LECTURE 2

STRONG APPROXIMATION FOR THIN MATRIX GROUPS AND DIOPHANTINE APPLICATIONS

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Affine Sieve

Γ a group of affine polynomial maps of affine n-space $\mathbb{A}^n$ which preserve $\mathbb{Z}^n$. Fix a $\in \mathbb{Z}^n$.

$$O := \Gamma \cdot a \text{ , the orbit of } a \text{ under } \Gamma.$$

$O \subset \mathbb{Z}^n$, $V := Zcl(O)$, the Zariski closure of O.

V is defined over $\mathbb{Q}$.

Diophantine analysis of O:

- Strong Approximation; for $q \geqslant 1$

$$O \xrightarrow{\text{red mod } q} V(\mathbb{Z}/q\mathbb{Z}).$$

What is the image?

• Sieving for primes or almost primes.

If $f \in \mathbb{Z}[x_1, x_2, \ldots x_n]$, not constant on O; is the set of $x \in O$ for which $f(x)$ is prime (or has at most a fixed number r prime factors) Zariski dense in V?

Examples of Γ and Orbits:

(1) Classical (automorphic forms)

$\Gamma \leqslant GL_3(\mathbb{Z})$ generated by

$$\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix},$$

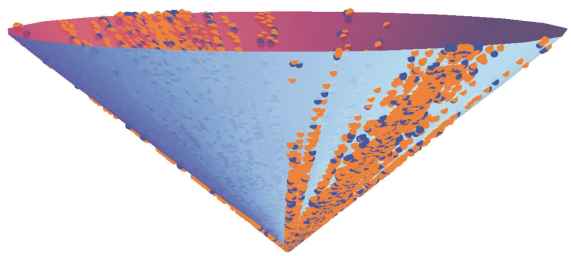
Γ is a finite index subgroup of $O_f(\mathbb{Z})$, where

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$$

Γ is an arithmetic group

$$O = \Gamma \cdot (3,4,5)$$

yields all (primitive) Pythagorean triples.



(2) Γ linear and "thin", not so classical:

$\Gamma = A \subset GL_4(\mathbb{Z})$ the Apollonian Group generated by the involutions S_1, S_2, S_3, S_4

$$\begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix}$$

S_j corresponds to switching the root x_j to its conjugate on the cone

$$F(x) = 0 \text{ , where}$$

$$F(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) \\ - (x_1 + x_2 + x_3 + x_4)^2.$$

$$A \leq O_F(\mathbb{Z})$$

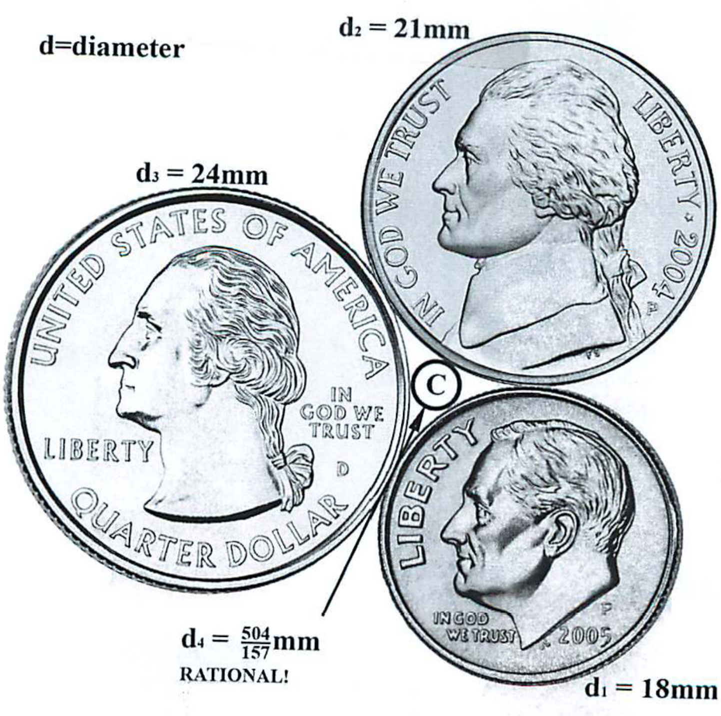
but while $Zcl(A) = O_F$, A is of infinite index in $O_F(\mathbb{Z})$, i.e. “thin".

