SPLITTING ALGEBRAS AND GYSIN HOMOMORPHISMS

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Abstract. We give several definitions of splitting algebras and give the main properties of such algebras.

1. Definition of splitting algebras

1.1 Definition. Let $A$ be a commutative ring with unit and let

$$p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$$

be a polynomial in the variable $T$ with coefficients in $A$. For $d = 1, 2, \ldots, n$ we have a covariant functor $\text{Split}^d_p$ from $A$-algebras to sets that maps a homomorphism $\varphi : A \to B$ to

$$\text{Split}^d_p(B) = \{ \text{splittings } (\varphi p)(T) = T^n - \varphi(c_1)T^{n-1} + \cdots + (-1)^n \varphi(c_n) \}
= (T - b_1) \cdots (T - b_d) s(T) \text{ over } B \text{ where}
\quad b_1, \ldots, b_d \text{ is an ordered sequence of roots},$$

and where the map

$$\text{Split}^d_p(\psi) : \text{Split}^d_p(B) \to \text{Split}^d_p(C)$$

corresponding to an $A$-algebra homomorphism $\psi : B \to C$ is defined by

$$\text{Split}^d_p(\psi)((\varphi p)(T)) = (T - \psi(b_1)) \cdots (T - \psi(b_n))(\psi q)(T).$$

An $A$-algebra $\text{Split}^d_A(p)$ that represents the functor $\text{Split}^d_p$ we call a $d$'th splitting algebra for $p(T)$ over $A$ and if

$$p(T) = (T - \xi_1) \cdots (T - \xi_d)p_{d+1}(T)$$

is the universal splitting over $\text{Split}^d_A(p)$ we call the ordered set of roots $\xi_1, \ldots, \xi_d$, the universal roots.

It is often more convenient to define splitting algebras in the following more concrete way.
1.2 Definition. For $d = 1, \ldots, n$ a $d$'th splitting algebra for the polynomial

$$p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$$

in the variable $T$ with coefficients in $A$ is an $A$-algebra $\text{Split}^d_A(p)$, over which $p(T)$, considered as a polynomial over $\text{Split}^d_A(p)$, splits as

$$p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T),$$

and where $\text{Split}^d_A(p)$ has the following universal property:

For every $A$-algebra $\varphi : A \to B$ over which we have a splitting

$$(\varphi p)(T) := T^n - \varphi(c_1) T^{n-1} + \cdots + (-1)^n \varphi(c_n) = (T - b_1) \cdots (T - b_d) s(T)$$

over $B$, there is a unique $A$-algebra homomorphism

$$\psi : \text{Split}^d_A(p) \to B$$

that satisfies $\psi(\xi_i) = b_i$ for $i = 1, \ldots, d$.

We let $\text{Split}^0_A(p) = A$ and $p(T) = p_1(T)$. The ordered set of roots $\xi_1, \ldots, \xi_d$ of $p(T)$ in $\text{Split}^d_A(p)$ we call the universal roots, and the splitting $p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T)$ we call the universal splitting.

2. The first construction of splitting algebras

The main properties of splitting algebras, including their existence, are contained in the following results.

2.1 Proposition. Let $\text{Split}_A^d(p)$ be a $d$'th splitting algebra for the polynomial $p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$ with coefficients in $A$, and with universal splitting

$$p(T) = (T - \xi_1) \cdots (T - \xi_d) p_{d+1}(T).$$

(1) The $A$-algebra $\text{Split}_A^d(p)$ is generated by $\xi_1, \ldots, \xi_d$. That is, $\text{Split}_A^d(p) = A[\xi_1, \ldots, \xi_d]$.

(2) The $\text{Split}_A^d(p)$-algebra $\text{Split}_A^d(p)[T]/(p_{d+1})$ is a $(d + 1)$'th splitting algebra for $p(T)$ over $A$ with universal roots $\xi_1, \ldots, \xi_d, \xi_{d+1}$ with $\xi_{d+1}$ the class of $T$.

Proof. (1) It is clear that the $A$-algebra $A[\xi_1, \ldots, \xi_d]$ has the universal property of a splitting algebra described in Definition 1.2. Since the universal property characterizes the $d$'th splitting algebra up to isomorphisms it follows that the homomorphism defined by the inclusion of $A[\xi_1, \ldots, \xi_d]$ in $\text{Split}_A^d(p)$ is an isomorphism. That is, we have $\text{Split}_A^d(p) = A[\xi_1, \ldots, \xi_d]$ as asserted.

(2) Let $\varphi : A \to B$ be an $A$-algebra over which we have a splitting

$$(\varphi p(T)) = (T - b_1) \cdots (T - b_{d+1}) s(T)$$
over $B$. We then have a unique $A$-algebra homomorphism $\chi : \text{Split}_d^d(p) \to B$ such that $\chi(\xi_i) = b_i$ for $i = 1, \ldots, d$. It follows from the equality

$$(\chi p)(T) = (T - \chi(\xi_1)) \cdots (T - \chi(\xi_d))(\chi p_{d+1})(T) = (T - b_1) \cdots (T - b_d)(T - b_{d+1})s(T)$$

that $(\chi p_{d+1})(T) = (T - b_{d+1})s(T)$ in the polynomial ring $B[T]$.

Let $\xi_{d+1}$ be the class of $T$ in $\text{Split}_d^d(p)[T]/(p_{d+1})$. Since $(\chi p_{d+1})(T)$ has the root $b_{d+1}$ in $B$ it follows from the universal property of residue algebras that we can extend $\chi$ uniquely to an $\text{Split}_d^d(p)$-algebra homomorphism $\chi : \text{Split}_d^d(p)[\xi_{d+1}] \to B$ such that $\chi(\xi_{d+1}) = b_{d+1}$. Thus we have an $A$-algebra homomorphism

$$\psi : A[\xi_1, \ldots, \xi_{d+1}] = A[\xi_1, \ldots, \xi_d][\xi_{d+1}] \to B$$

that takes the values $\psi(\xi_i) = b_i$ for $i = 1, \ldots, d + 1$, and, by the first part of the proposition, is uniquely determined by these values. Thus $A[\xi_1, \ldots, \xi_{d+1}]$ is a $(d+1)'$st splitting algebra for $p(T)$ over $A$.

