THE DEFECT OF TORAL LAPLACE EIGENFUNCTIONS AND ARITHMETIC RANDOM WAVES

PÄR KURLBERG[1], IGOR WIGMAN[2], AND NADAV YESHA[3]

ABSTRACT. We study the defect (or "signed area") distribution of standard toral Laplace eigenfunctions restricted to shrinking balls of radius above the Planck scale, either for deterministic eigenfunctions averaged w.r.t. the spatial variable, or in a random Gaussian scenario ("Arithmetic Random Waves"). In either case we exploit the associated symmetry of the eigenfunctions to show that the expectation (spatial or Gaussian) vanishes.

In the deterministic setting, we prove that the variance of the defect of flat eigenfunctions, restricted to balls shrinking above the Planck scale, vanishes for "most" energies. Hence the defect of eigenfunctions restricted to most of the said balls is small. We also construct "esoteric" eigenfunctions with large defect variance, by choosing our eigenfunctions so that to mimic the situation on the hexagonal torus, thus breaking the symmetries associated to the standard torus. In the random Gaussian setting, we establish various upper and lower bounds for the defect variance w.r.t. the Gaussian probability measure. A crucial ingredient in the proof of the lower bound is the use of Schmidt's subspace theorem.

1. INTRODUCTION

1.1. **Toral Laplace eigenfunctions and Arithmetic Random Waves.**

Toral Laplace eigenfunctions are an important model in Quantum Chaos since they capture many aspects of Laplace eigenfunction behaviour on generic manifolds. Toral eigenfunctions enjoy two significant privileges over the general case, making them attractive to address, in addition to their own sake, being Fourier sums with particular frequencies. First, its number theoretic ingredient makes them susceptible to methods borrowed from Analytic Number Theory. Second, their (slowly in 2 dimensions) growing spectral degeneracies allow for the study of the "typical" case, for example endowing the linear space of Laplace eigenfunctions with the same eigenvalue with a Gaussian probability measure (thus giving rise to "Arithmetic Random Waves").

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-torus, let

$$S := \{a^2 + b^2 : \ a, b \in \mathbb{Z}\}$$

be the set of all integers expressible as sum of two squares ("sequence of toral energies"), and for $n \in S$ let

$$N_n := r_2(n) = \# \left\{(a, b) \in \mathbb{Z}^2 : \ n = a^2 + b^2\right\}$$

be the number of ways to express n as sum of two squares. Then every function of the form

$$f_n(x) = \frac{1}{\sqrt{2N_n}} \sum_{\lambda\in\mathbb{Z}^2:\, \|\lambda\|^2=n} a_\lambda \cdot e(\langle x, \lambda\rangle) \tag{1.1}$$

with convenience only pre-factor $\frac{1}{\sqrt{2N_n}}$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$, $x = (x_1, x_2) \in \mathbb{T}^2$,

$$\langle x, \lambda\rangle = x_1\lambda_1 + x_2\lambda_2,$$

$e(y) := e^{2\pi i y}$, and $a_\lambda \in \mathbb{C}$ some complex coefficients subject to

$$a_{-\lambda} = \overline{a_\lambda}, \tag{1.2}$$

E-mail addresses: kurlberg@kth.se, igor.wigman@kcl.ac.uk, nyesha@univ.haifa.ac.il.
Date: July 11, 2021.

is a real-valued Laplace eigenfunction with eigenvalue $E = E_n = 4\pi^2 n$, i.e. it satisfies the Helmholtz equation

$$\Delta f_n + E f_n = 0. \tag{1.3}$$

Conversely, every real-valued function satisfying the equation (1.3) is necessarily of the form (1.1) for some $n \in S$, and $\{a_\lambda\}_{\|\lambda\|^2=n}$ as above.

Given $n \in S$, the linear space of functions (1.1) subject to (1.2) is of real dimension N_n. The sequence N_n is subject to large and erratic fluctuations. It is *on average* $N_n \sim \frac{1}{\kappa_{RL}} \cdot \sqrt{\log n}$ with $\kappa_{RL} > 0$ the Ramanujan–Landau constant [18], but its *normal order* is $N_n = \log n^{\log 2/2 + o(1)}$, i.e. for every $\epsilon > 0$, for "most" numbers $n \in S$, the inequality $\log n^{\log 2/2-\epsilon} < N_n < \log n^{\log 2/2+\epsilon}$ holds (cf. [13, §22.11]). Moreover,

$$N_n = O(n^\epsilon), \tag{1.4}$$

by an elementary argument.

We denote by

$$\mathcal{E}_n = \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \ \lambda_1^2 + \lambda_2^2 = n\}$$

the representations of n as sum of two squares, or, what is equivalent, $\mathcal{E}_n$ are all $\mathbb{Z}^2$-lattice points lying on the radius-$\sqrt{n}$ circle. One may endow this space with a probability measure by assuming that the $\{a_\lambda\}_{\lambda\in\mathcal{E}_n}$ are standard (complex) Gaussian[1] i.i.d. save to (1.2), turning $\{f_n\}_{n\in S}$ into a Gaussian *ensemble of random fields* [23, 26], all defined on $\mathbb{T}^2$, usually referred to as "Arithmetic Random Waves" [16]. Alternatively, f_n are unit variance stationary random fields on $\mathbb{T}^2$, uniquely defined via their covariance function

$$r_n(x) = r_n(y, x+y) := \mathbb{E}[f_n(y) \cdot f_n(x+y)] = \frac{1}{N_n} \sum_{\lambda\in\mathcal{E}_n} \cos(2\pi\langle\lambda, x\rangle). \tag{1.5}$$

1.2. **Defect.** The (total) defect of a smooth, not identically vanishing, function $g : \mathbb{T}^2 \to \mathbb{R}$, (called "signed area" in the physics literature) is

$$\mathcal{D}(g) := \text{Area}(g^{-1}(0, +\infty)) - \text{Area}(g^{-1}(-\infty, 0)) = \int_{\mathbb{T}^2} H(g(y))dy,$$

with $H(\cdot)$ denoting the sign function

$$H(y) := \begin{cases} 1 & y > 0, \\ 0 & y = 0, \\ -1 & y < 0. \end{cases} \tag{1.6}$$

The defect of Laplace eigenfunctions was first addressed in the physics literature [4] for random planar monochromatic waves. A precise asymptotic expression for the defect variance, and a Central Limit Theorem was established, along with generic nonlinear functionals, for the ensemble $\{T_l\}_{l\geq 1}$ of random Gaussian spherical harmonics [19, 20] with mathematical rigour. The $T_l : \mathcal{S}^2 \to \mathbb{R}$ is the important ensemble of spherical random fields defined by the covariance functions

$$\mathbb{E}[T_l(x) \cdot T_l(y)] = P_l(\cos(d(x, y))),$$

where $P_l(\cdot)$ are the Legendre polynomials and $d(\cdot, \cdot)$ is the spherical distance; $T_l(\cdot)$ scales asymptotically like Berry's Random Waves around every point of $\mathcal{S}^2$, the main findings of [19, 20] being consistent with [4], up to the said scaling. The sign distribution of Laplace eigenfunctions on closed surfaces in the high energy limit was further addressed in [22], where it was shown that if one restricts an eigenfunction to any disc centred at the nodal line, then the defect can be at most inversely logarithmically close to ± 1.