The orbits of A in $\mathbb{Z}^4$ corresponds to the curvatures of 4 mutually tangent circles in an integral Apollonian packing.

For example $O = A.(-11, 21, 24, 28)$

corresponds to:


UNITED STATES OF AMERICA
IN GOD WE TRUST
LIBERTY
D
QUARTER DOLLAR
IN GOD WE TRUST
LIBERTY · 2004
LIBERTY
IN GOD WE TRUST
2005


d=diameter
d₂ = 21mm
d₃ = 24mm
C
d₄ = 504/157 mm
RATIONAL!
d₁ = 18mm

Scale the picture by a factor of 252 and let $a(c) =$ curvature of the circle $c = 1/radius(c)$.

The curvatures are displayed. Note the outer one by convention has a negative sign.
By a theorem of Apollonius, place unique circles in the lunes.

The Diophantine miracle is the curvatures are integers!

Γ LINEAR

$\Gamma \leq GL_n(\mathbb{Z})$ (OR MORE GENERALLY S-INTEGRAL)

$G = Zcl(\Gamma)$, ZARISKI CLOSURE OF Γ, $\mathbb{Q}$ ALGEBRAIC GROUP

$\Gamma \leq G(\mathbb{Z})$

- Γ IS ARITHMETIC IF IT IS FINITE INDEX IN $G(\mathbb{Z})$ AND IT IS <u>THIN</u> IF NOT.

IF Γ IS ARITHMETIC THE DIOPHANTINE PROBLEMS FOR $\mathcal{O} = \Gamma . a$ BECOME THE USUAL ONES FOR $V(\mathbb{Z})$ $V = Zcl(\mathcal{O})$. IF THE ORBIT $G.a$ IS CLOSED, $V \cong G/H$ WITH H REDUCTIVE, AND G IS SEMISIMPLE, THEN $V(\mathbb{Z})$ CONSISTS OF FINITELY MANY Γ ORBITS (BOREL-HARISHCHANDRA).

USING THE THEORY OF ARITHMETIC GROUPS AND AUTOMORPHIC FORMS (AND ERGODIC THEORY) IE THE SPECTRAL DECOMPOSITION OF $L^2(\Gamma \backslash G(\mathbb{R}))$ UNDER THE RIGHT $G(\mathbb{R})$ ACTION, ALLOWS FOR A DIOPHANTINE ANALYSIS OF $\Gamma.a$.

GHOSH-GORODNIK AND NEVO HAVE OBTAINED QUATITATIVE RESULTS ON THE DIOPHANTINE APPROXIMATION PROBLEM (STRONG APPROXIMATION) IN THIS CONTEXT. IN PARTICULAR IN CERTAIN CASES WHERE THE "FULLY TEMPERED" VERSION OF THE RAMANUJAN CONJECTURES HOLD THE SHOW THAT ALMOST ALL POINTS OF $V(\mathbb{R})$ HAVE OPTIMALLY SHARP DIOPHANTINE EXPONENT.

- IF Γ IS THIN THE DIOPHANTINE PROBLEMS ARE MORE EXOTIC AND THE FAMILIAR TOOL, GONE.

TOOLS:

STRONG APPROXIMATION:

BASIC CASE: $SL_n(\mathbb{Z}) \xrightarrow{\text{onto}} SL_n(\mathbb{Z}/q\mathbb{Z})$

IF $\Gamma \leq SL_n(\mathbb{Z})$ AND IS ZARISKI DENSE IN SL_n (COULD BE THIN!)
WHAT ABOUT $\Gamma \longrightarrow SL_n(\mathbb{Z}/q\mathbb{Z})$?

THEOREM (MATHEWS-WEISFELER-VASERSTEIN, ALSO NORI, LARSEN-PINK):

THERE IS A FINITE SET $S = S(\Gamma)$ SUCH THAT FOR $(q, S) = 1$

$$\Gamma \xrightarrow{\text{mod } q} SL_n(\mathbb{Z}/q\mathbb{Z})$$ IS STILL ONTO!

MORE GENERALLY THE ABOVE IS TRUE WITH G REPLACING SL_n, G SEMI-SIMPLE AND SIMPLY CONNECTED.