2.2 Corollary. With the notation of the proposition we have

1. For each $d = 0, 1, \ldots, n$ there exists a $d'$th splitting algebra for $p(T)$ over $A$.
2. The splitting algebra $\text{Split}_d^d(p)$ is free as an $A$-module with a basis $\xi_1^{h_1} \cdots \xi_d^{h_d}$ for $0 \leq h_j \leq n - j$ and $j = 1, \ldots, d$.

Proof. (1) The existence of the $d'$th splitting algebra follows from assertion (2) of the proposition by induction on $d$, beginning at $\text{Split}_0^0(p) = A$.

(2) This assertion also follows from assertion (2) of the proposition by induction on $d$ since $\text{Split}_0^0(p) = A$, and since $\text{Split}_d^{d+1}(p) = \text{Split}_d^d(p)[\xi_{d+1}]$ is a free $\text{Split}_d^d(p)$-module with basis $1, \xi_{d+1}, \ldots, \xi_{d+1}^{n-d-1}$.

3. The second construction of splitting algebras

We give a proof of the existence of splitting algebras depending on the algebraic independence of elementary symmetric polynomials.

3.1 Notation. Let $T_1, \ldots, T_n$ be independent variables over the polynomial ring $A[T]$. For $i = 1, \ldots, n$ denote by $C_i = c_i(T_1, \ldots, T_n)$ the $i$'th elementary symmetric polynomial in $T_1, \ldots, T_n$, and for $i = 1, \ldots, n - d$ we denote by $D_i = c_i(T_{d+1}, \ldots, T_n)$ the $i$'th elementary symmetric polynomial in the variables $T_{d+1}, \ldots, T_n$.

Over the ring $A[T_1, \ldots, T_n]$ we have a splitting

$$T^n - C_1 T^{n-1} + \cdots + (-1)^n C_n = (T - T_1) \cdots (T - T_n) = (T - T_1) \cdots (T - T_d)(T^{n-d} - D_1 T^{n-d-1} + \cdots + (-1)^{n-d} D_{n-d}).$$

(3.1.1)

In particular, we have that the $A$-algebra $A[C_1, \ldots, C_n, T_1, \ldots, T_d]$ in $A[T_1, \ldots, T_n]$ is equal to the $A$-algebra $A[D_1, \ldots, D_{n-d}, T_1, \ldots, T_d]$. 

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3.2 Proposition. The residue algebra

\[ A[C_1, \ldots, C_n, T_1, \ldots, T_d]/(c_1 - C_1, \ldots, c_n - C_n) \]

is a \(d\)'th splitting algebra for \(p(T)\) over \(A\). If \(\xi_1, \ldots, \xi_d\) denote the classes of \(T_1, \ldots, T_d\) and \(q_1, \ldots, q_{n-d}\) denote the classes of \(D_1, \ldots, D_{n-d}\), the universal splitting is

\[ p(T) = (T - \xi_1) \cdots (T - \xi_d)(T^{n-d} - q_1T^{n-d-1} + \cdots + (-1)^{n-d}q_{n-d}). \quad (3.2.1) \]

→ Proof. The existence of the splitting (3.2.1) of \(p(T)\) follows from the equality (3.1.1), since \(c_i\) is the class of \(C_i\) for \(i = 1, \ldots, n\).

Let \(\varphi : A \to B\) be an \(A\)-algebra over which there is a splitting

\[ (\varphi p)(T) = (T - b_1) \cdots (T - b_d)(T^{n-d} - d_1T^{n-d-1} + \cdots + (-1)^{n-d}d_{n-d}). \]

Since \(T_1, \ldots, T_d, D_1, \ldots, D_{n-d}\) are algebraically independent over \(A\) we can define an \(A\)-algebra homomorphism

\[ \chi : A[C_1, \ldots, C_n, T_1, \ldots, T_d] = A[D_1, \ldots, D_{n-d}, T_1, \ldots, T_d] \to B \]

such that \(\chi(T_i) = b_i\) for \(i = 1, \ldots, d\) and \(\chi(D_i) = d_i\) for \(i = 1, \ldots, n - d\). It follows from (3.0.1) that

\[ T^n - \chi(T_1)T^{n-1} + \cdots + (-1)^n\chi(C_n) = (T - b_1) \cdots (T - b_d)(T^{n-d} - d_1T^{n-d-1} + \cdots + (-1)^{n-d}d_{n-d}). \quad (3.2.2) \]

In particular we must have that \(\chi(C_i) = \varphi(c_i)\) for \(i = 1, \ldots, n\). Consequently the homomorphism \(\chi\) induces an \(A\)-algebra homomorphism

\[ \psi : A[C_1, \ldots, C_n, T_1, \ldots, T_d]/(c_1 - C_1, \ldots, c_n - C_n) \to B \]

such that the conditions \(\psi(\xi_i) = b_i\) are fulfilled for \(i = 1, \ldots, n\). Since \(\psi(C_i) = \varphi(c_i)\) it follows from (3.2.2) that \(\psi\) is uniquely determined by these conditions. Hence the residue algebra of the proposition satisfies the universal properties of a \(d\)'th splitting algebra of Definition 1.2.

4. THE THIRD CONSTRUCTION OF SPLITTING ALGEBRAS

In this section we give an elementary and natural construction of splitting algebras based upon the division algorithm.

4.1 Notation. Let \(T_1, \ldots, T_d\) be algebraically independent variables over the polynomial ring \(A[T]\). The division algorithm used over \(A[T_1, \ldots, T_d]\) to the polynomial \(p(T)\) modulo the polynomial \(P(T) = (T - T_1) \cdots (T - T_d)\) gives

\[ p(T) = (T - T_1) \cdots (T - T_d)(T^{n-d} - q_1T^{n-d-1} + \cdots + (-1)^{n-d}q_{n-d}) + r_{d-1}T^{d-1} + r_{d-2}T^{d-2} + \cdots + r_0. \quad (4.1.1) \]
4.2 Proposition. The residue algebra

\[ A[T_1, \ldots, T_d]/(r_0, \ldots, r_{d-1}) \]

is a \(d \)’th splitting algebra for \( p(T) \) over \( A \). The classes \( \xi_1, \ldots, \xi_d \) of \( T_1, \ldots, T_d \) are the universal roots and the universal splitting is

\[ p(T) = (T - \xi_1) \cdots (T - \xi_d)p_{d+1}(T) \] (4.2.1)

where the coefficient of \( T^i \) in \( p_{d+1}(T) \) is the class of \( q_{n-d-i} \) for \( i = 0, \ldots, n - d - 1 \).