[1] We work under the convention that $a_\lambda = b_\lambda + ic_\lambda$, where the b_λ and c_λ are standard real-valued Gaussians.

We are interested in the defect of $f_n(\cdot)$, with $f_n(\cdot)$ defined as in (1.1). We claim that for *every* such function f_n, the corresponding defect

$$\mathcal{D}(f_n) \equiv 0 \tag{1.7}$$

vanishes, so the study of $\mathcal{D}(f_n)$ trivialises, and, accordingly, below we will pass to subdomains of $\mathbb{T}^2$. First, if n is odd, then for every $\lambda = (\lambda_1, \lambda_2) \in \mathcal{E}_n$, precisely one of λ_1 and λ_2 is odd. Hence, f_n changes its sign under the involution $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ mapping $\cdot \mapsto \cdot + (1/2, 1/2)$, i.e.

$$f_n(\tau x) = -f_n(x),$$

which readily implies $\mathcal{D}(f_n) = 0$. Otherwise, if n is even, we may assume w.l.o.g. that[2] $n \equiv 2(4)$, whence for all $\lambda \in \mathcal{E}_n$, both λ_1, λ_2 are odd, and then f_n changes its sign under the involution $\rho : \mathbb{T}^2 \to \mathbb{T}^2$ mapping $\cdot \mapsto \cdot + (1/2, 0)$ (or $\cdot \mapsto \cdot + (0, 1/2)$), also yielding $\mathcal{D}(f_n) = 0$.

It is therefore essential to pass to, possibly shrinking, *subdomains* of $\mathbb{T}^2$, most canonically, the radius-s discs $B_x(s) \subseteq \mathbb{T}^2$ centred at $x \in \mathbb{T}^2$, $0 < s < 1/2$, and $B(s) := B_0(s)$, with $s = s(n)$ allowed to depend on n, (possibly with $s(n) \to 0$). Since Quantum Chaos should exhibit itself above the Planck scale $s \gg \frac{1}{\sqrt{n}}$ [2], it makes sense to take, as an example, $s = n^{-1/2+\epsilon}$, or, perhaps, replace the ϵ-power of n with a slower growing function of n (such as a power of $\log n$). Our principal results concern the defect distribution corresponding to both individual deterministic cases, w.r.t. space average in section 1.3 below, and the Arithmetic Random Waves (random Gaussian toral eigenfunctions) in section 1.4 below.

1.3. Statement of principal results: spatial defect distribution.

We work with a sequence of *deterministic* eigenfunctions f_n of the form (1.1), and study the defect distribution of f_n restricted to $B_x(s)$, where x is *random uniform* on $\mathbb{T}^2$, and s is above the Planck scale. That is, given a function f_n of the form (1.1), $x \in \mathbb{T}^2$ and $s > 0$, we consider

$$Y_{f_n,s}(x) := \frac{1}{\pi s^2} \int\limits_{B_x(s)} H(f_n(y))dy, \tag{1.8}$$

the defect of f_n restricted to $B_x(s)$, with the normalisation making $Y_{f_n,s}(x)$ invariant w.r.t. homotheties. Such an approach was recently taken by Sarnak [27] and Humphries [14] for modular forms, and Granville–Wigman [12] and Wigman–Yesha [34] for toral Laplace eigenfunctions (1.1), in studying the *mass distribution* of the respective models, showing, in particular, that if there exist discs observing unproportionately large or small L^2-mass of f_n, then these are not "typical".

Our principal interest here is the distribution of the values of $Y_{f_n,s}(\cdot)$ in (1.8) as x distributes randomly uniformly on $\mathbb{T}^2$; we denote accordingly the "spatial defect expectation"

$$\mathbb{E}_{\mathbb{T}^2}[Y_{f_n,s}] := \int\limits_{\mathbb{T}^2} Y_{f_n,s}(x)dx,$$

and the "spatial defect variance"

$$\mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s}) := \int\limits_{\mathbb{T}^2} \left(Y_{f_n,s}(x) - \mathbb{E}_{\mathbb{T}^2}[Y_{f_n,s}]\right)^2 dx.$$

The degeneracy argument identical to the argument we used to establish (1.7) that the total defect of every function (1.1) vanishes, yields that, in general, the spatial defect expectation vanishes precisely, i.e. that

$$\mathbb{E}_{\mathbb{T}^2}[Y_{f_n,s}] = 0. \tag{1.9}$$

[2] Otherwise both the entries λ_1, λ_2 are even, which yields that f_n is invariant under the involutions $\cdot \mapsto \cdot + (1/2, 0)$ and $\cdot \mapsto \cdot + (0, 1/2)$, and we may pass from n to $n/4$.

In what follows, we will restrict ourselves to Bourgain's class [7] of eigenfunctions

$$\mathscr{B}_n = \left\{ f_n = \sum_{\lambda \in \mathcal{E}_n} a_\lambda \cdot e(\langle x, \lambda \rangle) : \forall \lambda \in \mathcal{E}_n, \ |a_\lambda| = 1 \text{ and } a_{-\lambda} = \overline{a_\lambda} \right\}.$$

Bourgain's class eigenfunctions are the first deterministic "implementation" of Berry's Random Wave Model [7]. Our first principal result asserts that for generic $n \in S$, and $f_n \in \mathscr{B}_n$ a Bourgain class function, the spatial defect variance vanishes uniformly for s slightly above the Planck scale. Since $Y_{f_n,s}$ is bounded, this is equivalent to the statement that, in the said scenario, the proportion of positive values of f_n in "most" discs of radius above the Planck scale is asymptotic to $1/2$ (see Lemma 3.8 below). It is likely that the proof of the principal result immediately below holds for a more general family of flat eigenfunctions of the type considered in [34] (an event of almost full Gaussian probability); despite that we abandon the possible generality for the sake of the elegance of presentation. That *some* flatness condition is *essential* for the defect variance vanishing is asserted in Theorem 1.2 to follow immediately after the announced principal result.