TWO NOVEL TOOLS INVOLVED ARE

(1) EXPANSION OR SUPERSTRONG APPROXIMATION

IT HAS ITS ROOTS IN THE GENERAL RAMANUJAN CONJECTURES IN THE THEORY OF AUTOMORPHIC FORMS.

$\Gamma \leq GL_n(\mathbb{Q})$ FINITELY GENERATED

LET S BE A SYMMETRIC SET OF GENERATORS ($s \in S \iff s^{-1} \in S$).

FOR $q \geq 1$ LET $\Gamma(q)$ BE THE KERNEL OF REDUCTION MOD q AND LET $X(q)$ BE THE $|S|$- REGULAR "CONGRUENCE GRAPH"

$(\Gamma/\Gamma(q), S)$, VERTICES $\Gamma/\Gamma(q)$

AND

$$x\Gamma(q) \xleftrightarrow{\text{JOINED}} sx\Gamma(q)$$

FOR $s \in S$.

THE CRITICAL FEATURE IS THAT THESE CONGRUENCE GRAPHS $X(q)$ FORM AN EXPANDER FAMILY AS $q \to \infty$.

THE MAIN EXPANSION THEOREM WHICH IS A CONSEQUENCE OF MANY ADVANCES FROM SPECIAL TO GENERAL AND CHRONOLOGICALLY

[XUE-S, GAMBURD, HELFGOTT, BOURGAIN-GAMBURD, BOURGAIN-GAMBURD-S, PYBER-SZABO, BREUILLARD-GREEN-TAO, VARJU]

THEOREM SUPERSTRONG APPROXIMATION (SALEHI-VARJU 2011)

THE CONGRUENCE GRAPHS $X(q)$ AS ABOVE FORM AN EXPANDER FAMILY IFF G^0 THE IDENTITY COMPONENT OF $G = Zcl(\Gamma)$ IS PERFECT,

I.E. $G^0 = [G^0, G^0]$.

THIS EXPANSION PROPERTY HAS MANY APPLICATIONS BESIDES THE DIOPHANTINE ORBIT METHOD:

- SIEVING IN GROUPS (RIVIN, LUBOTZKY MEIRI, KOWALSKI, ...)

- BETTI NUMBERS OF RANDOM 3-MANIFOLDS (KOWALSKI, DUNFIELD-THURSTON MODEL)

- HEEGARD GENUS OF HYPERBOLIC 3-MANIFOLDS (LACKENBY, LONG-LUBOTZKY-REID)

- LARGE DISTORTION FOR ISOTOPY CLASSES OF KNOTS IN S^3 (GROMOV-GUTH)

- GONALITY OF TOWERS OF CURVES (ZOGRAF, ELLENBERG-HALL-KOWALSKI TO DIOPHANTINE FINITENESS THEOREMS).

- UNIFORM LIMIT MULTIPLICITIES

AFFINE SIEVE:

$f \in \mathbb{Z}[x_1, x_2, \ldots x_n]$, $\mathcal{O} = \Gamma v$

WE SAY THAT $(\mathcal{O}, f)$ SATURATES IF THERE IS $r < \infty$ SUCH THAT

$\{x \in \mathcal{O} : f(x)$ HAS AT MOST r PRIME FACTORS$\}$

IS ZARISKI DENSE IN $Zcl(\mathcal{O})$.

- THE MINIMAL SUCH r IS THE SATURATION NUMBER $r_0(\mathcal{O}, f)$.

EXAMPLES (CLASSICAL):

1) $r_0(\mathbb{Z}, x(x+2)) = 2$ IFF TWIN PRIME
2) $r_0(\mathbb{Z}, x(x+2)) < \infty$ BRUN 1915
3) $r_0(\mathbb{Z}, x(x+2)) \leq 3$ CHEN 1973
4) $r_0(\mathbb{Z}, x(x+k)) = 2$ FOR SOME $k < \infty$ Y. ZHANG, 2013
5) GIVEN m THERE ARE $k_1 < k_2 \ldots < k_m$ SUCH THAT $r_0(\mathbb{Z}, (x+k_1)(x+k_2)\ldots(x+k_m)) = m$. J. MAYNARD 2013.