Proof. The existence of the splitting (4.1.1) follows from (4.0.1).

Let \( \varphi : A \rightarrow B \) be an \( A \)-algebra over which we have a splitting

\[ (\varphi p)(T) = (T - b_1) \cdots (T - b_d)(T^{n-d} - d_1T^{n-d-1} + \cdots + (-1)^{n-d-1}d_{n-d}) \] (4.2.2)

We define a homomorphism of \( A \)-algebras

\[ \chi : A[T_1, \ldots, T_d] \rightarrow B \]

by \( \chi(T_i) = b_i \) for \( i = 1, \ldots, d \). It follows from (4.0.1) that

\[ (\chi p)(T) = (T - b_1) \cdots (T - b_d) \]

\[ (T^{n-d} - \chi(q_1)T^{n-d-1} + \cdots + (-1)^{n-d}\chi(q_{n-d})) + \chi(r_{d-1})T^{d-1} + \cdots + \chi(r_0) \]

over \( B \). Hence it follows from (4.1.2) that we have equalities

\[ 0 = \chi(r_{d-1}) = \cdots = \chi(r_0) \quad \text{and} \quad \chi(q_i) = d_i \quad \text{for} \quad i = 0, \ldots, n - d - 1, \]

in \( B \). We consequently have that \( \chi \) factors via a unique \( A \)-algebra homomorphism

\[ \psi : A[T_1, \ldots, T_d]/(r_0, \ldots, r_{d-1}) \rightarrow B \]

such that \( \psi(\xi_i) = b_i \) for \( i = 1, \ldots, d \). Then

\[ (\varphi p)(T) = (T - b_1) \cdots (T - b_d)(\psi p_{d+1})(T) \]

over \( B \). We have thus that the residue algebra of the proposition satisfies the universal property of Definition 1.2.

5. Residues

We shall in this section introduce residues.

5.1 Notation. Let

\[ p(T) = T^n - c_1T^{n-1} + \cdots + (-1)^nc_n \]

be a polynomial in the variable \( T \) with coefficients in the ring \( A \). We define elements \( \ldots, s_{-2} = 0, s_{-1} = 0, s_0 = 1, s_1, s_2, \ldots \) in the ring \( A \) by the relation

\[ 1 = (1 - c_1T + \cdots + (-1)^nc_nT^n)(1 + s_1T + s_2T^2 + \cdots) \]

in the algebra of formal power series in \( T \) over \( A \).
5.2 Definition. Let \( g_i = \cdots + a_{i-1}T + a_{i0} + \frac{a_i}{T} + \frac{a_i}{T^2} + \cdots \) for \( i = 1, \ldots, d \) be formal Laurent series in the variable \( \frac{1}{T} \). We let
\[
\text{Res}(g_1, \ldots, g_d) = \det \left( \begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1d} \\
\vdots & \vdots & \ddots & \vdots \\
a_{d1} & a_{d2} & \cdots & a_{dd}
\end{array} \right).
\]

5.3 Proposition. We have that \( \text{Res}(g_1, \ldots, g_d) \) is multilinear and alternating in \( g_1, \ldots, g_d \), and it is zero if at least one of the \( g_i \) is a polynomial in \( T \).

Proof. All the assertions follow immediately from Definition 5.1.

5.4 Proposition. For all natural numbers \( h_1, \ldots, h_d \) we have
\[
\text{Res} \left( \frac{T^{h_1}}{p}, \ldots, \frac{T^{h_d}}{p} \right) = \det \left( \begin{array}{cccc}
s_{h_1-n+1} & s_{h_1-n+2} & \cdots & s_{h_1-n+d} \\
\vdots & \vdots & \ddots & \vdots \\
s_{h_d-n+1} & s_{h_d-n+2} & \cdots & s_{h_d-n+d}
\end{array} \right). \tag{5.4.1}
\]

In particular, when \( 0 \leq h_j \leq n - j \) for \( j = 1, \ldots, d \) we have
\[
\text{Res} \left( \frac{T^{h_1}}{p}, \ldots, \frac{T^{h_d}}{p} \right) = \begin{cases} 
1 & \text{when } h_j = n - j \text{ for } j = 1, \ldots, d \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. The first part of the proposition follows from Definition 5.1 and the equality
\[
\frac{T^h}{p} = T^{h-n} \left( 1 - c_1 \frac{1}{T} + \cdots + (-1)^d \frac{c_d}{T^d} \right) = T^{h-n} \left( 1 + \frac{s_1}{T} + \frac{s_2}{T^2} + \cdots \right)
\]
of formal Laurent series in \( \frac{1}{T} \).

The second part follows from the first since when \( 0 \leq h_j \leq n - j \) for \( j = 1, \ldots, d \) the \( d \times d \)-matrix \( (s_{h_i-n+j}) \) that gives \( \text{Res} \left( \frac{T^{h_1}}{p}, \ldots, \frac{T^{h_d}}{p} \right) \) is upper triangular. When at least one of the equalities \( h_j \leq n - j \) is strict there is a zero on the diagonal. Otherwise all the diagonal elements are one.

**TWO AUXILIARY RESULTS**

To prove the main result we need the following result on matrices. We are thankful to Bengt Ek (The Royal Institute of Technology, Stockholm) and Michael Shapiro (Michigan State University, East Lansing) for the elegant proof.

6.1 Lemma. Let \((a_{ij})\) and \((b_{ij})\) be two \( d \times d \)-matrices with coordinates in the ring \( A \).

1. For \( j = 0, 1, \ldots, d \) we have
\[
\sum \det \left( \begin{array}{cccc}
a_{11} & b_{11} & \cdots & b_{1j} \\
\vdots & \ddots & \ddots & \vdots \\
a_{d1} & \cdots & \cdots & b_{dj}
\end{array} \right) = \sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right). \tag{6.4.1}
\]

2. For \( j = 0, 1, \ldots, d \) we have
\[
\sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right) = \sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right). \tag{6.4.1}
\]

3. For \( j = 0, 1, \ldots, d \) we have
\[
\sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right) = \sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right). \tag{6.4.1}
\]

4. For \( j = 0, 1, \ldots, d \) we have
\[
\sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right) = \sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right). \tag{6.4.1}
\]

5. For \( j = 0, 1, \ldots, d \) we have
\[
\sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right) = \sum \det \left( \begin{array}{cccc}
a_{11} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
b_{i1} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{array} \right). \tag{6.4.1}
\]
where both sums are over all indices $1 \leq i_1 < \cdots < i_j \leq d$.

