Matematiska Institutionen, KTH

Tentamen SF1633, Differentialekvationer, den 20 december 2021 kl 08.00-12.00.

Examinator: Pär Kurlberg (08-7906582) samt Kevin Schnelli (08-7907202).

OBS: Inga hjälpmedel, utöver de bifogade formelbladen, är tillåtna på tentamensskrivningen. Formelblad finns efter tentalydelsen. För full poäng krävs korrekta och väl presenterade resonemang.

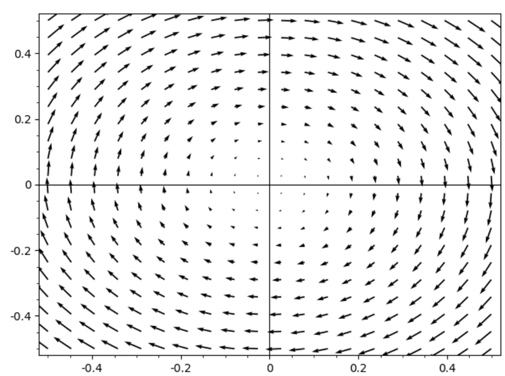
1. (4p) Para ihop de linjära systemen med motsvarande riktningsfält (du behöver inte motivera svaret.)

   (a) $\frac{dx}{dt} = 2\,y, \quad \frac{dy}{dt} = -2\,x.$

   (b) $\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = y.$

   (c) $\frac{dx}{dt} = -2\,y, \quad \frac{dy}{dt} = 2\,x.$

   (d) $\frac{dx}{dt} = x + 3\,y, \quad \frac{dy}{dt} = -3\,x + y.$



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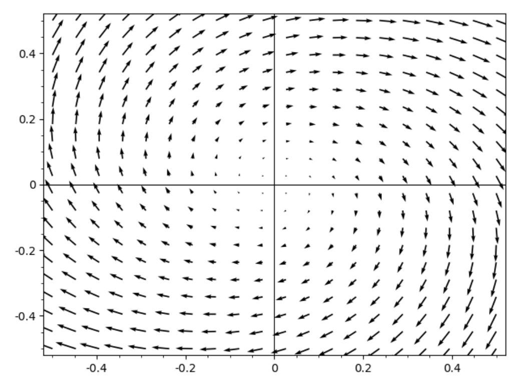

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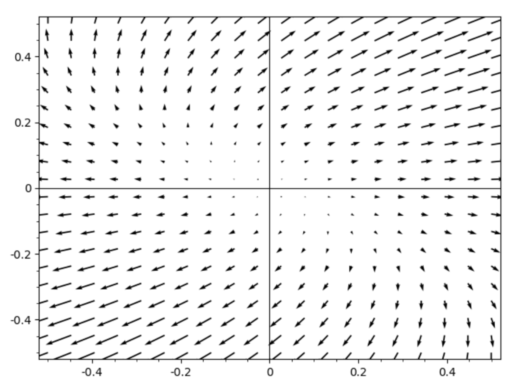

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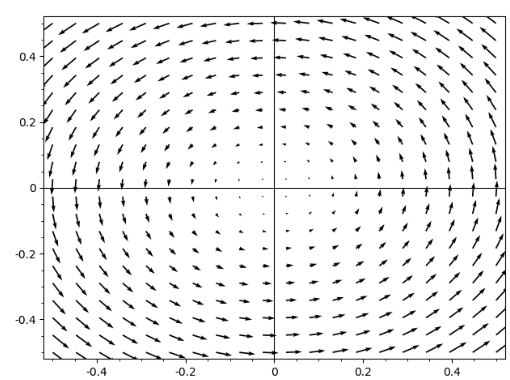

4

1,3,4,2

2. (4p) Lös begynnelsevärdesproblemet

$$\frac{dy}{dx} = (y-6)^2, \quad y(0) = 1$$

samt ange existensintervallet för lösningen.

Omskrivning som $\frac{dy}{(y-6)^2} = dx$ samt integrering ger

$$\frac{-1}{y-6} = x + C,$$

och $y(0) = 1$ ger $C = 1/5$ och vi får $y = 6 - 1/(x + 1/5)$ och existensintervallet ges av $I = (-1/5, \infty)$.