FUNDAMENTAL SATURATION THEOREM OF THE AFFINE SIEVE (SALEHI-S 2012):

Γ, f AS ABOVE $\mathcal{O} = \Gamma v \subset \mathbb{Z}^n$

IF $G = Zcl(\Gamma)$ IS LEVI-SEMISIMPLE (IE RAD(G) CONTAINS NO TORUS) THEN $r_0(\mathcal{O}, f) < \infty$.

- HEURISTIC ARGUMENTS SHOW THAT THE CONDITION ON RAD(G) IS PROBABLY NECESSARY FOR SATURATION!

- FOR EXAMPLES OF THE THEORY APPLIED TO LOCAL/GLOBAL PRINCIPLES FOR INTEGRAL APOLLONIAN PACKINGS, SEE

  E. FUCHS BAMS 2013.
  A. KONTOROVICH

UBIQUITY OF THIN MATRIX GROUPS?

- THERE IS NO DECISION PROCEDURE TO TELL WHETHER A GIVEN $A_1, \ldots, A_7$ IN $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ GENERATES A THIN GROUP OR NOT (MIHALOVA 1959).

- IN PRACTICE IF Γ IS IN FACT A CONGRUENCE SUBGROUP OF $G(\mathbb{Z})$ AND IS GIVEN IN TERMS OF GENERATORS, THEN ONE CAN VERIFY THIS BY PRODUCING GENERATORS. HOWEVER IF Γ IS THIN HOW CAN WE CERTIFY THIS?

- FOR A TRUE GROUP THEOREST, THIN IS THE RULE! GIVEN $A, B \in SL_n(\mathbb{Z})$ CHOSEN AT RANDOM, THEN $\Gamma = \langle A, B \rangle$ HAS $G = SL_n$, Γ IS FREE AND THIN. (AOUN, FUCHS).

UNIVERSAL QUANTUM GATE GROUPS:

THE PRIMARY GOLDEN GATE GROUP Γ WAS GENERATED BY

C CLIFFORD GROUP OF ORDER 24 IN $G = PU(2)$

AND

$$T = T_4 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

("$\pi/8$ - GATE")

THE KEY WAS THAT Γ IS ARITHMETIC.

WHAT IF INSTEAD WE ADD TO C A $\pi/2n$ GATE (IT WILL STILL BE UNIVERSAL)

$$T_n = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/n} \end{bmatrix} \quad ?$$

(asked and studied by FOREST, GOSSET, KLIUCHNIKOV, McKINON)

In (SARNAK LETTER 2015) I INDICATE A PROOF THAT UNLESS $n = 3, 4, 8, 12$ Γ WILL BE THIN!

HYPERBOLIC REFLECTION GROUPS (VINBERG):

$f(x_1, \dots, x_n)$ A RATIONAL QUADRATIC FORM OF SIGNATURE $(n-1, 1)$, $n \geq 5$.

$G = O_f$, $G(\mathbb{Z})$ ARITHMETIC.

$R_f(\mathbb{Z})$ THE (NORMAL) SUBGROUP OF $G(\mathbb{Z})$ GENERATED BY β'S WHICH INDUCE HYPERBOLIC REFLECTIONS ON $\mathbb{H}^{n-1}$! THEN EXCEPT FOR RARE CASES

$$\left| O_f(\mathbb{Z}) / R_f(\mathbb{Z}) \right| = \infty .$$

MONODROMY GROUPS: A NATURAL GEOMETRIC SOURCE OF FINITELY GENERATED SUBGROUPS OF $GL_n(\mathbb{Z})$ IS THE MONODROMY REPRESENTATION ON COHOMOLOGY OF A FAMILY OF ALGEBRAIC VARIETIES, VARIATONS OF HODGE STRUCTURES, MONODROMY OF LINEAR DIFFERENTIAL EQUATIONS, ...

- THE BASIC QUESTION AS TO WHETHER IN THE CASE OF VARIATION OF HODGE STRUCTURES THE MONODROMY Γ IS ARITHMETIC WAS POSED IN 1973 BY GRIFFITHS AND SCHMID.

- THEY SHOW THAT IF THE PERIOD MAP FROM THE PARAMETER SPACE S TO THE PERIOD DOMAIN D IS OPEN THE Γ IS ARITHMETIC.

