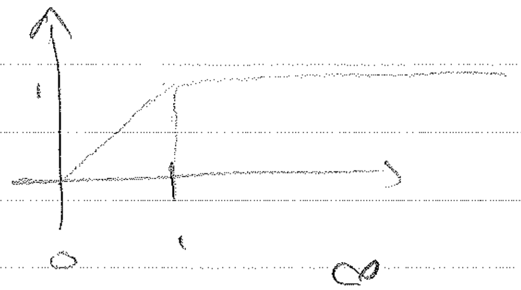
7.1.8 Bestäm $\mathcal{L}\{f(t)\}$ när

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & t \geq 1 \end{cases}$$



Lösning

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^1 t e^{-st}\,dt + \int_1^\infty e^{-st}\,dt$$

$$\int_0^1 t \cdot e^{-st}\,dt = \Big[t\left(-\tfrac{1}{s}\right)e^{-st}\Big]_{t=0}^{1} - \int_0^1 -\frac{1}{s}e^{-st}\,dt$$

Part. int: $\int u v' = uv - \int u'v$

$u = t$, $v' = e^{-st}$

$v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}e^{-s} + \frac{1}{s}\int_0^1 e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-s} + \frac{1}{s}\Big[\left(-\tfrac{1}{s}\right)e^{-st}\Big]_0^1$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}\left(e^{-s} - 1\right) = -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}$$

$$\int_1^\infty e^{-st}\,dt = \Big[-\tfrac{1}{s}e^{-st}\Big]_1^{t\to\infty} = \lim_{t\to\infty}\left(-\frac{1}{s}e^{-st}\right) - \left(-\frac{1}{s}e^{-s}\right)$$

$$= \frac{1}{s}e^{-s}$$

$$\mathcal{L}\{f(t)\}(s) = -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2} + \frac{1}{s}e^{-s} = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} = \underline{\underline{\frac{1}{s^2}\left(1 - e^{-s}\right)}}$$

7.1.15 Bestäm $\mathcal{L}\{e^{-t}\sin t\}$

$$\boxed{a=-1-s,\ b=1 \quad \int e^{au}\sin bu\, du}$$

$$\mathcal{L}\{e^{-t}\sin t\}(s) = \int_0^\infty e^{-t}\sin t\, e^{-st}\,dt = \int_0^\infty e^{-(1+s)t}\sin t\,dt$$

$$\underset{\text{tabell}}{\uparrow}= \left[\frac{e^{(-1-s)t}}{(-1-s)^2+1}\left((-1-s)\sin t - \cos t\right)\right]_0^\infty$$

$$= \lim_{t\to\infty}\left(\frac{\overset{\to 0}{e^{(-1-s)t}}}{(1+s)^2+1}\underbrace{(-1-s)\sin t - \cos t}_{\text{begränsad}}\right) - \left(\frac{1}{(1+s)^2+1}(0-1)\right)$$

$$= 0 + \frac{1}{s^2+2s+1+1} = \underline{\underline{\frac{1}{s^2+2s+2}}}$$

7.2.5 Invers Laplace-transform

Bestäm $\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$

Kom ihåg: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

och $\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s^3+3s^2+3s+1}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= 1 + 3\cdot t + 3\cdot\frac{t^2}{2} + \frac{t^3}{3!} = \underline{\underline{1+3t+\frac{3}{2}t^2+\frac{1}{6}t^3}}$$

Bestäm

7.2.15 $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$ Lösning: $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{-6}{s^2+3^2}\right\}$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = 2\cos(3t) - 2\sin(3t)$$

7.2.41 Lös begynnelsesvärde problemet med Laplace transform

$y'' + y = \sqrt{2}\sin(\sqrt{2}t)$, $y(0) = 10$, $y'(0) = 0$

Skriver $Y = \mathcal{L}\{y\}$, så att

$$\mathcal{L}\{y^{(n)}\} = s^n Y - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - y^{(n-1)}(0)$$

Ekvationen blir

$$s^2 Y - s\cdot 10 - 0 + Y = \sqrt{2}\cdot\frac{\sqrt{2}}{s^2+2} = \frac{2}{s^2+2}$$

Vi får

$$s^2 Y - 10s + Y = \frac{2}{s^2+2}$$

Löser för Y: $Y(s^2+1) - 10s = \frac{2}{s^2+2}$

$$Y = \frac{\frac{2}{s^2+2} + 10s\,\frac{(s^2+2)}{(s^2+2)}}{s^2+1} = \frac{2 + 10s^3 + 20s^2}{(s^2+1)(s^2+2)}$$

Partialbråksuppspaltning:

$$\frac{10s^3 + 20s^2 + 2}{(s^2+1)(s^2+2)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+2}$$

Måste bestämma a, b, c, d:
Gånger med $(s^2+1)(s^2+2)$ och får

$$10s^3 + 20s + 2 = (as+b)(s^2+2) + (cs+d)(s^2+1)$$
$$= as^3 + 2as + bs^2 + 2b + cs^3 + cs + ds^2 + d$$

