Matematiska Institutionen, KTH

Tentamen SF1692, Analytiska och numeriska metoder för ordinära differentialekvationer, den 20 december 2021 kl 08.00-13.00.

Examinator: Pär Kurlberg (08-7906582).

OBS: Inga hjälpmedel, utöver de bifogade formelbladen, är tillåtna på tentamensskrivningen. Formelblad finns efter tentalydelsen. För full poäng krävs korrekta och väl presenterade resonemang.

1. (4p) Betrakta ekvationen
$$y'' + ay' + by = 0 \tag{1}$$
där $a, b \in \mathbb{R}$. Finn alla a, b så att alla lösningar $y(t)$ till (1) går mot 0 då $t \to \infty$.

2. (4p) Låt $y(t)$ vara storleken av en population vid tiden t och antag att populationens utveckling i tiden beskrivs av ekvationen
$$y' = ry\ln(K/y)$$
där r och K är positiva konstanter. Populationen vid tiden $t = 0$ är y_0. Finn en formel för populationen $y(t)$ som en funktion av t. Ledtråd: använd substitutionen $y = Ke^u$.

3. (4p) Lös ekvationen
$$y'' - 4y' + 4y = (x+1)e^{2x}.$$

4. (4p) Låt $\alpha \in \mathbb{R}$ och betrakta begynnelsevärdesproblemet
$$y'' + y = \sin(\alpha t), \quad y(0) = 0, \quad y'(0) = 1.$$
Bestäm alla α så att lösningen $y(t)$ är begränsad då $t \to \infty$.

5. (4p) Systemet
$$\begin{cases} \frac{dx}{dt} = -y + x(1 - x^2 - y^2) \\ \frac{dy}{dt} = x + y(1 - x^2 - y^2) \end{cases}$$
har en kritisk punkt i $(0, 0)$. Avgör huruvida denna kritiska punkt är stabil eller inte. Ledtråd: polära koordinater.

6. (4p) Visa följande sats från boken: låt $y_1(x)$ och $y_2(x)$ vara linjärt oberoende lösningar till den homogena ekvationen
$$y'' + P(x)y' + Q(x)y = 0 \tag{2}$$
på intervallet $[a, b]$. Då är
$$c_1y_1(x) + c_2y_2(x) \tag{3}$$
den allmänna lösningen till ekvation (2), i meningen att varje lösning till (2) på $[a, b]$ fås ur (3) genom att välja värden på konstanterna c_1, c_2.

Anmärkning: du behöver inte visa att Wronskianen $W = W(y_1, y_2)$, där y_1, y_2 är två lösningar till (2), antingen är noll på hela $[a, b]$, eller nollskild på hela $[a, b]$.

7. (4p) Betrakta ekvationen

$$y'' + P(x)y' + Q(x)y = 0$$

där P, Q är kontinuerliga på något intervall I, och antag att y_1 är en icketrivial lösning till ekvationen på I. Visa att metoden “reduktion av ordning” resulterar i en lösning y_2 så att $\{y_1, y_2\}$ bildar en fundamental lösningsmängd.

Table of Integrals*

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \quad (1)$$

$$\int \frac{1}{x}dx = \ln|x| \quad (2)$$

$$\int udv = uv - \int vdu \quad (3)$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| \quad (4)$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a} \quad (5)$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1 \quad (6)$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \quad (7)$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1}x \quad (8)$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \quad (9)$$

$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2| \quad (10)$$

$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a} \quad (11)$$

$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2| \quad (12)$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \quad (13)$$

$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln|\frac{a+x}{b+x}|, \ a \neq b \quad (14)$$

$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x| \quad (15)$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln|ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \quad (16)$$

Integrals with Roots

$$\int \sqrt{x-a}dx = \frac{2}{3}(x-a)^{3/2} \quad (17)$$

$$\int \frac{1}{\sqrt{x\pm a}}dx = 2\sqrt{x\pm a} \quad (18)$$

$$\int \frac{1}{\sqrt{a-x}}dx = -2\sqrt{a-x} \quad (19)$$

$$\int x\sqrt{x-a}dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} \quad (20)$$

$$\int \sqrt{ax+b}dx = \left(\frac{2b}{3a}+\frac{2x}{3}\right)\sqrt{ax+b} \quad (21)$$

$$\int (ax+b)^{3/2}dx = \frac{2}{5a}(ax+b)^{5/2} \quad (22)$$

$$\int \frac{x}{\sqrt{x\pm a}}dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a} \quad (23)$$

$$\int \sqrt{\frac{x}{a-x}}dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a} \quad (24)$$

$$\int \sqrt{\frac{x}{a+x}}dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x}+\sqrt{x+a}\right] \quad (25)$$