ONE PARAMETER HYPERGEOMETRIC ${}_nF_{n-1}$:

$$\alpha, \beta \in \mathbb{Q}^n, \quad 0 \le \alpha_j < 1, \quad 0 \le \beta_j < 1$$

$$(*) \qquad Du = 0,$$

$$\theta = z\frac{d}{dz}$$

$$D = (\theta+\beta_1-1)(\theta+\beta_2-1)\cdots(\theta+\beta_n-1) - z(\theta+\alpha_1)\cdots(\theta+\alpha_n)$$

SOLUTIONS ARE

$$z^{1-\beta_i} {}_nF_{n-1}(1+\alpha_1-\beta_i, \ldots, 1+\alpha_n-\beta_i ; 1+\beta_1-\beta_i, \ldots \overset{\vee}{\ldots}, 1+\beta_n-\beta_i \mid z)$$

WHERE $\vee$ MEANS OMIT $1+\beta_i-\beta_i$ AND

$${}_nF_{n-1}(\rho_1, \ldots, \rho_n ; \eta_1, \ldots, \eta_{n-1} \mid z) = \sum_{k=0}^{\infty} \frac{(\rho_1)_k \cdots (\rho_n)_k}{(\eta_1)_k \cdots (\eta_{n-1})_k} \frac{z^k}{k!}$$

(*) IS SINGULAR AT $z=0, 1, \infty$ AND THE MONODROMY GROUP $H(\alpha,\beta)$ IS GOTTEN BY ANALYTIC CONTINUATION ALONG PATHS IN $\mathbb{P}^1 \setminus \{0,1,\infty\}$ OF A BASIS OF SOLUTIONS.

WE RESTRICT TO α, β SUCH THAT $H(\alpha,\beta)$ IS UP TO CONJUGATION IN $GL_n(\mathbb{Z})$.

BEUKERS AND HECKMAN COMPUTE

$$G = Zcl(H(\alpha, \beta))$$

EXPLICITLY IN TERMS OF α, β.

IN THIS SELF-DUAL SETTING G IS

(i) FINITE (SPORDIC LIST $n \leq 8$, ONE FAMILY $n > 9$)

(ii) O_n

(iii) Sp_n (ONLY OCCURS IF n EVEN).

VENKATARAMANA (2012): $n \geq 2$ EVEN

$$\alpha = \left(\frac{1}{2} + \frac{1}{n+1}, \frac{1}{2} + \frac{2}{n+1}, \ldots, \frac{1}{2} + \frac{n}{n+1}\right)$$

$$\beta = \left(0, \frac{1}{2} + \frac{1}{n}, \frac{1}{2} + \frac{2}{n}, \ldots, \frac{1}{2} + \frac{n+1}{n}\right)$$

THEN $G(\alpha, \beta) = Sp_n$ AND

$H(\alpha, \beta)$ IS ARITHMETIC!

THERE ARE 112 (α, β)'s GIVING $G(\alpha, \beta) = Sp_4$, ALL COMING FROM VARIATIONS OF INTEGRAL HODGE STRUCTURES (DORAN - MORGAN).

OF THESE MORE THAN HALF ARE ARITHMETIC (SINGH - VENKATARAMANA 2012)

14 CORRESPOND TO CALABI-YAU FAMILIES OF 3-FOLDS

EG: $\alpha = (0,0,0,0)$, $\beta = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

[DWORK FAMILY, CANDELAS ET AL]

BRAV-THOMAS (2012) SHOW THAT SEVEN OF THESE ARE THIN, WHILE SINGH THAT THE OTHER SEVEN ARE ARITHMETIC.

BRAV-THOMAS SHOW THAT THE GENERATORS OF $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$, A AND C ABOUT 0 AND 1 PLAY GENERALIZED PING-PONG ON A COMPLICATED POLYHEDRAL SUBSET OF $\mathbb{P}^3$.