Koll: $VL = dy/dx = 1/(x+1/5)^2$. $HL = (y-6)^2 = (-1/(x+1/5))^2 = 1/(x+1/5)^2$, dvs $VL = HL$.

3. (4p) Lös begynnelsevärdesproblemet

$$X'(t) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} X(t), \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Det karakteristiska polynomet till matrisen är $(1-\lambda)(-1-\lambda)+1 = \lambda^2 - 1 + 1$ som har en dubbelrot för $\lambda = 0$.

Lösning av $(A - 0 \cdot I)v_1$ ger $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Alla andra lösningar är på formen $t \cdot v_1$ och matrisen är därför ej diagonaliserbar.

Ekvationen $(A - 0 \cdot I)v_2 = v_1$ har (tex) lösningen $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Då v_1, v_2 är linjärt oberoende ges den allmänna lösningen av

$$c_1 X_1(t) + c_2 X_2(t)$$

där $X_1(t) = v_1$ och $X_2(t) = tv_1 + v_2$.

Initialvillkoret ger

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

och vi får $c_1 = -2, c_2 = 3$. Den sökta lösningen ges alltså av

$$X(t) = (-2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3(t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}). = 3t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Koll: $VL = X'(t) = 3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $HL = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} X(t) = 3t\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$. Alltså är $VL = HL$, och vi ser även att $X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

4. (4p) Betrakta systemet

$$\begin{cases} \frac{dx}{dt} = 33 - 10x - 3y + x^2 \\ \frac{dy}{dt} = -18 + 6x + 2y - xy. \end{cases}$$

Avgör stabiliteten hos den kritiska punkten $(4, 3)$.

Vi får

$$J_{x,y} = \begin{pmatrix} -10 + 2x & -3 \\ 6 - y & 2 - x \end{pmatrix}$$

och i synnerhet

$$J = J_{4,3} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}.$$

Eftersom $trace(J) = -4$ har åtminstone ett egenvärde realdel < 0, då $det(J) > 0$ ser vi att båda egenvärdena har negativ realdel. Systemet är alltså asymptotiskt stabilt.

5. (4p) Lös vågekvationen

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t \geq 0$$

med randvillkor

$$u(0, t) = u(\pi, t) = 0, \quad t > 0$$

samt

$$u(x, 0) = \begin{cases} x & \text{då } 0 \leq x \leq \pi/2, \\ \pi - x & \text{då } \pi/2 \leq x \leq \pi. \end{cases}$$

och

$$\frac{\partial u}{\partial t}(x, 0) = 0, \quad 0 \leq x \leq \pi.$$

Insättning av produktansatsen $u(x, t) = X(x)T(t)$ i ekvationen ger

$$X''/X = T''/T = -\lambda$$

för λ konstant.

Icketriviala lösningar till $X'' + \lambda X = 0$ samt $X(0) = X(\pi) = 0$ fås då $\lambda = w^2 > 0$ och är på formen $X(x) = c_1 \cos(wx) + c_2 \sin(wx)$. Randvillkoret $X(0) = 0$ ger $c_1 = 0$ och $X(\pi) = 0$ ger $w = n$ för heltal $n = 1, 2, 3 \ldots$.

Lösningar till $T'' + n^2 T = 0$ ges av $c_3 \sin(nt) + c_4 \cos(nt)$, och randvillkoret $\frac{\partial u}{\partial t}(x, 0) = 0$ ger $c_3 = 0$

Vi får produktlösningar på formen $u_n(x, t) = C_n \sin(nx) \cos(nt)$, och superposition ger en lösning på formen $u(x, t) = \sum_{n=1}^{\infty} C_n \sin(nx) \cos(nt)$ där vi kan bestämma C_n mha en sinusserie för den udda utvidgningen av

$$f(x) = \begin{cases} x & \text{då } 0 \leq x \leq \pi/2, \\ \pi - x & \text{då } \pi/2 \leq x \leq \pi. \end{cases}$$