(2) When

\[
\begin{pmatrix}
  b_{11} & \cdots & b_{1d}
  \\
  \vdots & \ddots & \vdots
  \\
  b_{d1} & \cdots & b_{dd}
\end{pmatrix}
= \begin{pmatrix}
  a_{12} & \cdots & a_{1d+1}
  \\
  \vdots & \ddots & \vdots
  \\
  a_{d2} & \cdots & a_{dd+1}
\end{pmatrix}
\]

(6.4.1)

for some elements $a_{1d+1}, \ldots, a_{dd+1}$ of $A$ we obtain for $j = 0, 1, \ldots, d$.

\[
\det\left(\begin{array}{cccc}
  a_{11} & \cdots & a_{1d-j} & a_{1d-j+2} & \cdots & a_{1d+1} \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
  a_{d1} & \cdots & a_{dd-j} & a_{dd-j+2} & \cdots & a_{dd+1}
\end{array}\right)
= \sum_{1 \leq i_1 < \cdots < i_j \leq d} \det\left(\begin{array}{cccc}
  a_{11} & \cdots & a_{1d} \\
  \vdots & \ddots & \vdots & \ddots \\
  a_{ij} & \cdots & a_{ijd+1} \\
  a_{i1} & \cdots & a_{idd}
\end{array}\right).
\]

(6.4.2)

(3) When

\[a_{i,d+1} - c_1 a_{i,d} + \cdots + (-1)^d c_d a_{i,1} = 0 \quad \text{for} \quad i = 1, \ldots, d\]

we have for $j = 1, \ldots, d$ the equality

\[
\det\left(\begin{array}{cccc}
  a_{11} & \cdots & a_{1d-j} & a_{1d-j+2} & \cdots & a_{1d+1} \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
  a_{d1} & \cdots & a_{dd-j} & a_{dd-j+2} & \cdots & a_{dd+1}
\end{array}\right)
= c_j \det\left(\begin{array}{cccc}
  a_{11} & \cdots & a_{1d} \\
  \vdots & \ddots & \vdots & \ddots \\
  a_{i1} & \cdots & a_{idd}
\end{array}\right).
\]

(6.4.3)

\[\rightarrow \quad \text{Proof.} \quad \text{We obtain the left and right sides of (6.4.1) by expanding the determinant of the } d \times d\text{-matrix } (a_{ij}) + T(b_{ij}) = (a_{ij} + Tb_{ij}) \text{ along “columns”, respectively “rows”, and comparing the coefficient of } T^j.\]

\[\rightarrow \quad \text{When the equality } 6.* \text{ holds the equality (6.4.2) follows from (6.4.1) because, on the left hand side of (6.4.1), the matrix of which we take the determinant has columns } i_k \text{ and } i_k + 1 \text{ equal when } i_k + 1 < i_{k+1}, \text{ and columns } i_j \text{ and } i_j + 1 \text{ are equal if } i_j < d.\]

\[\rightarrow \quad \text{To obtain } 6.*.* \text{ we substitute } c_1 a_{ikd} - c_2 a_{ikd-1} + \cdots + (-1)^{d+1} c_d a_{ikd+1} \text{ for } a_{ik,d+1} \text{ on the left hand side of } 6.*.*.\]

We are thankful to Bengt Ek (The Royal Institute of Technology, Stockholm) for the elegant proof.

\textbf{6.1 Lemma.} \textit{Let } $d$ \textit{be a natural number and let}

\[(x_{ij}(k)) = \begin{pmatrix}
  x_{11}(k) & \cdots & x_{1d}(k) \\
  \vdots & \ddots & \vdots \\
  x_{d1}(k) & \cdots & x_{dd}(k)
\end{pmatrix} \quad \text{for} \quad k = 0, 1, \ldots \]

be $d \times d$-matrices with coefficients in the ring $A$. For $k = 0, 1, \ldots$ we let $I_k$ be all multiindices $(k_1, \ldots, k_d)$ of natural numbers such that $k_1 + \cdots + k_d = k$.

(1) For $k = 0, 1, \ldots$ we have

\[
\sum \det\left(\begin{array}{cccc}
  x_{11}(k_1) & \cdots & x_{1d}(k_d) \\
  \vdots & \ddots & \vdots \\
  x_{d1}(k_1) & \cdots & x_{dd}(k_d)
\end{array}\right) = \sum \det\left(\begin{array}{cccc}
  x_{11}(k_1) & \cdots & x_{1d}(k_1) \\
  \vdots & \ddots & \vdots \\
  x_{d1}(k) & \cdots & x_{dd}(k)
\end{array}\right),
\]

(6.1.1)
where both sums are over all \((k_1, \ldots, k_d) \in \mathcal{I}_k\).

(2) Assume that for \(k = 0, 1, \ldots\) we have

\[
\begin{pmatrix}
  x_{11}(k) & \cdots & x_{1d}(k) \\
  \vdots & \ddots & \vdots \\
  x_{d1}(k) & \cdots & x_{dd}(k)
\end{pmatrix}
= \begin{pmatrix}
  x_{11+k} & \cdots & x_{1d+k} \\
  \vdots & \ddots & \vdots \\
  x_{d1+k} & \cdots & x_{dd+k}
\end{pmatrix}
\]  

(6.1.2)

for elements \(x_{ij}\) in \(A\). Then

\[
\det \begin{pmatrix}
  x_{11} & \cdots & x_{1d-1} & x_{1d+k} \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{d1} & \cdots & x_{dd-1} & x_{dd+k}
\end{pmatrix} = \sum_{(k_1, \ldots, k_d) \in \mathcal{I}_k} \det \begin{pmatrix}
  x_{11+k_1} & \cdots & x_{1d+k_1} \\
  \vdots & \ddots & \vdots \\
  x_{d1+k_d} & \cdots & x_{dd+k_d}
\end{pmatrix},
\]

(6.1.3)

for \(k = 0, 1, \ldots\).