Som ger: 1) $10 = a + c$
2) $0 = b + d$
3) $20 = 2a + c$
4) $2 = 2b + d$

3) − 2) ger: $10 = a$ och 1) ger $c = 0$

4) − 2) ger: $2 = b$ och 2) ger $d = -2$

Vi får
$$Y = \frac{10s+2}{s^2+1} - \frac{2}{s^2+2}$$

Tar $\mathcal{L}^{-1}$ och får

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{10s+2}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s^2+2}\right\}$$

$\boxed{2 = \sqrt{2}\sqrt{2}}$

$$= 10\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \sqrt{2}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\}$$

$$= \underline{\underline{10\cos t + 2\sin t - \sqrt{2}\sin\sqrt{2}t}}$$

7.3.3 Bestäm $\mathcal{L}\{t^3 e^{-2t}\}$

Vet att om $F(s) = \mathcal{L}\{f(t)\}$, då är $\mathcal{L}\{f(t)e^{at}\} = F(s-a)$.

$f(t) = t^3 \Rightarrow \mathcal{L}\{f(t)\} = \frac{3!}{s^4} = F(s)$ och

$$\mathcal{L}\{t^3 e^{-2t}\} = F(s-(-2)) = F(s+2) = \frac{3!}{(s+2)^4} = \underline{\underline{\frac{6}{(s+2)^4}}}.$$

7.3.15 Bestäm $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$

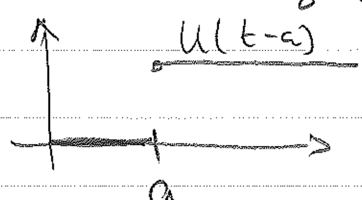
$$s^2+4s+5 = s^2+4s+4+1 = (s+2)^2+1$$

$$\frac{s+2}{(s+2)^2+1} = \frac{s}{(s+2)^2+1} + \frac{2}{(s+2)^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\frac{s+2}{(s+2)+1}}_{\mathcal{L}\{\cos(t)\}(s+2)}\right\} - 2\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{(s+2)^2+1}}_{\mathcal{L}\{\sin t\}(s+2)}\right\}$$

$$= \underline{\underline{e^{-2t}\cos t - 2e^{-2t}\sin t}}$$

7.3.39 Kom ihåg: $\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$



Och att $\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

Bestäm $\mathcal{L}\{t\,\mathcal{U}(t-2)\}$

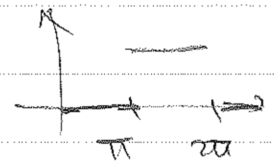
Sätt $g(t) = t$ då är

$$\mathcal{L}\{t\,\mathcal{U}(t-2)\} = e^{-2s}\mathcal{L}\{t+2\} = \underline{\underline{e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)}}$$

7.3.69 Lös begynnelsesvärdeproblemet

$$y''+y = f(t), \quad y(0)=0, \; y'(0)=1, \; f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1 & \pi \le t < 2\pi \\ 0 & 2\pi \le t \end{cases}$$

Ser att $f(t) = \mathcal{U}(t-\pi) - \mathcal{U}(t-2\pi) \xrightarrow{\text{graf}}$



Laplacetransform av ekvationen ger

$$(s^2Y-1)+Y = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s} = \frac{e^{-\pi s}-e^{-2\pi s}}{s}$$

$$\Downarrow$$

$$\underbrace{s^2Y+Y}_{(s^2+1)Y} - 1 = \frac{e^{-\pi s}-e^{-2\pi s}}{s}$$

$$Y = \frac{\frac{e^{-\pi s}-e^{-2\pi s}}{s}+1}{s^2+1} = \frac{e^{-\pi s}-e^{-2\pi s}+s}{s(s^2+1)}$$

Partialbrøksoppspalting :

$$\frac{1}{s(s^2+1)} = \frac{a}{s} + \frac{bs+c}{s^2+1}$$

$$\Rightarrow \quad 1 = (s^2+1)a + bs^2 + cs$$

Gir:
$$a = 1$$
$$c = 0$$
$$a + b = 0 \Rightarrow b = -a = -1$$

Så
$$\frac{1}{s(s^2+1)} = \frac{1}{s} + \frac{-s}{s^2+1}$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s} - e^{-2\pi s} + s}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s(s^2+1)}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right\} + \sin t$$

$$= \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{s}{s^2+1}\right\} + \sin t$$

$$= u(t-\pi) - \cos(t+\pi)u(t-\pi) - u(t-2\pi) + \cos(t+2\pi)u(t+2\pi) + \sin t$$

$$= u(t-\pi)\left(1 - \cos(t+\pi)\right) + u(t-2\pi)\left(\cos(t+2\pi) - 1\right) + \sin t$$