Vi får

$$C_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\left(\int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(\left[\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi/2} + \left[\frac{-\pi\cos nx}{n} + \frac{x\cos(nx)}{n} - \frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right)$$

$$= \frac{2}{\pi}\frac{2\sin(\pi n/2)}{n^2} = \frac{4}{\pi}\cdot\frac{\sin(\pi n/2)}{n^2}.$$

Således ges en lösning av

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\pi n/2)}{n^2}\sin(nx)\cos(nt)$$

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6. (4p) Lös ekvationen

$$y'' + y = \sin t$$

med begynnelsevärdena

$$y(0) = 0, \quad y'(0) = 1.$$

Laplacetransformering ger

$$s^2Y - 1 + Y = \frac{1}{s^2+1}$$

som ger $Y(s^2+1) = 1 + \frac{1}{s^2+1}$ och vi får

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2}$$

Tabell ger $\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$ samt $\mathcal{L}(t\cos(t)) = \frac{s^2-1}{(s^2+1)^2}$ och eftersom

$$\frac{1}{(s^2+1)^2} = \frac{1}{2}\left(\frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\right)$$

ger inverstransform

$$y(t) = \frac{3}{2}\sin t - \frac{1}{2}t\cos(t)$$

Koll: $y' = (3/2)\cos t - (\cos t - t\sin t)/2$ och $y'' = -(1/2)\sin t + 1/2t\cos t$, och vi får $VL = y'' + y == -(1/2)\sin t + 1/2t\cos t + 3/2\sin t - 1/2t\cos t = \sin t$ samt $HL = \sin t$. Vi får även $y(0) = 0$ samt $y'(0) = 3/2\cos 0 - 1/2\cos 0 = 1$.

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7. (4p) Bestäm värdet av serien

$$4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n).$$

Ledtråd: serien kan tolkas som Fourierserien för en 2π-periodisk funktion $f(t)$ där $f(t) = p(t)$ för $|t| \leq \pi$ och p är ett polynom av lågt gradtal.

Eftersom serien är en cosinusserie måste polynomet vara jämt, dvs på formen $a + bt^2 + \ldots$. Vi räknar ut Fourierserien för t^2 på intervallet $[-\pi, \pi]$ och får (eftersom funktionen är jämn)

$$a_0 = \frac{2}{\pi} \int_0^\pi t^2 \, dt = \frac{2}{\pi}[t^3/3]_0^\pi = 2\pi^2/3$$

och tabell ger

$$a_n = \frac{2}{\pi} \int_0^\pi t^2 \cos(nt) \, dt = \frac{2}{\pi}[\frac{2t\cos(nt)}{n^2} + \frac{n^2t^2 - 2}{n^3} \sin(nt)]_0^\pi = \frac{2}{\pi} \frac{2\pi\cos(\pi n)}{n^2} = \frac{4}{n^2}(-1)^n$$

På intervallet $[-\pi, \pi]$ får vi då

$$t^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nt)$$

Då t^2 ger en styckvist kontinuerlig funktion på $[-\pi, \pi]$ (och samma gäller för derivatan) konvergerar Fourierserien mot 1^2 för $t = 1$, och vi ser att

$$4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n) = 1 - \pi^2/3.$$

Formelblad om Fourierserier

En Fourierserie för en funktion f definierad på intervallet $(-p, p)$ ges av

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right)$$

där

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx,$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx,$$

och

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx.$$

Table of Integrals*

Basic Forms

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \tag{1}$$

$$\int \frac{1}{x}dx = \ln |x| \tag{2}$$

$$\int udv = uv - \int vdu \tag{3}$$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln |ax+b| \tag{4}$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2}dx = -\frac{1}{x+a} \tag{5}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1}, n \neq -1 \tag{6}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \tag{7}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x \tag{8}$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \tag{9}$$

$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln |a^2+x^2| \tag{10}$$

$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a} \tag{11}$$

$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2 \ln |a^2+x^2| \tag{12}$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \tag{13}$$

$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln |\frac{a+x}{b+x}|, \ a \neq b \tag{14}$$

$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln |a+x| \tag{15}$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln |ax^2+bx+c| - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \tag{16}$$