Theorem 1.1. *There exists a sequence $S' \subseteq S$ of relative density*[3] *1, so that for all $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ sufficiently large, so that for all $n > n_0$ with $n \in S'$,*

$$\mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s}) < \epsilon$$

holds uniformly for all $f_n \in \mathscr{B}_n$, $s > R/\sqrt{n}$. Equivalently,

$$\lim_{\substack{R\to\infty \\ n\to\infty,\, n\in S'}} \sup_{\substack{s>R/\sqrt{n} \\ f_n \in \mathscr{B}_n}} \mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s}) = 0.$$

The arithmetic conditions on the sequence S' as postulated in Theorem 1.1 will be explicated in section 2.2 below, as part of Theorem 2.5. Finally, the result on the flatness being of essence for the spatial defect variance vanishing announced above is stated, with radii vanishing *arbitrarily* slowly, or even for *fixed* sufficiently small radii.

Theorem 1.2. *There exists a (thin) sequence $S'' \subseteq S$, a deterministic sequence $\{f_n\}_{n\in S''}$ of eigenfunctions* (1.1)*, and numbers $\gamma, \epsilon_0 > 0$, so that the inequality*

$$\liminf_{n\in S''} \mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,\Psi(n)}) > \epsilon_0$$

holds for every function $\Psi : \mathbb{Z}_{>0} \to (0, \min(\gamma, 1/2))$, subject to $\Psi(n)n^{1/2} \to \infty$.

1.4. **Statement of principal results: defect variance for Arithmetic Random Waves.** Rather than averaging w.r.t. the spatial variable, we can take $f_n(\cdot)$ to be the Arithmetic Random Waves (i.e. the random Gaussian model associated to (1.1)), and denote

$$\mathcal{D}_{n;s} := \frac{1}{\pi s^2} \int_{B(s)} H(f_n(y))dy. \tag{1.10}$$

We observe that, by the stationarity of f_n, the law of $\mathcal{D}_{n;s}$ is independent of the centre of the disc, so that, in what follows, we will assume that the disc on the r.h.s. of (1.10) is *centred*. Since, for a given $y \in \mathbb{T}^2$, the law of $f_n(y)$ is symmetric around the origin, and $H(\cdot)$ is odd, we have $\mathbb{E}[H(f_n(y))] \equiv 0$, and, by inverting the integral on the r.h.s. of (1.10), it is evident that for every $n \in S$ and $s > 0$,

$$\mathbb{E}[\mathcal{D}_{n;s}] = 0. \tag{1.11}$$

---

[3] A subset $S' \subseteq S$ is of relative density κ in S, if

$$\lim_{X\to\infty} \frac{\#S'(X)}{\#S(X)} = \kappa,$$

where for $\mathcal{A} \subseteq \mathbb{N}$ we define $\mathcal{A}(X) := \{n \le X : n \in \mathcal{A}\}$.

Our next principal result asserts that $\mathrm{Var}(\mathcal{D}_{n;s}) \to 0$ as long as the ball radius is above the Planck scale, i.e. $s \cdot \sqrt{n} \to \infty$.

Theorem 1.3. *Fix $\epsilon > 0$ sufficiently small. For every $0 < \delta < 4\epsilon$ one has*

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right) \ll \frac{1}{N_n^{\delta}}$$

uniformly for all $s > n^{-1/2+\epsilon}$. Equivalently,

$$\sup_{n\in S}\sup_{s>n^{-1/2+\epsilon}} \mathrm{Var}\left(\mathcal{D}_{n;s}\right)\cdot N_n^{\delta} < +\infty.$$

If one is willing to excise a *thin* sequence of energies, that is, a subsequence S_1 of S whose relative asymptotic density in S is 0, so that whatever generic energy levels are remaining satisfy certain arithmetic conditions explicated in Theorem 2.6 of section 2.4 below, then the asserted rate of decay is significantly more rapid, namely, faster than polynomial in N_n.

Theorem 1.4. *For every $\epsilon > 0$ there exists a subsequence $S''' = S'''(\epsilon) \subseteq S$ of energy levels of relative density 1, so that, along $n \in S'''$, the inequality*

$$\sup_{s>n^{-1/2+\epsilon}} \mathrm{Var}\left(\mathcal{D}_{n;s}\right) \ll_A \frac{1}{N_n^{A}}, \tag{1.12}$$

holds for every $A > 0$.

To the other end, we claim the following *lower* bound for $\mathrm{Var}(\mathcal{D}_{n;s})$ above the Planck scale, valid for all $n \in S$.

Theorem 1.5. *Let $s = s(n)$ be a sequence of radii so that $T := s \cdot \sqrt{n} \to \infty$.*
a. For every $\delta > 0$ there exists a sufficiently large number $A = A(\delta)$ so that

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right) \gg_{\delta} \frac{1}{N_n^{A}\cdot T^{3+\delta}}. \tag{1.13}$$

b. If, in addition, $2\pi T$ is uniformly bounded away from the zeros of the Bessel J_1 function, then

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right) \gg \frac{1}{T^{3}}. \tag{1.14}$$

For comparison of the generic upper bound (1.12) with the lower bounds (1.13) and (1.14) (restricted to the regime $s > n^{-1/2+\epsilon}$ all the said bounds hold) one should bear in mind (1.4), i.e. that every arbitrarily small positive power of n dominates every power of N_n. It is well known that at infinity, the zeros of the Bessel J_1 function are asymptotic to the arithmetic sequence

$$\left\{\frac{\pi}{4} + \pi\cdot m\right\}_{m\geq 1}. \tag{1.15}$$

The a fortiori meaning of the condition postulated by Theorem 1.5b is that $2\pi T$ is bounded away by at least $\epsilon_0 > 0$ from the said sequence (1.15), whence the conclusions apply (with constants depending on ϵ_0). In particular, given a scaling sequence $s(n)$, after making a small perturbation (of order $O(n^{-1/2})$), we can ensure that the lower bound in (1.14) holds.

Acknowledgements. We are indebted to Zeév Rudnick for many stimulating discussions, and his comments on an earlier version of this manuscript, in particular, pertaining to Lemma 5.3 on Diophantine approximations. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013), ERC grant agreement nº 335141 (I.W. and N.Y.). P.K. was partially supported by the Swedish Research Council (2016-03701).

2. Outline of the paper

2.1. **Number theoretic preliminaries.** Before we will be able to explain the essence of our arguments we will be required to bring forward some arithmetic aspects of the lattice points $\mathcal{E}_n$.

