SF1633, #18

(Mer egenskaper för Laplace-transformen)

Derivator av Laplace-transformen

Låt $F(s) = \mathcal{L}\{f(t)\}(s)$. Då är

$$\frac{d}{ds} F(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st} \cdot f(t)\,dt \right) =$$

$$[\text{försiktig!}] = \int_0^\infty \left(\frac{d}{ds}\left(e^{-st} f(t)\right)\right) dt$$

$$= \int_0^\infty \left(e^{-st}(-t) \cdot f(t)\right) dt =$$

$$= -\int_0^\infty e^{-st}\left(t \cdot f(t)\right) dt$$

$$- \mathcal{L}\{t \cdot f(t)\}.$$

Upprepad användning:

$$\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n}\left(F(s)\right)$$

Ex: Beräkna $\mathcal{L}\{t \cdot \cos(at)\}$

$(n=1)$

Vi får: $\mathcal{L}\{t \cdot \cos(at)\} =$

$$= -\frac{d}{ds}\Big(\underbrace{\mathcal{L}\{\cos(at)\}}\Big) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$\left(\text{tabell} \Rightarrow \frac{s}{s^2+a}\right)$$

$$= -\left(\frac{1}{s^2+a^2} - \frac{s\cdot(2s)}{(s^2+a^2)^2}\right)$$

$$= -\left(\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right) = -\frac{a^2-s^2}{(s^2+a^2)^2}$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}.$$

Laplace transform av integral

$t \qquad t$

Notera: $\int_0 f(\tau)\,d\tau = \int_0 f(\tau)\cdot 1\,d\tau$

$$= \int_0^t f(\tau)\,\mathbb{1}(t-\tau)\,d\tau$$

"ett-funktionen"

$$= (f * \mathbb{1})(t)$$

$$\therefore\ \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \mathcal{L}\{f * 1\}$$

$$= \mathcal{L}\{f\}\cdot\mathcal{L}\{1\} = F(s)\cdot\frac{1}{s}$$

$$\therefore\ \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = F(s)\cdot\frac{1}{s}$$

Anm: Mycket enkel "formel" för integraler!

Sfr: $\mathcal{L}\{f'(t)\} = s\cdot F(s) - f(0)$

"B.V."

Coolt: $\mathcal{L}$ "gör om"

derivering till multiplikation med s (samt B.V. $-f(0)$),

integral "görs om" till division med s.

System av linj. diff. ekv. mha Laplace (7.6)

Laplace-transformen funkar även för <u>system</u>.

<u>Ex</u>: Lös BVP

$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = x \end{cases}$$

$x(0) = 1, \quad y(0) = 0.$

"$+\delta(t-1)$"

"$+\delta(t-2)$"

Just nu: ignorera för enklare räkning i "inbrint..."

<u>Lösn.</u>: Laplace

på båda ekvationerna

[illegible]
kommer
strax"

ger:

(Kom ihåg: $\mathcal{L}\{f'(t)\} = s \cdot F(s) - f(0)$)

$$\begin{cases} s \cdot X(s) - \underbrace{1}_{x(0)} = -Y(s) & (1) \\ s \cdot Y(s) - \underbrace{0}_{y(0)} = X(s) & (2) \end{cases}$$

$$\Rightarrow \quad s^2 \cdot Y(s) - 1 = -Y(s)$$

$$\Rightarrow (s^2+1) \cdot Y(s) = 1.$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1}$$

$$(2) \Rightarrow \quad X(s) = s \cdot Y(s) = s \cdot \frac{1}{s^2+1} = \frac{s}{s^2+1}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \cos(t)$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \sin(t).$$

Hurra!

Ide: lös ut antingen $X(s)$ eller $Y(s)$ och transformera tillbaka.

Ex (mer komplicerat, med δ)

Lös BVP

$$\begin{cases} \dfrac{dx}{dt} = -y + \delta(t-1) \\ \dfrac{dy}{dt} = x \end{cases}$$

$x(0) = 0, \quad y(0) = 0.$

Lösning: Laplace på båda ekvationerna $\Longrightarrow$

$$\begin{cases} s \cdot X(s) = -Y(s) + 1 \cdot e^{-s} \end{cases}$$

$$s \cdot Y(s) = X(s)$$

Igen: $s^2 \cdot Y(s) = -Y(s) + e^{-s}$

$$\Rightarrow Y(s)(s^2+1) = e^{-s}$$

$$\Rightarrow Y(s) = \frac{e^{-s}}{s^2+1}$$

sin(t)