(3) Let

\[
R_k = \sum_{(k_1, \ldots, k_d) \in \mathcal{I}_k} \det \begin{pmatrix}
  x_{11+k_1} & \cdots & x_{1d+k_1} \\
  \vdots & \ddots & \vdots \\
  x_{d1+k_d} & \cdots & x_{dd+k_d}
\end{pmatrix},
\]

and let \(c_1, \ldots, c_n\) be elements in \(A\). Assume that

\[
x_{id+k} - c_1x_{id+k-1} + \cdots + (-1)^n c_n x_{id+k-n} = 0
\]

(6.1.4)

for \(i = 1, \ldots, d\), and for \(k = n - d + 1, n - d + 2, \ldots\) Then we have

\[
R_k - c_1 R_{k-1} + \cdots + (-1)^k c_k R_0 = 0 \quad \text{for} \quad k = n - d + 1, \ldots, n.
\]

(6.1.5)

Proof. (1) For all permutations \(\sigma\) of \(\{1, \ldots, d\}\), the monomial \(x_{\sigma(1)}(k_1) \cdots x_{\sigma(d)}(k_d)\) on the left hand side of (6.1.1) is equal to the monomial \(x_{\sigma(1)}(k_1') \cdots x_{\sigma(d)}(k_d')\) on the right hand side in (6.1.1) when \(k_i = k_i'\) for \(i = 1, \ldots, d\). Hence (6.1.1) holds.

(2) Under the assumption (6.1.2) the left hand side of (6.1.1) becomes the coefficient of \(T^k\) in the determinant of the matrix

\[
\begin{pmatrix}
  x_{11} + x_{12} T + x_{13} T^2 + \cdots & x_{1d} + x_{1d+1} T + x_{1d+2} T^2 + \cdots \\
  \vdots & \ddots & \vdots \\
  x_{d1} + x_{d2} T + x_{d3} T^2 + \cdots & x_{dd} + x_{dd+1} T + x_{dd+2} T^2 + \cdots
\end{pmatrix}.
\]

(6.1.6)

For \(1, 2, \ldots, d - 1\) we subtract \(T\) times the \((i + 1)'\)st column in this matrix from the \(i\)'th column. The result is that the \(i\)'th column has of coordinates \(x_{1i}, \ldots, x_{di}\). Thus the determinant of the matrix (6.1.6) thus is the determinant of

\[
\begin{pmatrix}
  x_{11} & \cdots & x_{1d-1} & x_{1d} + x_{1d+1} T + x_{1d+2} T^2 + \cdots \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{d1} & \cdots & x_{dd-1} & x_{dd} + x_{dd+1} T + x_{dd+2} T^2 + \cdots
\end{pmatrix}.
\]

(6.1.7)

The coefficient of \(T^k\) in the determinant of the matrix (6.1.7) is the determinant to the left of (6.1.3), which is thus equal to the left hand side of (6.1.1). Finally the
right hand side of (6.1.3) is, under the assumption (6.1.2) equal to the right hand side of (6.1.1). Thus equation (6.1.3) follows from (6.1.1).

(3) From (6.1.3) it follows that the left hand side of (6.1.5) is
\[
\begin{align*}
\det \begin{pmatrix}
  x_{11} & \cdots & x_{1d-k} & x_{1d+k} \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{d1} & \cdots & x_{dd-k} & x_{dd+k}
\end{pmatrix} - c_1 \det \begin{pmatrix}
  x_{11} & \cdots & x_{1d-k} & x_{1d+k-1} \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{d1} & \cdots & x_{dd-k} & x_{dd+k-1}
\end{pmatrix} + \cdots \\
+ (-1)^k c_k \det \begin{pmatrix}
  x_{11} & \cdots & x_{1d-k} & x_{1d+k-1}+\cdots+(-1)^k c_k x_{1d} \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{d1} & \cdots & x_{dd-k} & x_{dd+k-1}+\cdots+(-1)^k c_k x_{dd}
\end{pmatrix} \\
= \det \begin{pmatrix}
  x_{11} & \cdots & x_{1d-k} & x_{1d+k-1}+\cdots+(-1)^n c_n x_{1d+k-n} \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{d1} & \cdots & x_{dd-k} & x_{dd+k-1}+\cdots+(-1)^n c_n x_{dd+k-n}
\end{pmatrix}.
\end{align*}
\]

The right hand side of (6.1.8) is equal to
\[
= \det \begin{pmatrix}
  x_{11} & \cdots & x_{1d-k} & x_{1d+k-1}+\cdots+(-1)^n c_n x_{1d+k-n} \\
  \vdots & \ddots & \vdots & \vdots \\
  x_{d1} & \cdots & x_{dd-k} & x_{dd+k-1}+\cdots+(-1)^n c_n x_{dd+k-n}
\end{pmatrix}.
\]

when \( k = n - d + 1, n - d + 2, \ldots, n \). It follows from the recursion formula (6.1.4) that the right hand side of (6.1.8) is zero for \( k = n - d + 1, \ldots, n \). Equation (6.1.5) thus follows from (6.1.3).

### SOME RESULTS ON RESIDUES

7.5 Proposition. Let \( g_i = \cdots + \frac{a_{1i}}{T} + \frac{a_{2i}}{T^2} + \cdots + \frac{a_{di}}{T^d} + \cdots \) for \( i = 1, \ldots, d \) be formal Laurent series in \( \frac{1}{T} \), and let \( g_{d+1} = \cdots + \frac{a_{1d+1}}{T} + \frac{a_{2d+1}}{T^2} + \cdots + \frac{a_{d+1}}{T^d} + \cdots \). Then
\[
\text{Res}(g_1, \ldots, g_{d+1}) = \sum_{j=0}^{d} (-1)^j a_{d-j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq d} \text{Res}(g_1, \ldots, Tg_{i_1}, \ldots, Tg_{i_j}, \ldots, g_d).
\]