2.1.1. *Angular equidistribution of lattice points.* First, we are interested in the *angular* distribution of $\mathcal{E}_n$. To this end we define the sequence

$$\nu_n := \frac{1}{N_n} \sum_{\lambda \in \mathcal{E}_n} \delta_{\lambda/\sqrt{n}}$$

of probability measures on $\mathcal{S}^1 \subseteq \mathbb{R}^2$, indexed by $n \in S$. It is well-known [15, 10, 11] that generically the angles of $\mathcal{E}_n$ are equidistributed, i.e. along a sequence $\{n\} \subseteq S$ of relative density 1,

$$\nu_n \Rightarrow \frac{d\theta}{2\pi}, \tag{2.1}$$

where, as usual, " $\Rightarrow''$ stands for weak-$*$ convergence of probability measures, and $\frac{d\theta}{2\pi}$ is the normalised arc-length measure on the unit circle. However, even under the (generic) assumption $N_n \to \infty$, there exist sequences $\{n\} \subseteq S$ so that $\nu_n \Rightarrow \tau$ with τ different than $\frac{d\theta}{2\pi}$; by definition, τ can be any "attainable" probability measure on $\mathcal{S}^1$, e.g. the Cilleruelo measure [9]

$$\tau = \frac{1}{4}\left(\delta_{\pm 1} + \delta_{\pm i}\right),$$

or "intermediate" measures (e.g. measures supported on Cantor set, cf. [16]); for a partial classification see [17, 29].

Definition 2.1. For a sequence $\{n\} \subseteq S$ we say that $\mathcal{E}_n$ are asymptotically equidistributed if (2.1) holds.

2.1.2. *Spectral correlations and quasi-correlations.* One of the key ingredients in [16] was controlling the size of length-6 "spectral correlations set". Given $l \geq 3$, the length-l spectral correlation set of the torus is the set

$$\mathscr{P}_n(l) := \left\{ (\lambda^1, \ldots, \lambda^l) \in \mathcal{E}_n^l : \sum_{j=1}^{l} \lambda^j = 0 \right\} \tag{2.2}$$

of l-tuples of lattice points in $\mathcal{E}_n$ summing up to 0. If n is divisible by 4, then it forces all entries of $\lambda \in \mathcal{E}_n$ to be even, and hence $\lambda/2 \in \mathcal{E}_{n/4}$, and in this case one can keep replacing n by $n/4$ until n is no longer divisible by 4. Assuming n is not divisible by 4 in the first place, $\lambda \in \mathcal{E}_n$ forces that the number of odd entries among λ_1, λ_2 is either 1 or 2 for n odd or even respectively (independent of $\lambda \in \mathcal{E}_n$), and in either case, for l odd, the correlation sets

$$\mathscr{P}_n(l) = \varnothing \tag{2.3}$$

are all empty [8] by a congruence obstruction modulo 2, an argument similar to the one yielding (1.7). Otherwise, for l even, the number of length-l correlations

$$\frac{1}{N_n^l} \cdot \#\mathscr{P}_n(l) = \int_{\mathbb{T}^2} r_n(x)^l dx$$

is equal to the (normalized) moments of the covariance function (1.5) of the Arithmetic Random Waves.

Since for $l = 2k$, all the "diagonal" tuples $(\lambda^1, -\lambda^1, \ldots, \lambda^k, -\lambda^k)$ and their permutations are in $\mathscr{P}_n(l)$, it implies the inequality

$$\#\mathscr{P}_n(l) \gg N_n^k.$$

Conversely, Bombieri–Bourgain [6] proved, among other things, that, given $l = 2k$ even, the inequality

$$\#\mathscr{P}_n(l) \ll_l N_n^k \tag{2.4}$$

holds for a generic sequence $\{n\} \subseteq S$; by invoking the usual diagonal argument, (2.4) holds for *all* l even, along a generic sequence $\{n\} \subseteq S$.

Definition 2.2 (Correlation-tame sequences of energies)**.** We say sequence $S' \subseteq S$ is correlation-tame, if for every $l = 2k \geq 6$ even, the inequality (2.4) holds true.

In fact, Bombieri–Bourgain [6] proved a stronger property satisfied by the correlations of $\mathcal{E}_n$, with n generic, i.e. that a generic sequence in S satisfies the following axiom $\mathcal{F}(\gamma)$ for some $0 < \gamma < 1/2$.

Definition 2.3 (Axiom $\mathcal{F}(\gamma)$)**.**

(1) For $l \geq 4$, $n \in S$, we say that $(\lambda^1, \ldots, \lambda^l) \in \mathcal{E}_n^l$ is a minimal correlation, if $\sum\limits_{j=1}^{l} \lambda^j = 0$ and no proper subsum of $\sum\limits_{j=1}^{l} \lambda^j$ vanishes.

(2) For $0 < \gamma < 1/2$ we say that a sequence $\{n\} \subseteq S$ satisfies the axiom $\mathcal{F}(\gamma)$, if for every $l \geq 4$, the number of length-l minimal correlations of $\mathcal{E}_n$ is at most $N_n^{\gamma \cdot l}$ for n sufficiently big.

As we will deal with moments of $r_n(\cdot)$ restricted to shrinking balls, we will find that, for our purposes, the relevant notion is that of *quasi-correlations* [5] (see (2.11) below). Given $n \in S$, $\epsilon > 0$ and $l \geq 2$, the length-l quasi-correlation set is[4]

$$\mathcal{C}_n(l, \epsilon) := \left\{ (\lambda^1, \ldots, \lambda^l) \in \mathcal{E}_n^l : \ 0 < \left\| \sum_{j=1}^{l} \lambda^j \right\| < n^{1/2 - \epsilon} \right\};$$

note that, by the definition, $\mathscr{P}_n(l)$ and $\mathcal{C}_n(l, \epsilon)$ are disjoint. It was shown [5, Theorem 1.4] that, given $l \geq 2$ and $\epsilon > 0$, the length-l quasi-correlation set is empty $\mathcal{C}_n(l, \epsilon) = \varnothing$ along a generic sequence $\{n\} \subseteq S$, and, as it is the case of the correlation set, by a diagonal argument, we may choose a density-1 subsequence $\{n\} \subseteq S$, so that along that sequence, for *every* $l \geq 2$,

$$\mathcal{C}_n(l, \epsilon) = \varnothing$$

holds true for n sufficiently big (depending on l).

Definition 2.4 (Axiom $\mathcal{A}(\epsilon)$ on sequences of energies)**.** Given $\epsilon > 0$ we say that a sequence $S' \subseteq S$ satisfies the axiom[5] $\mathcal{A}(\epsilon)$, if for every $l \geq 2$, the equality $\mathcal{C}_n(l, \epsilon) = \varnothing$ holds for n sufficiently big.

