module isomorphism. The assertions on the $A[T_1, \dots, T_n]^{\mathrm{sym}}$-module structure are obvious.

The determinant formula follows since $f_1 \wedge \cdots \wedge f_n$ and $\operatorname{Res}(\frac{f_1}{P}, \dots, \frac{f_n}{P})T^{n-1} \wedge \cdots \wedge T^0$ map to the same element by the map (8.1.1) since $\operatorname{Res}(\frac{T^{n-1}}{P}, \dots, \frac{T^0}{P}) = 1$ by Proposition 2.3.

8.2 Theorem. *Let $p(T)$ be a monic polynomial of degree n in $A[T]$ and let $\operatorname{Fact}_A^d(p)$ be a factorization algebra for $p(T)$ with universal splitting $p(T) = p_1(T)p_2(T)$ where $p_1(T)$ has degree d. Moreover, let $A[\xi] = A[T]/(p)$ with ξ the class of T modulo $p(T)$. Then we have an A-module isomorphism*

$$\bigwedge^d A[\xi] \to \operatorname{Fact}_A^d(p) \tag{8.2.1}$$

that maps $f_1(\xi) \wedge \cdots \wedge f_d(\xi)$ to $\operatorname{Res}(\frac{f_1}{p_1}, \dots, \frac{f_d}{p_1})$.

The isomorphism (8.2.1) induces a $\operatorname{Fact}_A^d(p)$-module structure on $\bigwedge^d A[\xi]$ that makes the canonical homomorphism $\bigwedge^d A[T] \to \bigwedge^d A[\xi]$ to an $A[T_1, \dots, T_d]^{\mathrm{sym}}$-module homomorphism when we consider $\bigwedge^d A[\xi]$ as an $A[T_1, \dots, T_d]^{\mathrm{sym}}$-module via the surjection $\varphi : A[T_1, \dots, T_d]^{\mathrm{sym}} \to \operatorname{Fact}_A^d(p_1)$ that is determined by the equation $\varphi((T - T_1) \cdots (T - T_d)) = p_1(T)$.

With this $\mathrm{Fact}^d_A(p)$*-module structure on* $\bigwedge^d A[\xi]$ *we have a determinantal formula*

$$f_1(\xi) \wedge \cdots \wedge f_d(\xi) = \mathrm{Res}\left(\frac{f_1}{p_1}, \ldots, \frac{f_d}{p_1}\right) \xi^{d-1} \wedge \cdots \wedge \xi^0.$$

Proof. Let $T_1, \ldots, T_d$ be algebraically independent elements over A and let

$$P_1(T) = (T - T_1) \cdots (T - T_d) = T^d - C'_1 T^{d-1} + \cdots + (-1)^d C'_d.$$

By Corollary 4.3 (3) we obtain that $A[T_1, \ldots, T_d]^{\mathrm{sym}} = A[C'_1, \ldots, C'_d]$, and from Theorem 8.1 we obtain an $A[T_1, \ldots, T_d]^{\mathrm{sym}}$-module homomorphism

$$\chi : \bigwedge^d A[T] \to A[T_1, \ldots, T_d]^{\mathrm{sym}}$$

determined by $\chi(f_1 \wedge \cdots \wedge f_d) = \mathrm{Res}(\frac{f_1}{P_1}, \ldots, \frac{f_d}{P_1})$ for all $f_1, \ldots, f_d$ in $A[T]$.

Let $\mathrm{Split}_A(p) = A[\xi_1, \ldots, \xi_n]$ be a splitting algebra for $p(T)$ over A with universal roots $\xi_1, \ldots, \xi_d$, and write

$$p_1(T) = (T - \xi_1) \cdots (T - \xi_d) = T^d - c'_1 T^{d-1} + \cdots + (-1)^d c'_d.$$

It follows from Corollary 5.11 that the algebra $A[c'_1, \ldots, c'_d]$ in $A[\xi_1, \ldots, \xi_d]$ is a d'th factorization algebra $\mathrm{Fact}^d_A(p)$ for $p(T)$ over A with universal splitting $p(T) = p_1(T)p_2(T)$, where $p_2(T) = (T - \xi_{d+1}) \cdots (T - \xi_n)$.