Integrals with Roots

$$\int \sqrt{x-a}dx = \frac{2}{3}(x-a)^{3/2} \tag{17}$$

$$\int \frac{1}{\sqrt{x\pm a}}dx = 2\sqrt{x\pm a} \tag{18}$$

$$\int \frac{1}{\sqrt{a-x}}dx = -2\sqrt{a-x} \tag{19}$$

$$\int x\sqrt{x-a}dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} \tag{20}$$

$$\int \sqrt{ax+b}dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} \tag{21}$$

$$\int (ax+b)^{3/2}dx = \frac{2}{5a}(ax+b)^{5/2} \tag{22}$$

$$\int \frac{x}{\sqrt{x\pm a}}dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a} \tag{23}$$

$$\int \sqrt{\frac{x}{a-x}}dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a} \tag{24}$$

$$\int \sqrt{\frac{x}{a+x}}dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x} + \sqrt{x+a}\right] \tag{25}$$

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2+abx+3a^2x^2)\sqrt{ax+b} \tag{26}$$

$$\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}}\left[(2ax+b)\sqrt{ax(ax+b)} - b^2\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|\right] \tag{27}$$

$$\int \sqrt{x^3(ax+b)}dx = \left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)} + \frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right| \tag{28}$$

$$\int \sqrt{x^2\pm a^2}dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{29}$$

$$\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} \tag{30}$$

$$\int x\sqrt{x^2\pm a^2}dx = \frac{1}{3}\left(x^2\pm a^2\right)^{3/2} \tag{31}$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{32}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} \tag{33}$$

$$\int \frac{x}{\sqrt{x^2\pm a^2}}dx = \sqrt{x^2\pm a^2} \tag{34}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2} \tag{35}$$

$$\int \frac{x^2}{\sqrt{x^2\pm a^2}}dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \mp \frac{1}{2}a^2\ln\left|x+\sqrt{x^2\pm a^2}\right| \tag{36}$$

$$\int \sqrt{ax^2+bx+c}dx = \frac{b+2ax}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx^{+}c)}\right| \tag{37}$$

$$\int x\sqrt{ax^2+bx+c} = \frac{1}{48a^{5/2}}\left(2\sqrt{a}\sqrt{ax^2+bx+c} \times \left(-3b^2+2abx+8a(c+ax^2)\right) + 3(b^3-4abc)\ln\left|b+2ax+2\sqrt{a}\sqrt{ax^2+bx+c}\right|\right) \tag{38}$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{39}$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{40}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \tag{41}$$

Integrals with Logarithms

$$\int \ln ax dx = x\ln ax - x \tag{42}$$

$$\int \frac{\ln ax}{x}dx = \frac{1}{2}(\ln ax)^2 \tag{43}$$

$$\int \ln(ax+b)dx = \left(x+\frac{b}{a}\right)\ln(ax+b) - x, a\neq 0 \tag{44}$$

$$\int \ln(x^2+a^2)\,\mathrm{dx} = x\ln(x^2+a^2) + 2a\tan^{-1}\frac{x}{a} - 2x \tag{45}$$

$$\int \ln(x^2-a^2)\,\mathrm{dx} = x\ln(x^2-a^2) + a\ln\frac{x+a}{x-a} - 2x \tag{46}$$

$$\int \ln\left(ax^2+bx+c\right)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a}+x\right)\ln\left(ax^2+bx+c\right) \tag{47}$$

$$\int x\ln(ax+b)dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln(ax+b) \tag{48}$$

$$\int x\ln\left(a^2-b^2x^2\right)dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2-\frac{a^2}{b^2}\right)\ln\left(a^2-b^2x^2\right) \tag{49}$$