2.2. **Outline of the proofs for spatial fluctuations (Theorem 1.1).** By a simple manipulation with the defect definition (1.8) and interchanging the order of integration it is straightforward to derive the expression

$$\mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s}) = \frac{1}{(\pi s^2)^2} \int\limits_{\mathbb{T}^2 \times \mathbb{T}^2} H(f_n(y)) H(f_n(z)) \cdot s^2 W(\|y - z\|/s) dy dz \tag{2.5}$$

for the spatial defect variance, where W is a certain weight function ("circle-circle intersection function") supported on $[0, 2]$, and is C^1 on $(0, 2)$. It is conceivable that the asymptotic vanishing of $\mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s})$ follows by a direct analysis of the r.h.s. of (2.5). However it seems very difficult, as the appearance of $H(\cdot)$ on the r.h.s. of (2.5) does not allow us to capitalise on the special additive structure (1.1) of f_n, especially, in light of the discontinuity of $H(\cdot)$ at the origin (so, for example, Taylor expanding $H(\cdot)$ around the origin is problematic).

---

[4] Mind the slight abuse of notation as compared to [5]

[5] Mind again an abuse of notation compared to [5]

We abandon such a direct approach, and instead notice that, since the random variable $Y_{f_n,s}$ is bounded (by 1), the variance $\mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s})$ asymptotically vanishing is equivalent to $Y_{f_n,s}$ asymptotically vanishing with high probability (i.e. for "most" of the ball centres on the torus), and recall that, under certain flatness conditions on f_n (certainly satisfied by all $f_n \in \mathscr{B}_n$) and arithmetic conditions on n (in the spirit of the ones given in section 2.1.2 above), $f_n(\cdot)$ exhibits [7, 8] Gaussian spatial value distribution when averaged over the whole torus. Using these "de-randomisation" techniques we will be able to prove the result to follow immediately; unlike the results of [7, 8] (and [30]), this is a second-order result, i.e. concerning variance (as opposed to a first order one concerning expectation). Moreover, since, unlike [7, 8], the Gaussian input for Theorem 2.5 is not inherently contained within its statement, it seems that a more direct approach might be possible for proving Theorem 2.5. Recall axiom $\mathcal{F}(\gamma)$ in Definition 2.3, and lattice points equidistribution in Definition 2.1.

Theorem 2.5 (A variant of Theorem 1.1 with control over S')**.** *Let $S' \subseteq S$ be a sequence of energy levels satisfying the axiom $\mathcal{F}(\gamma)$ for some $\gamma \in (0, 1/2)$, and assume further that the corresponding $\mathcal{E}_n$ are asymptotically equidistributed. Then the conclusions of Theorem 1.1 apply along S', i.e.*

$$\lim_{\substack{R\to\infty \\ n\to\infty,\, n\in S'}} \sup_{\substack{s>R/\sqrt{n} \\ f_n\in\mathscr{B}_n}} \mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s}) = 0. \tag{2.6}$$

Theorem 1.1 is a direct consequence of Theorem 2.5, because axiom $\mathcal{F}(\gamma)$ holds with some $\gamma \in (0, 1/2)$ for "generic" $n \in S$, and $\mathcal{E}_n$ is asymptotically distributed for "generic" $n \in S$ in the sense of Definition 2.3. The proof of Theorem 2.5 proceeds in three steps. First, we reduce proving (2.6) uniformly for $s > R/\sqrt{n}$ to proving for $s = R/\sqrt{n}$ only, via an analogue of the Integral-Geometric Sandwich, first introduced in [33, 21], adapted to our settings. Next, we exploit the said spatial Gaussianity of $f_n(\cdot)$ in order to reduce the variance vanishing to the analogous result for the limit random field, which, by the equidistribution assumption for $\mathcal{E}_n$ of Theorem 2.5, is the Gaussian random field of planar isotropic monochromatic waves (it is "Berry's Random Wave Model", uniquely defined by its covariance function $J_0(\|x\|)$).

It then remains to evaluate the variance of the defect for the limit Gaussian random field restricted to a compact domain (e.g. the unit square), which, in spirit, is already contained in [19] (and predicted by [4]), where a rapid decay rate is asserted. This result is the only use of the equidistribution assumption, and it should be not too technically demanding to remove this assumption, as long as some non-degeneracy for the limit Gaussian field is imposed (i.e. our techniques do not allow for including the case where the distribution of angles approximates the most degenerate "Cilleruelo" case), though it benefits us in no way if we are only interested in a density-1 sequences of energy levels. Our main result (2.6) is ineffective in terms of rate of decay for $\mathrm{Var}_{\mathbb{T}^2}(Y_{f_n,s})$, as the convergence of the spatial distribution of f_n to the Gaussian is ineffective.

2.3. **Outline of constructing functions with non-vanishing defect variance (Theorem 1.2).** The prevailing symmetry obstruction, dictating that for the standard torus, the total defect of any Laplace eigenfunction vanishes precisely does not persist for the non-standard tori. We exploit the hexagonal torus, so that to construct a single Laplace eigenfunction with total defect non-vanishing, and scale it to obtain a sequence of eigenfunctions of arbitrarily high energy, with defect growing on large fragments of the torus, above the Planck scale. We then mimic that situation on the standard torus, by appealing to the Pell equation $x^2 - 3y^2 = 1$, yielding solutions approximating the hexagonal toral eigenfunctions on the standard torus.

2.4. **Outline of the proofs for Arithmetic Random Waves (theorems 1.3-1.5).** Here we assume that $\{f_n\}_{n\in S}$ are the (Gaussian) Arithmetic Random Waves. Since it is possible to derive the identity

$$\mathbb{E}[H(f_n(x)) \cdot H(f_n(y))] = \frac{2}{\pi} \arcsin(r_n(x-y)), \tag{2.7}$$

(cf. the proof of Lemma 3.5) a straightforward manipulation with the definition (1.10) of $\mathcal{D}_{n;s}$ and inverting the order of integration, upon bearing in mind the stationarity of f_n, yields the following *precise* expression for the defect variance:

$$\mathrm{Var}(\mathcal{D}_{n;s}) = \frac{2}{\pi^3 s^4} \int\limits_{B(s)\times B(s)} \arcsin(r_n(x-y))dxdy. \tag{2.8}$$