Proof. Expand the determinant \( \text{Res}(g_1, \ldots, g_{d+1}) \) along the last row. We obtain
\[
\text{Res}(g_1, \ldots, g_{d+1}) = \sum_{j=0}^{d} (-1)^j a_{d-j+1} A_j
\]
where \( A_j \) is the determinant to the left of (7.4.2). The proposition is thus a consequence of (7.4.2) since \( Tg_k = \cdots + \frac{a_{1k}}{T} + \frac{a_{2k}}{T^2} + \cdots + \frac{a_{k+1}}{T^d} + \cdots \).
7.1 Notation. Let \( A[T_1, \ldots, T_d] \) be the polynomial ring in the independent variables \( T_1, \ldots, T_d \) over \( A \). Moreover, let \( 1 \leq d \leq n \) be integers and write
\[
P(T) = (T - T_1) \cdots (T - T_d)
\]
in \( A[T_1, \ldots, T_d] \), and
\[
p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n
\]
in \( A[T] \). We denote by \( U_i = s_i(T_1, \ldots, T_d) \) be the \( i \)'th complete symmetric polynomial in \( T_1, \ldots, T_d \) for \( i = 0, 1, \ldots \), and we let \( U_i = 0 \) for \( i < 0 \). For \( d = 1, \ldots, n \) we write
\[
V_j = U_{n-d+j} - c_1 U_{n-d+j-1} + \cdots + (-1)^n c_n U_{n-d+j-n} = U_{n-d+j} - c_1 U_{n-d+j-1} + \cdots + (-1)^{n-d+j} c_{n-d+j} U_0,
\]
for \( j = 1, \ldots, d \). Then
\[
\frac{p}{P} = T^{n-d} \left( 1 - \frac{c_1}{T} + \cdots + (-1)^n \frac{c_n}{T^n} \right) \left( 1 + \frac{U_1}{T} + \frac{U_2}{T^2} + \cdots \right) = \cdots + \frac{V_1}{T} + \frac{V_2}{T^2} + \cdots + \frac{V_d}{T^d} + \cdots \quad (7.0.1)
\]

7.1 Proposition. Let
\[
R_p : A[T_1, \ldots, T_d] \to A
\]
be the \( A \)-linear homomorphism determined by
\[
R_p(f_1(T_1) \cdots f_d(T_d)) = \text{Res} \left( \frac{f_1}{p}, \ldots, \frac{f_d}{p} \right)
\]
for all polynomials \( f_1(T), \ldots, f_d(T) \) in \( A[T] \). Then we have, for all natural numbers \( h_1, \ldots, h_d \), that
\[
R_p(U_{n-d+j} T_1^{h_1} \cdots T_d^{h_d}) - c_1 R_p(U_{n-d+j-1} T_1^{h_1} \cdots T_d^{h_d}) + \cdots + (-1)^{n-d+j} c_{n-d+j} R_p(U_0 T_1^{h_1} \cdots T_d^{h_d}) = 0
\]
for \( j = 1, \ldots, d \).

Proof. Let \( \mathcal{I}_k \) consist of all multiindices \( (k_1, \ldots, k_d) \) of natural numbers such that \( k_1 + \cdots + k_d = k \). Then, for all natural numbers \( h_1, \ldots, h_d \) we have
\[
R_p(U_{n-d+j} T_1^{h_1} \cdots T_d^{h_d}) = \sum \text{Res} \left( \frac{T_1^{h_1+k_1}}{p}, \ldots, \frac{T_d^{h_d+k_d}}{p} \right),
\]
where the sum is over all indices \( (k_1, \ldots, k_d) \in \mathcal{I}_{n-d+j} \). Let \( x_1 = s_{h_1-n+i}, \ldots, x_d = x_{k_d+1} \) for all integers \( i \). From formula (5.3.1) we obtain
\[
\text{Res} \left( \frac{T_1^{h_1+k_1}}{p}, \ldots, \frac{T_d^{h_d+k_d}}{p} \right) = \begin{pmatrix} x_{1+k_1+1} & \cdots & x_{k_1+d} \\ \vdots & \ddots & \vdots \\ x_{1+k_d+1} & \cdots & x_{k_d+d} \end{pmatrix}.
\]

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Thus, in the notation of Lemma 7.1 and with \( k = n - d + j \) we have

\[
R_{n-d+j} = \sum_{(k_1, \ldots, k_d) \in \mathcal{I}_{n-d+j}} \text{Res}(\frac{T^{h_1+k_1}}{p}, \ldots, \frac{T^{h_d+k_d}}{p}).
\]

Moreover we have

\[x_{in+j} - c_1 x_{in+j-1} + \cdots + (-1)^n c_n x_{ij} = s_{hi+j} - c_1 s_{hi+j-1} + \cdots + (-1)^n c_n s_{hi+n+j}\]

for \( j = 1, 2, \ldots \). Thus it follows from Lemma 7.1 (7.1.5) with \( k = n - d + j \) that

\[R_{n-d+j} = c_1 R_{n-d+j-1} + \cdots + (-1)^{n-d+j} c_{n-d+j} R_0 \text{ for } j = 1, \ldots, d.\]

We obtain from (8.1.1) that

\[R_p(U_{n-d+j} T_1^{h_1} \cdots T_d^{h_d}) = R_{n-d+j}\]

and we have proved the proposition.

7.2 Proposition. Let

\[p(T) = P(T)q(T) + r_{d-1} T^{d-1} + \cdots + r_0 T^0\]

be the result of using the division algorithm in \( A[T_1, \ldots, T_d] \) to the polynomial \( p(T) \) modulo \( P(T) \). Then

\[r_{d-i} = \text{Res}\left(\frac{T^{d-1}}{P}, \frac{T^{d-i+1}}{P}, \ldots, \frac{T^0}{P}\right)\]

for \( i = 1, \ldots, d \).

In particular, the ideal in \( A[T_1, \ldots, T_d] \) generated by the elements \( r_0, r_1, \ldots, r_{d-1} \) is generated by the elements

\[V_j = U_{n-d+j} - c_1 U_{n-d+j-1} + \cdots + (-1)^{n-d+j} c_{n-d+j} U_0\]

for \( j = 1, \ldots, d \).

Proof. We have by Proposition 5.3 with \( p = P \) that \( \text{Res}(\frac{T^{d-1}}{P}, \ldots, \frac{T^0}{P}) = 1 \). The first part of the proposition follows since \( \frac{P}{T} = q(T) + r_{d-1} \frac{T^{d-1}}{P} + \cdots + r_0 \frac{T^0}{P} \), and since \( \text{Res} \) is \( A[T_1, \ldots, T_d] \)-linear, alternating and zero if one of the factors is a polynomial.