Integrals with Exponentials

$$\int e^{ax}dx = \frac{1}{a}e^{ax} \tag{50}$$

$$\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}\left(i\sqrt{ax}\right), \quad \text{where } \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{51}$$

$$\int xe^x dx = (x-1)e^x \tag{52}$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \tag{53}$$

$$\int x^2e^x dx = \left(x^2-2x+2\right)e^x \tag{54}$$

$$\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} \tag{55}$$

$$\int x^3e^x dx = \left(x^3-3x^2+6x-6\right)e^x \tag{56}$$

$$\int x^ne^{ax}\,\mathrm{dx} = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{dx} \tag{57}$$

$$\int x^ne^{ax}\,\mathrm{dx} = \frac{(-1)^n}{a^{n+1}}\Gamma[1+n,-ax], \quad \text{where } \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,\mathrm{dt} \tag{58}$$

$$\int e^{ax^2}\,\mathrm{dx} = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(ix\sqrt{a}\right) \tag{59}$$

$$\int e^{-ax^2}\,\mathrm{dx} = \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(x\sqrt{a}\right) \tag{60}$$

$$\int xe^{-ax^2}\,\mathrm{dx} = -\frac{1}{2a}e^{-ax^2} \tag{61}$$

$$\int x^2e^{-ax^2}\,\mathrm{dx} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\mathrm{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2} \tag{62}$$

Integrals with Trigonometric Functions

$$\int \sin axdx = -\frac{1}{a}\cos ax \tag{63}$$

$$\int \sin^2 axdx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{64}$$

$$\int \sin^n axdx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \tag{65}$$

$$\int \sin^3 axdx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} \tag{66}$$

$$\int \cos axdx = \frac{1}{a}\sin ax \tag{67}$$

$$\int \cos^2 axdx = \frac{x}{2} + \frac{\sin 2ax}{4a} \tag{68}$$

$$\int \cos^p axdx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \tag{69}$$

$$\int \cos^3 axdx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} \tag{70}$$

$$\int \cos ax \sin bxdx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \tag{71}$$

$$\int \sin^2 ax \cos bxdx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \tag{72}$$

$$\int \sin^2 x \cos xdx = \frac{1}{3}\sin^3 x \tag{73}$$

$$\int \cos^2 ax \sin bxdx = \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} - \frac{\cos[(2a+b)x]}{4(2a+b)} \tag{74}$$

$$\int \cos^2 ax \sin axdx = -\frac{1}{3a}\cos^3 ax \tag{75}$$

$$\int \sin^2 ax \cos^2 bxdx = \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)} \tag{76}$$

$$\int \sin^2 ax \cos^2 axdx = \frac{x}{8} - \frac{\sin 4ax}{32a} \tag{77}$$

$$\int \tan axdx = -\frac{1}{a}\ln\cos ax \tag{78}$$

$$\int \tan^2 axdx = -x + \frac{1}{a}\tan ax \tag{79}$$

$$\int \tan^n axdx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \tag{80}$$

$$\int \tan^3 axdx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax \tag{81}$$

$$\int \sec xdx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) \tag{82}$$

$$\int \sec^2 axdx = \frac{1}{a}\tan ax \tag{83}$$

$$\int \sec^3 x \, \mathrm{d}x = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| \tag{84}$$

$$\int \sec x \tan xdx = \sec x \tag{85}$$

$$\int \sec^2 x \tan xdx = \frac{1}{2}\sec^2 x \tag{86}$$

$$\int \sec^n x \tan xdx = \frac{1}{n}\sec^n x, n \neq 0 \tag{87}$$

$$\int \csc xdx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| + C \tag{88}$$

$$\int \csc^2 axdx = -\frac{1}{a}\cot ax \tag{89}$$

$$\int \csc^3 xdx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| \tag{90}$$

$$\int \csc^n x \cot xdx = -\frac{1}{n}\csc^n x, n \neq 0 \tag{91}$$

$$\int \sec x \csc xdx = \ln|\tan x| \tag{92}$$

Products of Trigonometric Functions and Monomials

$$\int x \cos xdx = \cos x + x \sin x \tag{93}$$

$$\int x \cos axdx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax \tag{94}$$

$$\int x^2 \cos xdx = 2x\cos x + \left(x^2 - 2\right)\sin x \tag{95}$$

$$\int x^2 \cos axdx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax \tag{96}$$

$$\int x^n cosxdx = -\frac{1}{2}(i)^{n+1}\left[\Gamma(n+1, -ix) + (-1)^n\Gamma(n+1, ix)\right] \tag{97}$$

$$\int x^n cosaxdx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n\Gamma(n+1, -iax) - \Gamma(n+1, ixa)\right] \tag{98}$$