Now we Taylor expand the arcsine around the origin (note that the series converges absolutely at the endpoints $t = \pm 1$)

$$\arcsin(t) = \sum_{k=0}^{\infty} a_k t^{2k+1}, \tag{2.9}$$

where all the (explicit) $a_k > 0$ are *positive*, and substitute into (2.8) to relate between the defect variance and the moments of the covariance function restricted to $B(s)$:

$$\mathrm{Var}(\mathcal{D}_{n;s}) = \frac{2}{\pi^3 s^4} \sum_{k=1}^{\infty} a_k \cdot \int\limits_{B(s)\times B(s)} r_n(x-y)^{2k+1}dxdy. \tag{2.10}$$

We may in turn exploit the additive structure (1.5) to relate the said *odd* moments of $r_n(\cdot)$ to the spectral correlations (and, implicitly, the quasi-correlations) defined in section 2.1.2:

$$\int\limits_{B(s)\times B(s)} r_n(x-y)^{2k+1}dxdy = \frac{s^2}{N_n^{2k+1}} \sum_{(\lambda^1,\ldots,\lambda^{2k+1})\notin \mathscr{P}_n(2k+1)} \frac{J_1(2\pi s\cdot \|\lambda^1+\ldots+\lambda^{2k+1}\|)^2}{\|\lambda^1+\ldots+\lambda^{2k+1}\|^2}, \tag{2.11}$$

with $J_1(\cdot)$ the Bessel J function of the first order, so that to relate the defect variance to the spectral correlations and quasi-correlations (where, to obtain (2.11), we separate the diagonal and use the observation (2.3)). One may then substitute (2.11) into (2.10) to obtain a more explicit expression for $\mathrm{Var}(\mathcal{D}_{n;s})$, an absolutely convergent infinite series over all $(2k+1)$-tuples of lattice points. If we assume further, that $s = n^{-1/2+\epsilon}$ (say), and a sequence $\{n\} \subseteq S$ satisfies the $\mathcal{A}(\delta)$ axiom with some $\delta < \epsilon$, then all the summands on the r.h.s. of (2.11) are formally decaying like a (small) power of n, faster than any power of N_n (see (1.4)).

There is a subtlety with this outlined approach though, as controlling the decay rate in this infinite series uniformly seems difficult. Instead, we will only control finitely many summands and bound the contribution of the higher moments. With this approach, we will encounter the odd moments of the absolute value $|r_n(\cdot)|$ of the covariance rather than the moments of the covariance, that we will reduce to a moment of higher order via Cauchy–Schwarz. Theorem 1.3 is the result of such an application when capping the series at the first degree Taylor approximation of the arcsine (2.9), whereas Theorem 1.4 caps it at an arbitrarily high degree Taylor approximation, depending on the required $A > 0$ in (1.12), while also appealing to the correlation-tame property of a generic sequence of energies. We will be able to prove the following result, which, since the claimed sequence S''' is generic, thanks to the results mentioned in section 2.1.2, clearly implies Theorem 1.4.

Theorem 2.6 (Theorem 1.4 with control over $S'''(\epsilon)$)**.** *Let $\epsilon > 0$ be given, and assume that $S''' \subseteq S$ is a sequence of energy levels satisfying the axiom $\mathcal{A}(\delta)$ with some $\delta < \epsilon$, and is correlation-tame. Then the conclusions of Theorem 1.4 hold, i.e. along $n \in S'''$,*

$$\sup_{s>n^{-1/2+\epsilon}} \mathrm{Var}(\mathcal{D}_{n;s}) \ll \frac{1}{N_n^A},$$

for every $A > 0$.

For the lower bounds in Theorem 1.5 one also starts from (2.10) and (2.11). Indeed, since the Taylor coefficients a_k in (2.10) are all positive, and, in hindsight, so are all the moments (2.11) of

$r_n(\cdot)$, it is sufficient to bound any of these from below. If $T := s \cdot \sqrt{n}$ happens to be bounded away from zeros of the Bessel J_1 function, this readily yields the bound (1.14) of Theorem 1.5. Most of our argument takes upon the opposite situation when T approaches one of the Bessel J_1 zeros, whence we need to rule out the, a priori unlikely, possibility of all the terms

$$2\pi s \cdot \left\| \sum_{j=1}^{2k+1} \lambda^j \right\|$$

conspiring around the Bessel zeros. To resolve this situation we exploit the higher order Taylor approximates, whence appealing to the deep W. Schmidt's *simultaneous Diophantine approximation* theorem [32], for example, approximating $\sqrt{5}$ by rational numbers for $k = 1$ or $\sqrt{13}$ and $\sqrt{17}$ for $k = 2$; to attain $\frac{1}{T^{3+\delta}}$ as in (1.13), with $\delta > 0$ arbitrarily small, we will need to focus on arbitrarily high k; here Schmidt's result is crucial.

2.5. **Outline of the paper.** In section 3 Bourgain's de-randomization method will be invoked to prove Theorem 1.1 dealing with the spatial defect variance vanishing for the flat functions. Then a sequence of "esoteric" non-flat functions with spatial defect variance non-vanishing will be constructed in section 4, by first constructing eigenfunctions with the analogous properties defined on the *hexagonal* torus (as opposed to the standard torus). Finally, Section 5 is dedicated to giving the proofs for all the results concerning the defect of the Arithmetic Random Waves (theorems 1.3-1.5), appealing among the rest to Diophantine approximations.

3. Spatial defect distribution: proof of Theorem 1.1

Recall that Theorem 1.1 follows at once from its more explicit variant, Theorem 2.5, whose proof is the ultimate goal of this section.

3.1. **Proof of Theorem 2.5.** The following proposition is seemingly weaker, or less general, compared to Theorem 2.5, as it only allows for radii $s = \frac{R}{\sqrt{n}}$ with $R \to \infty$ growing *slowly*, instead of a uniform statement for *all* $s > R/\sqrt{n}$ as in (2.6). However, we will be able to infer the more general result, using the elegant Integral-Geometric Sandwich in Proposition 3.2 below, inspired to high extent by its counterpart introduced by Nazarov–Sodin [33, Lemma 1] for the sake of counting the number of nodal components (see also [28, Lemma 3.7] and [21, Lemma 1]). It seems a priori *counter-intuitive* that it is "easier" to first establish the spatial defect variance vanishing for smaller radii than bigger ones. Our explanation of the said *surprise* is that the asymptotic Gaussianity w.r.t. the spatial variable holds at the Planck scale only (or logarithmically above it [31]), rather than at *all* scales above it.