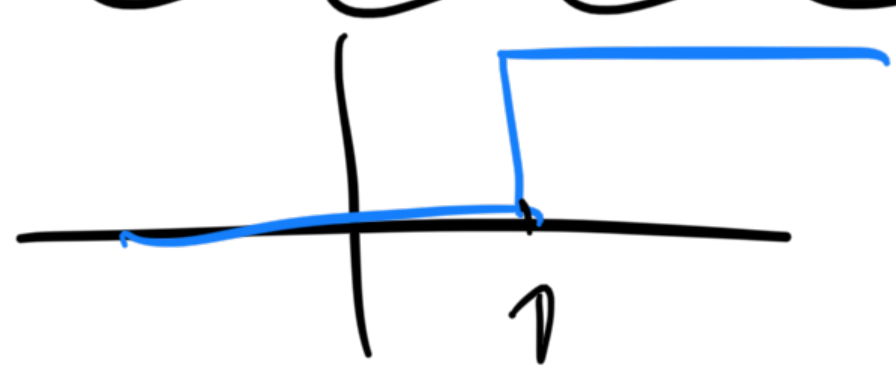
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{1}{s^2+1}\right\}$$

$$= u_1(t) \cdot \sin(t-1)$$

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$$

"<u>Regel</u>"

$u_1(t)$



$\delta(t-1)$

Dessutom: $X(s) = s \cdot Y(s)$

$$= \frac{s \cdot e^{-s}}{s^2+1}$$

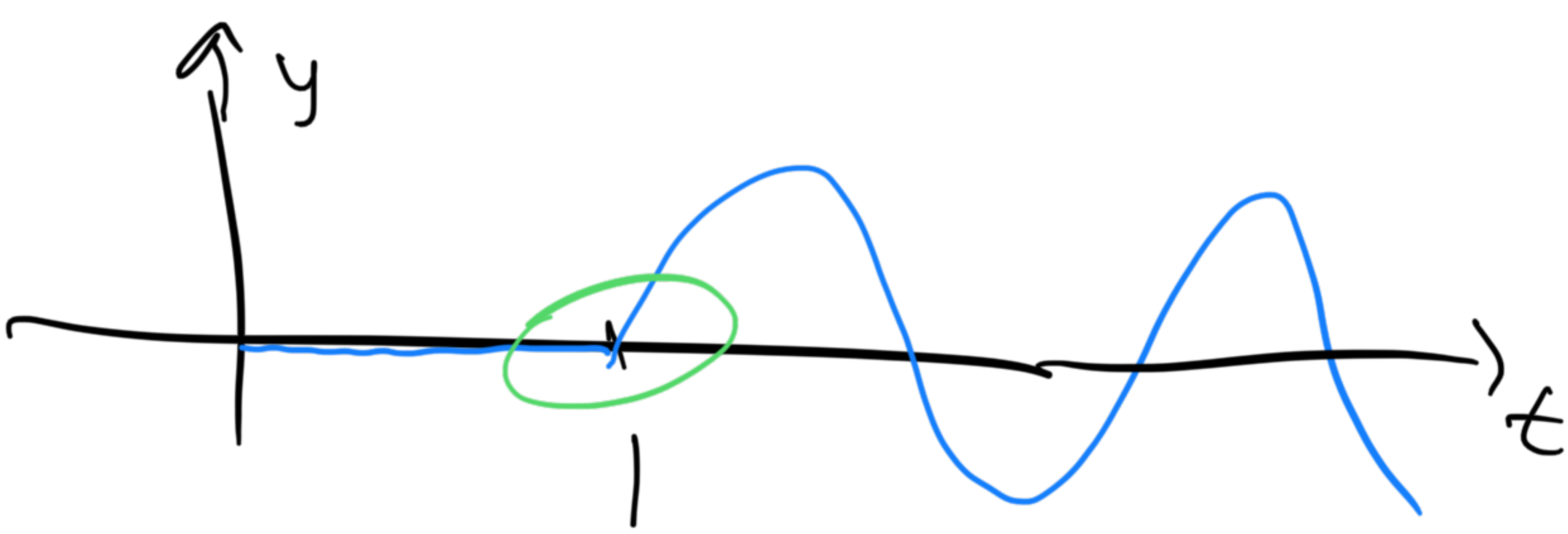
$\cos(t)$

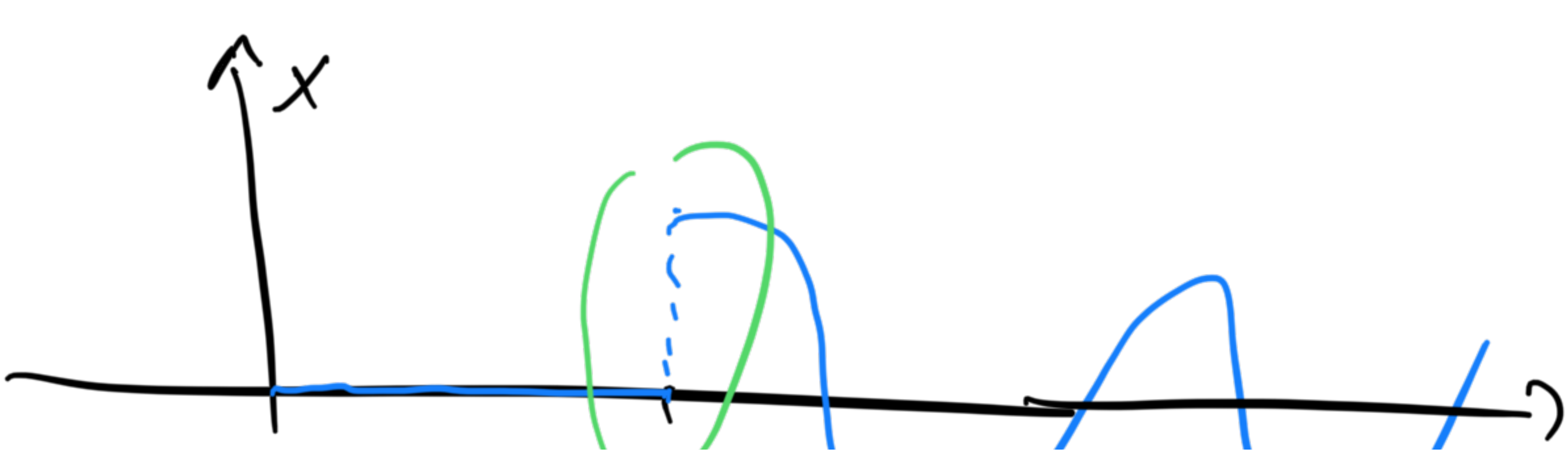
$$\Rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} \cdot e^{-s}\right\}$$

$$= u_1(t) \cdot \cos(t-1)$$

$$\therefore \quad y(t) = u_1(t) \cdot \sin(t-1)$$

$$x(t) = u_1(t) \cdot \cos(t-1)$$





$$\begin{cases} x' = -y + \delta(t-1) \\ y' = x \end{cases} \qquad x(0) = y(0) = 0$$

Q: varför uppenbart att

$x(t) = y(t) = 0$ för $0 \leq t < 1$?

Sv: "innan" $t = 1$ have vi

trivialt system: $\begin{cases} x' = -y \\ y' = x \end{cases}$

och $x(0) = y(0) = 0 \implies$

trivial funktion fram till $t = 1$.

Mer komplicerat system:

Ex: Lös

$$\begin{cases} \dfrac{dx}{dt} = 4x - 2y + 2u_1(t) \\ \dfrac{dy}{dt} = \dots \end{cases}$$

$$\frac{dy}{dt} = 3x - y + u_1(t)$$

med B.U. $x(0)=0$, $y(0)=\frac{1}{2}$.

Laplace transform $\Longrightarrow$

$$\begin{cases} s\cdot X(s) = 4\cdot X(s) - 2Y(s) + 2\cdot\frac{1}{s}\cdot e^{-s} \\ sY(s) - \frac{1}{2} = 3X(s) - Y(s) + \frac{1}{s}\cdot e^{-s} \end{cases}$$

$\Longrightarrow$

$$(*)\begin{cases} (s-4)\cdot X(s) + 2Y(s) = \frac{2}{s}\cdot e^{-s} \\ -3X(s) + (s+1)Y(s) = \frac{1}{s}e^{-s} + \frac{1}{2} \end{cases}$$