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b} \quad (26)$$

$$\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right|\right] \quad (27)$$

$$\int \sqrt{x^3(ax+b)}dx = \left[\frac{b}{12a}-\frac{b^2}{8a^2x}+\frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x}+\sqrt{a(ax+b)}\right| \quad (28)$$

$$\int \sqrt{x^2\pm a^2}dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \quad (29)$$

$$\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} \quad (30)$$

$$\int x\sqrt{x^2\pm a^2}dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2} \quad (31)$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| \quad (32)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} \quad (33)$$

$$\int \frac{x}{\sqrt{x^2\pm a^2}}dx = \sqrt{x^2\pm a^2} \quad (34)$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2} \quad (35)$$

$$\int \frac{x^2}{\sqrt{x^2\pm a^2}}dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \mp \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \quad (36)$$

$$\int \sqrt{ax^2+bx+c}dx = \frac{b+2ax}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx^+c)}\right| \quad (37)$$

$$\int x\sqrt{ax^2+bx+c} = \frac{1}{48a^{5/2}}\left(2\sqrt{a}\sqrt{ax^2+bx+c} \times \left(-3b^2+2abx+8a(c+ax^2)\right) + 3(b^3-4abc)\ln\left|b+2ax+2\sqrt{a}\sqrt{ax^2+bx+c}\right|\right) \quad (38)$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \quad (39)$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \quad (40)$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \quad (41)$$

Integrals with Logarithms

$$\int \ln ax dx = x\ln ax - x \quad (42)$$

$$\int \frac{\ln ax}{x}dx = \frac{1}{2}(\ln ax)^2 \quad (43)$$

$$\int \ln(ax+b)dx = \left(x+\frac{b}{a}\right)\ln(ax+b) - x, a\neq 0 \quad (44)$$

$$\int \ln(x^2+a^2)\,\mathrm{d}x = x\ln(x^2+a^2) + 2a\tan^{-1}\frac{x}{a} - 2x \quad (45)$$

$$\int \ln(x^2-a^2)\,\mathrm{d}x = x\ln(x^2-a^2) + a\ln\frac{x+a}{x-a} - 2x \quad (46)$$

$$\int \ln\left(ax^2+bx+c\right)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a}+x\right)\ln\left(ax^2+bx+c\right) \quad (47)$$

$$\int x\ln(ax+b)dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln(ax+b) \quad (48)$$

$$\int x\ln\left(a^2-b^2x^2\right)dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2-\frac{a^2}{b^2}\right)\ln\left(a^2-b^2x^2\right) \quad (49)$$

Integrals with Exponentials

$$\int e^{ax}dx = \frac{1}{a}e^{ax} \quad (50)$$

$$\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}\left(i\sqrt{ax}\right), \text{ where } \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \quad (51)$$

$$\int xe^x dx = (x-1)e^x \quad (52)$$

$$\int xe^{ax}dx = \left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax} \quad (53)$$

$$\int x^2e^x dx = \left(x^2-2x+2\right)e^x \quad (54)$$

$$\int x^2e^{ax}dx = \left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)e^{ax} \quad (55)$$

$$\int x^3e^x dx = \left(x^3-3x^2+6x-6\right)e^x \quad (56)$$

$$\int x^ne^{ax}\,\mathrm{d}x = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x \quad (57)$$

$$\int x^ne^{ax}\,\mathrm{d}x = \frac{(-1)^n}{a^{n+1}}\Gamma[1+n,-ax], \text{ where } \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}t \quad (58)$$

$$\int e^{ax^2}\,\mathrm{d}x = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(ix\sqrt{a}\right) \quad (59)$$

$$\int e^{-ax^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(x\sqrt{a}\right) \quad (60)$$

$$\int xe^{-ax^2}\,\mathrm{d}x = -\frac{1}{2a}e^{-ax^2} \quad (61)$$

$$\int x^2e^{-ax^2}\,\mathrm{d}x = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\mathrm{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2} \quad (62)$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a}\cos ax \tag{63}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{64}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \tag{65}$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} \tag{66}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax \tag{67}$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \tag{68}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \tag{69}$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} \tag{70}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \tag{71}$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \tag{72}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x \tag{73}$$

$$\int \cos^2 ax \sin bx dx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} \tag{74}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a}\cos^3 ax \tag{75}$$

$$\int \sin^2 ax \cos^2 bx dx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \tag{76}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \tag{77}$$

$$\int \tan ax dx = -\frac{1}{a}\ln \cos ax \tag{78}$$

$$\int \tan^2 ax dx = -x + \frac{1}{a}\tan ax \tag{79}$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \tag{80}$$

$$\int \tan^3 ax dx = \frac{1}{a}\ln \cos ax + \frac{1}{2a}\sec^2 ax \tag{81}$$

$$\int \sec x dx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) \tag{82}$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax \tag{83}$$

$$\int \sec^3 x \, \mathrm{d}x = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| \tag{84}$$