HYPERBOLIC HYPERGEOMETRICS (FUCHS-MEIRI-S) 2013

(α,β) IS HHM IF $G(\alpha,\beta)$ IS ORTHOGONAL AND OF SIGNATURE $(n-1,1)$. (IN THIS CASE n IS ODD)

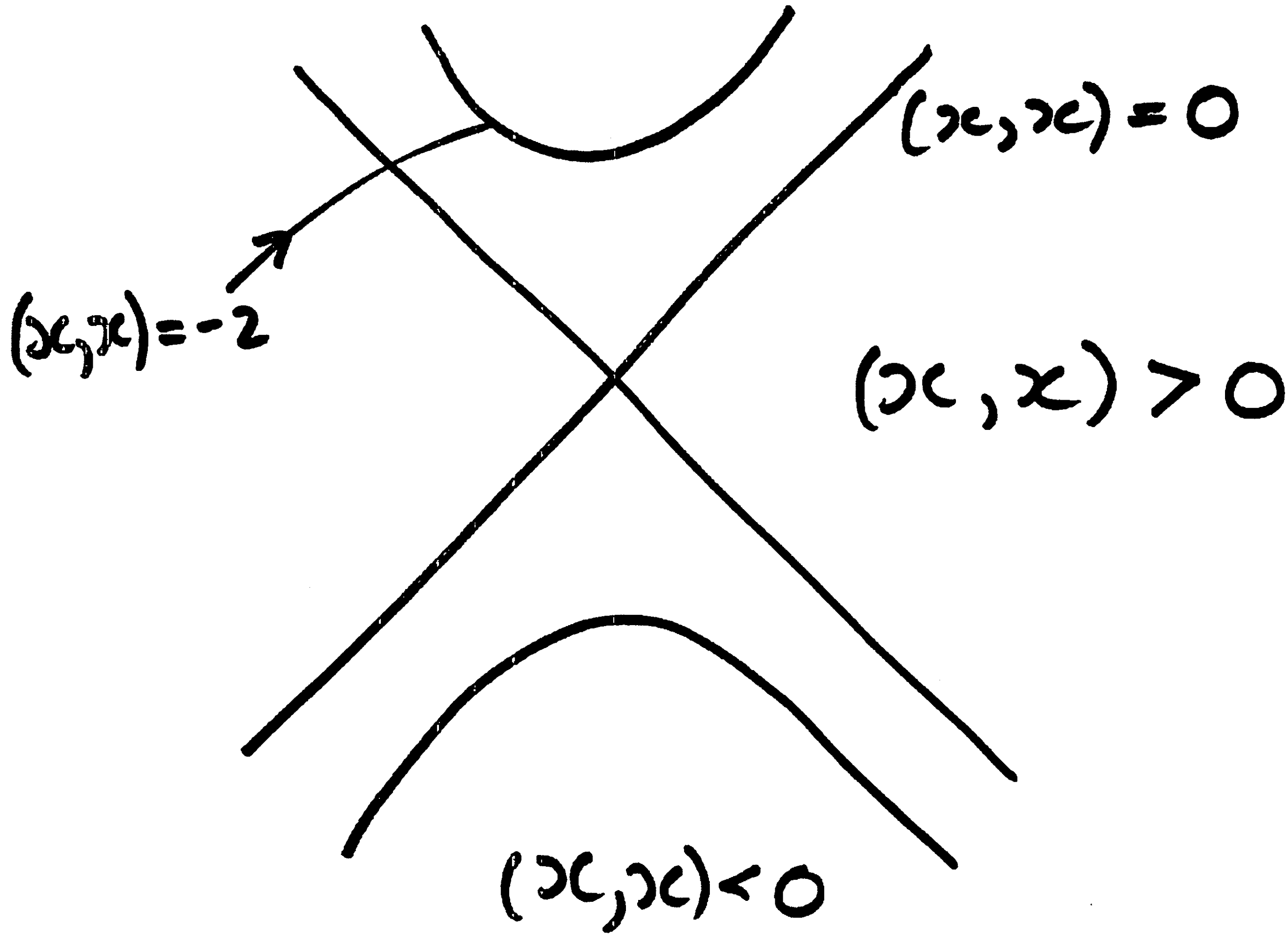
THEOREM 1 (F-M-S)

WITH THE EXCEPTION OF AN EXPLICIT (LONG) LIST OF FINITELY MANY (α,β)'s ALL WITH $n \leq 9$, ALL HHM's COME IN SEVEN INFINITE PARAMETRIC FAMILIES.