[Kramers regel: $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$

$\Longleftrightarrow$ $x = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

och $y = \begin{vmatrix} a & e \\ c & f \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ c & d \end{vmatrix}$]

För (*) får vi: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} s-4 & 2 \\ -3 & s+1 \end{vmatrix}$

$= (s-4)(s+1)+6 = s^2-3s+2 = (s-1)(s-2)$

och:

$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = \begin{vmatrix} \frac{2}{s}e^{-s} & 2 \\ \frac{1}{2}+\frac{1}{s}e^{-s} & s+1 \end{vmatrix} =$

$\frac{2}{s}(s+1)e^{-s} - 2\left(\frac{1}{2}+\frac{1}{s}e^{-s}\right) =$

$= \frac{2}{s}(s+1)e^{-s} - 1 - \frac{2}{s}e^{-s} =$

$= 2e^{-s} - 1$

$\therefore X = \frac{|\ldots|}{|\ldots|} = \frac{-1+2e^{-s}}{(s-1)(s-2)} =$

$= \frac{-1}{(s-1)(s-2)} + \frac{2 \cdot e^{-s}}{(s-1)(s-2)}$

Notera: $\mathcal{L}^{-1}\left\{\frac{+1}{(s-1)(s-2)}\right\} =$ snabb. part. bråksuppd.

$\mathcal{L}^{-1}\left\{\frac{1}{\ldots} - \frac{1}{\ldots}\right\} = e^{2t} - e^{t}$

$$\mathcal{L}\{\ s-2 \qquad s-1\ \}$$

tabell!

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s-1)(s-2)}\right\} = u_1(t)\cdot\left(e^{2(t-1)} - e^{t-1}\right)$$

$$\therefore\ x(t) = -\left(e^{2t}-e^{t}\right) + 2u_1(t)\left(e^{2(t-1)}-e^{t-1}\right)$$

$$= e^{t} - e^{2t} + 2u_1(t)\left(e^{2(t-1)}-e^{t-1}\right)$$

(Forts. följer....)