$$\int \sec x \tan x dx = \sec x \tag{85}$$

$$\int \sec^2 x \tan x dx = \frac{1}{2}\sec^2 x \tag{86}$$

$$\int \sec^n x \tan x dx = \frac{1}{n}\sec^n x, n \neq 0 \tag{87}$$

$$\int \csc x dx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| + C \tag{88}$$

$$\int \csc^2 ax dx = -\frac{1}{a}\cot ax \tag{89}$$

$$\int \csc^3 x dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| \tag{90}$$

$$\int \csc^n x \cot x dx = -\frac{1}{n}\csc^n x, n \neq 0 \tag{91}$$

$$\int \sec x \csc x dx = \ln|\tan x| \tag{92}$$

Products of Trigonometric Functions and Monomials

$$\int x \cos x dx = \cos x + x \sin x \tag{93}$$

$$\int x \cos ax dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax \tag{94}$$

$$\int x^2 \cos x dx = 2x\cos x + \left(x^2 - 2\right)\sin x \tag{95}$$

$$\int x^2 \cos ax dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax \tag{96}$$

$$\int x^n cosx dx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1, -ix) + (-1)^n\Gamma(n+1, ix)\right] \tag{97}$$

$$\int x^n cosax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1, -iax) - \Gamma(n+1, ixa)\right] \tag{98}$$

$$\int x \sin x dx = -x\cos x + \sin x \tag{99}$$

$$\int x \sin ax dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} \tag{100}$$

$$\int x^2 \sin x dx = \left(2 - x^2\right)\cos x + 2x\sin x \tag{101}$$

$$\int x^2 \sin ax dx = \frac{2 - a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2} \tag{102}$$

$$\int x^n \sin x dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1, -ix) - (-1)^n\Gamma(n+1, -ix)\right] \tag{103}$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x) \tag{104}$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax) \tag{105}$$

$$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) \tag{106}$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax) \tag{107}$$

$$\int xe^x \sin x dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x) \tag{108}$$

$$\int xe^x \cos x dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x) \tag{109}$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a}\sinh ax \tag{110}$$

$$\int e^{ax} \cosh bx dx = \begin{cases} \frac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} + \frac{x}{2} & a = b \end{cases} \tag{111}$$

$$\int \sinh ax dx = \frac{1}{a}\cosh ax \tag{112}$$

$$\int e^{ax} \sinh bx dx = \begin{cases} \frac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] & a \neq b \\ \frac{e^{2ax}}{4a} - \frac{x}{2} & a = b \end{cases} \tag{113}$$

$$\int e^{ax} \tanh bx dx = \begin{cases} \frac{e^{(a+2b)x}}{(a+2b)}{}_2F_1\left[1+\frac{a}{2b}, 1, 2+\frac{a}{2b}, -e^{2bx}\right] \\ \quad -\frac{1}{a}e^{ax}{}_2F_1\left[\frac{a}{2b}, 1, 1E, -e^{2bx}\right] & a \neq b \\ \frac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \tag{114}$$

$$\int \tanh ax \, dx = \frac{1}{a}\ln \cosh ax \tag{115}$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2+b^2}\left[a\sin ax \cosh bx + b\cos ax \sinh bx\right] \tag{116}$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b\cos ax \cosh bx + a\sin ax \sinh bx\right] \tag{117}$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2+b^2}\left[-a\cos ax \cosh bx + b\sin ax \sinh bx\right] \tag{118}$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b\cosh bx \sin ax - a\cos ax \sinh bx\right] \tag{119}$$

$$\int \sinh ax \cosh ax dx = \frac{1}{4a}\left[-2ax + \sinh 2ax\right] \tag{120}$$

$$\int \sinh ax \cosh bx dx = \frac{1}{b^2-a^2}\left[b\cosh bx \sinh ax - a\cosh ax \sinh bx\right] \tag{121}$$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$\delta(t)$	1	(5)
$\delta(t-t_0)$	e^{-st_0}	(6)
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$	(7)
$f'(t)$	$sF(s) - f(0)$	(8)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) -$	
	$\cdots - f^{(n-1)}(0)$	(9)
$\displaystyle\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(10)
$t^n \;\; (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	(11)
$t^x \;\; (x \geq -1 \in \mathbb{R})$	$\dfrac{\Gamma(x+1)}{s^{x+1}}$	(12)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(13)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(14)
e^{at}	$\dfrac{1}{s-a}$	(15)
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	(16)
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	(17)
$\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$	(18)
$\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$	(19)
te^{at}	$\dfrac{1}{(s-a)^2}$	(20)
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	(21)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(22)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(23)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(24)
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	(25)
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	(26)
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	(27)
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	(28)
$t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$	(29)
$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$	(30)
$\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$	(31)
$\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	(32)
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$	(33)