Since \( \frac{T^{d-i}}{P} = \frac{U_0}{T^i} + \frac{U_1}{T^{i+1}} + \cdots \) for \( i = 1, \ldots, d \) it follows from (8.1.1) and (7.2.1) that

\[r_{d-i} = \begin{pmatrix} U_0 & \cdots & U_{i-2} & U_{i-1} & U_i & \cdots & U_{d-1} \\ \vdots & \cdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & U_0 & U_1 & U_2 & \cdots & U_{d-i+1} \\ V_1 & V_{i-1} & V_i & V_{i+1} & \cdots & V_d \\ 0 & \cdots & 0 & U_0 & \cdots & U_{d-i-1} \\ \vdots & \cdots & \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & U_0 \end{pmatrix} \]

Hence we have \( r_{d-1} = V_1, r_{d-2} = V_2 + W_2, \ldots, r_0 = V_d + W_d \), where \( W_i \) lies in the ideal in \( A[T_1, \ldots, T_d] \) generated by \( V_1, V_2, \ldots, V_i \).
8. First proof of the main theorem

We give a proof based upon the first construction of splitting algebras.

8.1 Theorem. Let \( p(T) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n \) be a polynomial in the variable \( T \) with coefficients in the ring \( A \), and let \( \text{Split}_A^d(p) = A[\xi_1, \ldots, \xi_d] \) be a \( d \)'th splitting algebra for \( p(T) \) over \( A \) with universal splitting

\[
p(T) = (T - \xi_1) \cdots (T - \xi_d)p_{d+1}(T).
\]

Then there is an \( A \)-module homomorphism

\[
A[\xi_1, \ldots, \xi_d] \to A
\]
determined by mapping the element \( f_1(\xi_1) \cdots f_d(\xi_d) \) to \( \text{Res}(\frac{f_1}{p}, \ldots, \frac{f_d}{p}) \) for all polynomials \( f_1(T), \ldots, f_d(T) \) in \( A[T] \).

In particular, on the basis of Corollary 2.3 for the \( A \)-module \( A[\xi_1, \ldots, \xi_d] \), it takes the value

\[
\begin{cases}
1 \quad \text{on} \quad \xi^{n-1} \cdots \xi^{n-\ell} \\
0 \quad \text{on} \quad \xi^{h_1} \cdots \xi^{h_d} \quad \text{where} \quad 0 \leq h_j \leq n - j \quad \text{for} \quad j = 1, \ldots, d.
\end{cases}
\]

Proof. We prove the assertion by induction on \( d \). For \( d = 0 \) it is clear. Assume that we have an \( A \)-module homomorphism

\[
u : A[\xi_1, \ldots, \xi_d] \to A
\]
satisfying the properties of the theorem. We shall show how to obtain an \( A \)-linear homomorphism

\[
v : A[\xi_1, \ldots, \xi_{d+1}] \to A
\]
satisfying the properties of the theorem for \( d + 1 \).

Since \( \text{Res}(g) = 0 \) when \( g(T) \) is a polynomial, we have, with the notation of Proposition 2.1 an \( A \)-module homomorphism

\[
\partial' : A[\xi_1, \ldots, \xi_{d+1}] = A[\xi_1, \ldots, \xi_d][T]/(p_{d+1}) \to A[\xi_1, \ldots, \xi_d]
\]
such that \( \partial'(\frac{f_{d+1}(\xi_{d+1})}{p_{d+1}}) = \text{Res}(\frac{f_{d+1}}{p_{d+1}}) \) for all \( f_{d+1}(T) \) in \( A[T] \).

We now show that the composite \( A \)-module homomorphism

\[
v = u\partial' : A[\xi_1, \ldots, \xi_{d+1}] \to A
\]
has the properties of the homomorphism of the theorem with \( d + 1 \) instead of \( d \). Since \( v = u\partial' \) is linear in \( f_{d+1}(T) \) we can assume that \( f_{d+1}(T) = T^i \) for some non-negative integer \( i \).
Let $t_0, \ldots, t_d$ be the elementary symmetric polynomials in variables $T_1, \ldots, T_d$ evaluated at the elements $\xi_1, \ldots, \xi_d$. We have

$$\frac{T^i}{p_{d+1}} = \frac{T^i(T - \xi_1) \cdots (T - \xi_d)}{p} = T^{i+d-n} \left(1 - \frac{t_1}{T} + \cdots + (-1)^d \frac{t_d}{T^d}\right) \left(1 + \frac{s_1}{T} + \frac{s_2}{T^2} + \cdots\right)$$

and thus

$$\text{Res} \left(\frac{T^i}{p_{d+1}}\right) = s_{i+d-n+1} - t_1 s_{i+d-n} + \cdots + (-1)^d t_d s_{i+d-n-d+1}.$$

Consequently

$$f_1(\xi_1) \cdots f_d(\xi_d) \text{Res} \left(\frac{f_{d+1}}{p_{d+1}}\right) = \sum_{j=0}^d (-1)^j s_{i+d-n+1-j} t_j f_1(\xi_1) \cdots f_d(\xi_d).$$

Since $t_j = \sum_{1 \leq i_1 < \cdots < i_j \leq d} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_j}$ we obtain by the induction assumption

$$u(t_j f_1(\xi_1) \cdots f_d(\xi_d)) = \sum_{1 \leq i_1 < \cdots < i_j \leq d} \text{Res} \left(\frac{f_1}{p}, \ldots, \frac{T f_{i_1}}{p}, \ldots, \frac{T f_{i_j}}{p}, \ldots, \frac{f_d}{p}\right).$$

Consequently

$$u(f_1(\xi_1) \cdots f_d(\xi_d) \partial' (f_{d+1}(\xi_{d+1}))) = u(f_1(\xi_1) \cdots f_d(\xi_d) \text{Res}(\frac{f_{d+1}}{T_{d+1}}))$$

$$= \sum_{j=0}^d (-1)^j s_{i+d-n+1-j} \sum_{1 \leq i_1 < \cdots < i_j \leq d} \text{Res} \left(\frac{f_1}{p}, \ldots, \frac{T f_{i_1}}{p}, \ldots, \frac{T f_{i_j}}{p}, \ldots, \frac{f_d}{p}\right).$$

By Proposition 5.5 the right hand side of the latter equation is $\text{Res}(\frac{f_1}{p}, \ldots, \frac{f_{d+1}}{p})$ with $f_{d+1} = T^i$. Thus $v = u\partial'$ gives an $A$-module homomorphism as asserted in the theorem.