By Lemma 6.1 we have that $C'_1, \ldots, C'_d$ are algebraically independent over A. Thus we have an A-algebra homomorphism

$$\psi : A[C'_1, \ldots, C'_d] \to A[c'_1, \ldots, c'_d] = \mathrm{Fact}^d_A(p)$$

defined by $\psi(C'_i) = c'_i$ for $i = 1, \ldots, d$. We obtain that

$$p(T) = p_1(T)p_2(T) = (T^d - c'_1 T^{d-1} + \cdots + (-1)^d c'_d)p_2(T) = {}^{\psi}P_1(T)p_2(T)$$

in $\mathrm{Fact}^d_A(p)$. Consequently it follows from Lemma 2.2 that, for all $f_1, \ldots, f_d$ in $A[T]$ and all natural numbers h, we have

$$\begin{aligned}\psi\chi(T^h p \wedge f_2 \wedge \cdots \wedge f_d) &= \psi\,\mathrm{Res}\left(\frac{T^h p}{P_1}, \frac{f_2}{P_1}, \ldots, \frac{f_d}{P_1}\right) \\ &= \mathrm{Res}\left(\frac{{}^{\psi}(T^h p)}{{}^{\psi}P_1}, \frac{{}^{\psi}f_2}{{}^{\psi}P_1}, \ldots, \frac{{}^{\psi}f_d}{{}^{\psi}P_1}\right) = \mathrm{Res}\left(T^h p_2, \frac{f_2}{p_1}, \ldots \frac{f_d}{p_1}\right) = 0,\end{aligned}$$

where we consider $p(T)$ as an element in either of the algebras of the inclusion $A[T] \subseteq A[C'_1, \ldots, C'_d][T]$ or in $A[c'_1, \ldots, c'_d][T]$, in accordance with the convention in 5.8. Thus $\psi\chi : \bigwedge^d A[T] \to \mathrm{Fact}^d_A(p)$ is zero on the elements of the form $T^h p \wedge f_2 \wedge \cdots \wedge f_d$. However these elements generate the kernel of the canonical map $\bigwedge^d A[T] \to \bigwedge^d A[\xi]$, since $A[T]$ is the direct sum, as an A-module, of the kernel $(p) = Ap + ATp + AT^2p + \cdots$ of the homomorphism $A[T] \to A[\xi]$ and the A-module $A + AT +$

$\cdots + AT^{n-1}$. Consequently the homomorphism $\psi\chi$ factors via the canonical map $\bigwedge^d A[T] \to \bigwedge^d A[\xi]$ in the A-module homomorphism (8.2.1). This homomorphism maps the A-module basis $\xi^{h_1} \wedge \cdots \wedge \xi^{h_d}$ with $n > h_1 > \cdots > h_d \geq 0$ for $\bigwedge^d A[\xi]$ to the elements $\mathrm{Res}(\frac{T^{h_1}}{P_1}, \ldots, \frac{T^{h_d}}{P_1})$ with $n > h_1 > \cdots > h_d \geq 0$ that form an A-module basis for $\mathrm{Fact}^d_A(p)$ by Corollary 7.2. Consequently (8.2.1) is an A-module isomorphism.

The assertions on the $\mathrm{Fact}^d_A(p)$-module structure follow easily from the commutative diagram

$$\begin{array}{ccc}
\bigwedge^d A[T] & \longrightarrow & A[T_1, \ldots, T_d]^{\mathrm{sym}} \\
\downarrow & & \downarrow \psi \\
\bigwedge^d A[\xi] & \longrightarrow & \mathrm{Fact}^d_A(p)
\end{array}$$

where the upper and lower maps are the isomorphisms (8.1.1), respectively (8.2.1), and where the left homomorphism is the canonical map.

The determinantal formula follows since the elements $f_1(\xi) \wedge \cdots \wedge f_d(\xi)$ and $\mathrm{Res}(\frac{f_1}{p_1}, \ldots, \frac{f_d}{p_1})\xi^{d-1} \wedge \cdots \wedge \xi^0$ map to the same element by the map (8.2.1) because $\mathrm{Res}(\frac{T^{d-1}}{p_1}, \cdots, \frac{T^0}{p_1}) = 1$ by Proposition 2.3.

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