FOR THE HHM's WE GIVE A ROBUST OBSTRUCTION TO $H(\alpha,\beta)$ BEING ARITHMETIC, THAT IS A CERTIFICATE FOR $H(\alpha,\beta)$ TO BE THIN.

f A RATIONAL QUADRATIC FORM

$f(x) = (x,x)$ INTEGRAL ON A LATTICE L



$\{(x,x) = -2 : x_1 > 0\} = \mathbb{H}^{n-1}$

IF $(v,v) \neq 0$, $v \in L$ THEN THE LINEAR REFLECTION

$r_v(y) = y - \frac{2(v,y)}{(v,v)} v$, IS IN $O(L)$ IF $(v,v) = \pm 2$.

- IF $(v,v) > 0$ THEN τ_v INDUCES A HYPERBOLIC REFLECTION ON $\mathbb{H}^{n-1}$.
- IF $(v,v) < 0$ THEN $\tau_v \in O_f$ INDUCES A CARTAN INVOLUTION ON $\mathbb{H}^{n-1}$.

KEY POINT: FOR HHM'S

$$H(\alpha,\beta) = \langle A, B \rangle$$

A LOCAL MONODROMY ABOUT 0

B LOCAL MONODROMY ABOUT ∞

AND $C = A^{-1}B$ IS A CARTAN INVOLUTION.

UP TO COMMENSURABILITY $H(\alpha,\beta)$ IS GENERATED BY THE CARTAN INVOLUTIONS

$$A^k C A^{-k}, \quad k \in \mathbb{Z}.$$

$$R_2(L) := \{ v \in L : (v,v) = 2 \}$$

THE INTEGRAL ROOT VECTORS GIVING HYPERBOLIC REFLECTIONS

$$R_{-2}(L) := \{ v \in L : (v,v) = -2 \}$$

THE INTEGRAL ROOT VECTORS GIVING CARTAN INVOLUTIONS.

ACCORDING TO VINBERG/NIKULIN EXCEPT FOR SPECIAL L'S

$$| O(L)/R_2(L) | < \infty .$$

LET $\Delta \subset R_{-2}(L)$

WE GIVE A CONDITION UNDER WHICH $\langle \tau_v : v \in \Delta \rangle$ HAS FINITE IMAGE IN $O(L)/R_2(L)$.

MINIMAL DISTANCE GRAPH $X(L)$:

THE VERTICES OF $X(L)$ ARE THE CARTAN ROOTS $R_{-2}(L)$ AND JOIN v TO w IF $(v,w) = -3$ (MINIMAL DISTANCE THEY CAN BE)

PROPOSITION IF Δ IS CONTAINED IN A CONNECTED COMPONENT OF $X(L)$ THEN $\langle T_v : v \in \Delta \rangle$ HAS FINITE IMAGE IN $O(L)/R_2(L)$.

WITH THIS WE CAN SHOW THAT MOST OF THE HHM'S ARE THIN.

THEOREM n ODD

$$\alpha=\left(0,\frac{1}{n+1},\frac{2}{n+1},\ldots,\frac{n-1}{2(n+1)},\frac{n+3}{2(n+1)},\ldots,\frac{n}{n+1}\right),\ \beta=\left(\frac{1}{2},\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n}\right)$$

AND

$$\alpha=\left(\frac{1}{2},\frac{1}{2n-2},\frac{3}{2n-2},\ldots,\frac{2n-3}{2n-2}\right),\ \beta=\left(0,0,0,\frac{1}{n-2},\ldots,\frac{n-3}{n-2}\right)$$

ARE HYPERBOLIC HYPERGEOMETRIC AND ARE ARITHMETIC IF $n=3$ AND THIN $n\geq 5$.

CONJECTURE THERE ARE ONLY FINITELY MANY HHM'S WHICH ARE ARITHMETIC.

- H. PARK (THESIS 2013) SHOWS THAT THE HHM

$$\alpha=\left(0,\frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}\right),\ \beta=\left(\frac{1}{5},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{4}{5}\right)$$

IS GEOMETRICALLY FINITE (AND THIN).

REFERENCES TO MOST OF THE ABOVE CAN BE FOUND IN THE SURVEY

"NOTES ON THIN MATRIX GROUPS"

P. SARNAK

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