Proposition 3.1 (Planck scale spatial defect distribution)**.** *Let $S' \subseteq S$ be any sequence of energy levels satisfying the assumptions of Theorem 2.5. Then for every $\epsilon > 0$ there exists $R_0 = R_0(\epsilon) > 0$ sufficiently large so that for all $R > R_0$ there exists a number $n_0 = n_0(R, \epsilon)$ sufficiently large so that for all $n > n_0$, the inequality*

$$\mathrm{Var}_{\mathbb{T}^2}(Y_{f_n, R/\sqrt{n}}) < \epsilon$$

holds uniformly for $f_n \in \mathscr{B}_n$. Equivalently,

$$\lim_{R\to\infty} \limsup_{\substack{n\to\infty \\ n\in S'}} \sup_{f_n\in\mathscr{B}_n} \mathrm{Var}_{\mathbb{T}^2}(Y_{f_n, R/\sqrt{n}}) = 0. \tag{3.1}$$

The following proposition asserts the aforementioned Integral-Geometric Sandwich; unlike the original inequality, it contains an error term. Recall that the local (normalized) defect of a function an eigenfunction f_n as in (1.1) restricted to a radius-s ball around $x \in \mathbb{T}^2$ is given by (1.8).

Proposition 3.2 (Integral-Geometric Sandwich). *For every f_n of the form* (1.1)*, and $0 < r_1 < r_2$, the asymptotic estimate*

$$Y_{f_n,r_2}(x) = \frac{1}{\pi r_2^2} \int\limits_{B_x(r_2)} Y_{f_n,r_1}(y)dy + O\left(\frac{r_1}{r_2}\right) \tag{3.2}$$

holds, with constant associated to the 'O'-notation absolute.

Proof of Theorem 2.5 assuming propositions 3.1-3.2. Let $\epsilon > 0$ be given. First, we apply Proposition 3.1 to obtain a number $R_0 = R_0(\epsilon)$ so that for all $R > R_0$ there exists a number $n_0 = n_0(R, \epsilon)$ so that for $n > n_0$ with $n \in S'$, one has

$$\mathrm{Var}_{\mathbb{T}^2}\left(Y_{f_n,R/\sqrt{n}}\right) < \frac{\epsilon^2}{4}, \tag{3.3}$$

uniformly for all $f_n \in \mathscr{B}_n$. We define

$$R = R(\epsilon) := (R_0 + 1)^2, \tag{3.4}$$

and claim that with this choice of R, the conclusion of Theorem 2.5 holds, where the corresponding $n_0 = n_0(R_0 + 1, \epsilon)$, depending on ϵ only, is the one we received as the output from the application above of Proposition 3.1. For this particular choice of the parameters, the inequality (3.3) reads

$$\mathrm{Var}_{\mathbb{T}^2}\left(Y_{f_n,(R_0+1)/\sqrt{n}}\right) < \frac{\epsilon^2}{4}, \tag{3.5}$$

valid for all $n \in S'$, $n > n_0$ and $f_n \in \mathscr{B}_n$. To validate our claim we are to prove that for all $n > n_0$ with $n \in S'$, the inequality

$$\mathrm{Var}_{\mathbb{T}^2}\left(Y_{f_n,s}\right) < \epsilon \tag{3.6}$$

holds for all $s > \frac{R}{\sqrt{n}}$.

Now, we invoke the Integral-Geometric Sandwich of Proposition 3.2, with $r_2 = s > R/\sqrt{n}$ and

$$r_1 = \frac{R_0 + 1}{\sqrt{n}} < \frac{r_2}{R_0 + 1}, \tag{3.7}$$

by (3.4). Hence (3.2) reads

$$\begin{aligned} Y_{f_n,s}(x) = Y_{f_n,r_2}(x) &= \frac{1}{\pi s^2} \int\limits_{B_x(s)} Y_{f_n,(R_0+1)/\sqrt{n}}(y)dy + O\left(\frac{r_1}{s}\right) \\ &= \frac{1}{\pi s^2} \int\limits_{B_x(s)} Y_{f_n,(R_0+1)/\sqrt{n}}(y)dy + O\left(\frac{1}{R_0}\right), \end{aligned} \tag{3.8}$$

thanks to (3.7). We assume that R_0 is sufficiently large so that the error term on the r.h.s. of (3.8) is $O\left(\frac{1}{R_0}\right) < \frac{\epsilon}{2}$, take the absolute value of both sides of (3.8), and apply the triangle inequality to conclude that

$$|Y_{f_n,s}(x)| \le \frac{1}{\pi s^2} \int\limits_{B_x(s)} \left|Y_{f_n,(R_0+1)/\sqrt{n}}(y)\right| dy + \frac{\epsilon}{2}. \tag{3.9}$$

We then integrate both sides of (3.9) w.r.t. $x \in \mathbb{T}^2$ to yield

$$\int\limits_{\mathbb{T}^2} |Y_{f_n,s}(x)|dx \le \int\limits_{\mathbb{T}^2} \left|Y_{f_n,(R_0+1)/\sqrt{n}}(y)\right| dy + \frac{\epsilon}{2},$$

and invoke (3.5) together with Cauchy–Schwarz inequality, which gives (recalling that the spatial expectation vanishes identically, see (1.9))

$$\int_{\mathbb{T}^2} |Y_{f_n,s}(x)|dx \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{3.10}$$

Finally, the inequality (3.10) certainly implies (3.6), since $|Y_{f_n,s}(x)| \le 1$ (again, upon recalling (1.9)), which, as it was mentioned above, is sufficient to infer the statement of Theorem 2.5.

□

3.2. **Integral-Geometric Sandwich: Proof of Proposition 3.2.**

Proof. We start with the integral on the r.h.s. of (3.2), and use the definition (1.8) to write

$$\frac{1}{\pi r_2^2} \int_{B_x(r_2)} Y_{f_n,r_1}(y)dy = \frac{1}{\pi r_2^2} \int_{B_x(r_2)} \frac{1}{\pi r_1^2} \int_{B_y(r_1)} H(f_n(z))dzdy, \tag{3.11}$$

and aim at reversing the order of the integrals on the r.h.s. of (3.11). We have

$$\frac{1}{\pi r_2^2} \int_{B_x(r_2)} Y_{f_n,r_1}(y)dy = \frac{1}{\pi r_2^2} \int_{B_x(r_2+r_1)} H(f_n(z)) \cdot \frac{1}{\pi r_1^2} \operatorname{Vol}\left(B_z(r_1) \cap B_x(r_2)\right) dz. \tag{3.12}$$