The asserted values on the $A$-module basis of $A[\xi_1, \ldots, \xi_{d+1}]$ follow from Proposition 5.3.

9. Second proof of the main theorem

9.1 Theorem. Let

$$R_p: A[T_1, \ldots, T_n] \rightarrow A$$

be the $A$-module homomorphism determined by

$$R_p(f_1(T_1) \cdots f_n(T_n)) = \text{Res}(\frac{f_1}{p}, \ldots, \frac{f_n}{p})$$
for all polynomials \( f_1(T), \ldots, f_n(T) \) in \( A[T] \). Then \( R_p \) vanishes on the ideal in \( A[T_1, \ldots, T_n] \) generated by the polynomials \( c_1 - C_1, \ldots, c_n - C_n \).

**Proof.** We must prove that for all natural numbers \( h_1, \ldots, h_n \) we have

\[
R_p(T_1^{h_1} \cdots T_n^{h_n} C_j) = c_j R_p(T_1^{h_1} \cdots T_n^{h_n}) \quad \text{for} \quad j = 1, \ldots, n.
\]

However

\[
R_p(T_1^{h_1} \cdots T_n^{h_n} c_j) = \sum_{i < i_1 < \cdots < i_j \leq d} R_p\left( \frac{T_1^{h_1}}{p}, \ldots, \frac{T_{i_1}^{h_{i_1}}}{p}, \ldots, \frac{T_{i_j}^{h_{i_j}}}{p}, \ldots, \frac{T_n^{h_n}}{p} \right) \quad (9.\ast)
\]

Let \( a_{1i} = s_{h_1 - n + i}, \ldots, a_{ni} = s_{h_n - n + i} \) for all integers \( i \). From (5.3.1) we obtain

\[
\text{Res}\left( \frac{T_1^{h_1}}{p}, \ldots, \frac{T_{i_1}^{h_{i_1}}}{p}, \ldots, \frac{T_{i_j}^{h_{i_j}}}{p}, \ldots, \frac{T_n^{h_n}}{p} \right) = \sum_{1 \leq i_1 < \cdots < i_j \leq d} \det \begin{pmatrix}
    a_{i_1} & \ldots & a_{i_{n-1}} & a_{i_{n-j}} & \ldots & a_{i_{n+1}} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    a_{i_n} & \ldots & a_{i_{n-j}} & a_{i_{n-j+2}} & \ldots & a_{i_{n+1}}
\end{pmatrix} \quad (9.\ast)
\]

From Lemma 5.4 (2) we obtain that the right hand side of (9.\ast) becomes

\[
\text{det} \begin{pmatrix}
    a_{i_1} & \ldots & a_{i_{n-1}} & a_{i_{n-j}} & \ldots & a_{i_{n+1}} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    a_{i_n} & \ldots & a_{i_{n-j}} & a_{i_{n-j+2}} & \ldots & a_{i_{n+1}}
\end{pmatrix} \quad (9.\ast\ast)
\]

Since \( a_{in+1} - c_1 a_{in} + \cdots + (-1)^n c_n a_{i1} = s_{h_1 + 1} - c_1 s_{h_i} + \cdots + (-1)^n c_n s_{h_i - n + 1} = 0 \) for \( i = 1, \ldots, d \) we obtain from Lemma 5.4 (3) that (9.\ast\ast) is equal to

\[
\sum_{1 \leq i_1 < \cdots < i_j \leq d} \det \begin{pmatrix}
    s_{h_1 - n + 1} & \ldots & s_{h_1 - n + n} \\
    \vdots & \ddots & \vdots \\
    s_{h_n - n + 1} & \ldots & s_{h_n - n + n}
\end{pmatrix} = c_j \text{Res}\left( \frac{T_1^{h_1}}{p}, \ldots, \frac{T_n^{h_n}}{p} \right)
\]

that we wanted to prove.

10. **Third proof of the main theorem**

We shall give a proof of the main theorem based upon the third construction of Section 4.

8. **Residues and symmetric functions (Legges i annet kapittel)**

8.3 **Theorem.** Let

\[
R_p : A[T_1, \ldots, T_d] \to A
\]

be the \( A \)-module homomorphism that is determined by

\[
R_p(f_1(T_1) \cdots f_d(T_d)) = \text{Res}\left( \frac{f_1}{p}, \ldots, \frac{f_d}{p} \right)
\]
for all collections of polynomials \( f_1(T), \ldots, f_d(T) \) in \( A[T] \). Then \( R_p \) is zero on the ideal generated by the elements \( r_{d-i} = \text{Res}(\frac{T^{d-1}}{p}, \ldots, \frac{T^{d-i+1}}{p}, f_1, \frac{T^{d-i-1}}{p}, \ldots, \frac{T}{p}) \) for \( i = 1, \ldots, d \).

\[ \rightarrow \quad \text{Proof.} \quad \text{It follows from Proposition 8.2 that it suffices to show that } R_p \text{ is zero on the ideal generated by } V_j \text{ for } j = 1, \ldots, d. \text{ That is, it is zero on the elements } V_j T_1^{h_1} \cdots T_d^{h_d} \text{ for all natural numbers } h_1, \ldots, h_d, \text{ and } j = 1, \ldots, d. \text{ In other words, it suffices to show that }
\]
\[
R_p(U_{n-d+j} T_1^{h_1} \cdots T_d^{h_d}) - c_1 R_p(U_{n-d+j-1} T_1^{h_1} \cdots T_d^{h_d}) + \ldots + (-1)^{n-d+j} c_{n-d+j} R_p(U_0 T_1^{h_1} \cdots T_d^{h_d}) = 0 \quad \text{for } j = 1, \ldots, d.
\]

\[ \rightarrow \quad \text{However, this follows from Proposition 8.1.} \]

\[ \text{References} \]