$$\int x \sin xdx = -x\cos x + \sin x \tag{99}$$

$$\int x \sin axdx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} \tag{100}$$

$$\int x^2 \sin xdx = \left(2 - x^2\right)\cos x + 2x\sin x \tag{101}$$

$$\int x^2 \sin axdx = \frac{2 - a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2} \tag{102}$$

$$\int x^n \sin xdx = -\frac{1}{2}(i)^n\left[\Gamma(n+1, -ix) - (-1)^n\Gamma(n+1, -ix)\right] \tag{103}$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin xdx = \frac{1}{2}e^x(\sin x - \cos x) \tag{104}$$

$$\int e^{bx} \sin axdx = \frac{1}{a^2+b^2}e^{bx}(b\sin ax - a\cos ax) \tag{105}$$

$$\int e^x \cos xdx = \frac{1}{2}e^x(\sin x + \cos x) \tag{106}$$

$$\int e^{bx} \cos axdx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax) \tag{107}$$

$$\int xe^x \sin xdx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x) \tag{108}$$

$$\int xe^x \cos xdx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x) \tag{109}$$

Integrals of Hyperbolic Functions

$$\int \cosh axdx = \frac{1}{a}\sinh ax \tag{110}$$

$$\int e^{ax}\cosh bxdx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[a\cosh bx - b\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} & a = b \end{cases} \tag{111}$$

$$\int \sinh axdx = \frac{1}{a}\cosh ax \tag{112}$$

$$\int e^{ax}\sinh bxdx = \begin{cases} \dfrac{e^{ax}}{a^2-b^2}[-b\cosh bx + a\sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} & a = b \end{cases} \tag{113}$$

$$\int e^{ax}\tanh bxdx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)}{}_2F_1\left[1+\dfrac{a}{2b}, 1, 2+\dfrac{a}{2b}, -e^{2bx}\right] - \dfrac{1}{a}e^{ax}{}_2F_1\left[\dfrac{a}{2b}, 1, 1E, -e^{2bx}\right] & a \neq b \\ \dfrac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \tag{114}$$

$$\int \tanh ax \, dx = \frac{1}{a}\ln\cosh ax \tag{115}$$

$$\int \cos ax \cosh bxdx = \frac{1}{a^2+b^2}\left[a\sin ax \cosh bx + b\cos ax \sinh bx\right] \tag{116}$$

$$\int \cos ax \sinh bxdx = \frac{1}{a^2+b^2}\left[b\cos ax \cosh bx + a\sin ax \sinh bx\right] \tag{117}$$

$$\int \sin ax \cosh bxdx = \frac{1}{a^2+b^2}\left[-a\cos ax \cosh bx + b\sin ax \sinh bx\right] \tag{118}$$

$$\int \sin ax \sinh bxdx = \frac{1}{a^2+b^2}\left[b\cosh bx \sin ax - a\cos ax \sinh bx\right] \tag{119}$$

$$\int \sinh ax \cosh axdx = \frac{1}{4a}\left[-2ax + \sinh 2ax\right] \tag{120}$$

$$\int \sinh ax \cosh bxdx = \frac{1}{b^2-a^2}\left[b\cosh bx \sinh ax - a\cosh ax \sinh bx\right] \tag{121}$$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$\delta(t)$	1	(5)
$\delta(t-t_0)$	e^{-st_0}	(6)
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$	(7)
$f'(t)$	$sF(s) - f(0)$	(8)
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	(9)
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(10)
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	(11)
$t^x \ (x \geq -1 \in \mathbb{R})$	$\dfrac{\Gamma(x+1)}{s^{x+1}}$	(12)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(13)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(14)
e^{at}	$\dfrac{1}{s-a}$	(15)
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	(16)
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	(17)
$\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$	(18)
$\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$	(19)
te^{at}	$\dfrac{1}{(s-a)^2}$	(20)
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	(21)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(22)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(23)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(24)
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	(25)
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	(26)
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	(27)
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	(28)
$t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$	(29)
$\dfrac{\sin at}{t}$	$\arctan \dfrac{a}{s}$	(30)
$\dfrac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$	(31)
$\dfrac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	(32)
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$	(33)