Now, upon denoting

$$V_{x,z}(r_2, r_1) := \frac{1}{\pi r_1^2} \cdot \operatorname{Vol}\left(B_z(r_1) \cap B_x(r_2)\right),$$

the equality (3.12) reads

$$\frac{1}{\pi r_2^2} \int_{B_x(r_2)} Y_{f_n,r_1}(y)dy = \frac{1}{\pi r_2^2} \int_{B_x(r_2+r_1)} H(f_n(z)) \cdot V_{x,z}(r_2, r_1)dz, \tag{3.13}$$

and we notice that

$$0 \le V_{\cdot,z}(\cdot, r_1) \le \frac{1}{\pi r_1^2} \operatorname{Vol}(B_z(r_1)) = 1, \tag{3.14}$$

and, in addition, if $z \in B_x(r_2 - r_1)$, then $V(z) = 1$. We then separate the range of integration in (3.13) into $B_x(r_2 - r_1)$ and its complement to write

(3.15)

$$\begin{aligned} \frac{1}{\pi r_2^2} \int_{B_x(r_2)} Y_{f_n,r_1}(y)dy &= \frac{1}{\pi r_2^2} \int_{B_x(r_2-r_1)} H(f_n(z))dz + O\left(\frac{1}{\pi r_2^2} \operatorname{Vol}(B_x(r_2 + r_1) \setminus B_x(r_2 - r_1))\right) \\ &= \frac{1}{\pi r_2^2} \int_{B_x(r_2)} H(f_n(z))dz + O\left(\frac{1}{\pi r_2^2} \operatorname{Vol}(B_x(r_2 + r_1) \setminus B_x(r_2 - r_1))\right) \\ &= Y_{f_n,r_2}(x) + O\left(\frac{1}{\pi r_2^2} \operatorname{Vol}(B_x(r_2 + r_1) \setminus B_x(r_2 - r_1))\right) \end{aligned}$$

thanks to (3.14), $|H(\cdot)| \le 1$, and the definition (1.8) of $Y_{f_n,r_2}(x)$. Now the statement (3.2) of Proposition 3.2 finally follows from substituting the estimate

$$\frac{1}{\pi r_2^2} \operatorname{Vol}(B_x(r_2 + r_1) \setminus B_x(r_2 - r_1)) = O\left(\frac{r_2 r_1}{r_2^2}\right) = O\left(\frac{r_1}{r_2}\right)$$

into (3.15).

□

3.3. **Auxiliary results towards the proof of Proposition 3.1.** We denote Berry's random monochromatic isotropic waves $g : \mathbb{R}^2 \to \mathbb{R}$ defined on a probability space $(\Omega, \Sigma, \mathcal{P}r)$, i.e. for $\omega \in \Omega$ the corresponding sample function $g(\cdot) = g_\omega(\cdot)$ are distributed as a centred Gaussian random field uniquely determined via Kolmogorov's Theorem by its covariance function

$$r_g(|x-y|) := \mathbb{E}[g(x) \cdot g(y)] = J_0(\|x-y\|), \tag{3.16}$$

where J_0 is the Bessel J function of order 0. Proposition 3.3 immediately below asserts that locally, the functions $f_n \in \mathscr{B}_n$, appropriately scaled, converge to $g(\cdot)$ around a random spatial variable on the torus, understood as random fields. It is the heart of Bourgain's de-randomization method, originally in [7], and is a restatement of what turned out to be the key technical propositions in [8], in the precise form used in that manuscript. To state this result, given a function $f_n \in \mathscr{B}_n$, we introduce the function $F_{x;R}(y) : [-1,1]^2 \to \mathbb{R}$ to be

$$F_{x;R}(y) = f_n\left(x + \frac{R}{\sqrt{n}}y\right), \tag{3.17}$$

and think of $F_{x;R}(\cdot)$ as a random field, as $x \in \mathbb{T}^2$ varies randomly uniformly on the torus. In what follows we will obtain a sequence of random fields $g^n : \mathbb{R}^2 \to \mathbb{R}$, that will converge in suitable sense to g, and we will denote their scaled version

$$g^n_{\omega;R}(\cdot) := g^n_\omega(\cdot R),$$

that will be compared to the scaled version of g

$$g_{\omega;R}(\cdot) := g_\omega(\cdot R). \tag{3.18}$$

Proposition 3.3 ([8, Propositions 3.2-3.3])**.** *Let $S' \subseteq S$ be a sequence of energy levels satisfying the assumptions of Theorem 2.5. Then there exists a sequence of Gaussian stationary random fields $\{g^n\}_{n\in S'}$, converging in law to g as $n \to \infty$, with the following property. For every $R > 0$, $\epsilon > 0$ and $\eta > 0$, there exists $n_0 = n_0(R;\eta,\epsilon)$ sufficiently large so that for all $n \in S'$ with $n > n_0$ and $f_n \in \mathscr{B}_n$, there exists an event $\Omega' = \Omega'(n; f_n, R; \eta, \epsilon) \subseteq \Omega$ of high probability $\mathcal{P}r(\Omega') > 1-\epsilon$ and a measure preserving map $\tau : \Omega' \to \mathbb{T}^2$ so that $\operatorname{meas}(\tau(\Omega')) > 1-\epsilon$, and for all $\omega \in \Omega'$, one has*

$$\|g^n_{\omega;R} - F_{\tau(\omega);R}\|_{C^1([-1,1]^2)} < \eta. \tag{3.19}$$

Since, as mentioned above, Proposition 3.3 was proved in [8], there is no need to reprove it in this manuscript. Once the reduction to the Gaussian random field was performed within Proposition 3.3, replacing $g^n(\cdot)$ with Berry's $g(\cdot)$ in (3.19) is completely standard. That is, it is possible to couple $g^n(\cdot)$ with $g(\cdot)$ so that $\|g^n_{\omega;R} - g_{\omega;R}\|_{C^1([-1,1]^2)}$ is arbitrarily small for n sufficiently large, see e.g. [33, Lemma 4]. Together with (3.19) and the triangle inequality it yields the following corollary.

Corollary 3.4. *Let $S' \subseteq S$ be a sequence of energy levels satisfying the assumptions of Theorem 2.5. Then for every $R > 0$, $\epsilon > 0$ and $\eta > 0$, there exists $n_0 = n_0(R;\eta,\epsilon)$ sufficiently large so that for all $n \in S'$ with $n > n_0$ and $f_n \in \mathscr{B}_n$, there exists an event $\Omega' = \Omega'(n; f_n, R; \eta, \epsilon) \subseteq \Omega$ of high probability $\mathcal{P}r(\Omega') > 1-\epsilon$ and a measure preserving map $\tau : \Omega' \to \mathbb{T}^2$ so that $\operatorname{meas}(\tau(\Omega')) > 1-\epsilon$, and for all $\omega \in \Omega'$, one has*

$$\|g_{\omega;R} - F_{\tau(\omega);R}\|_{C^1([-1,1]^2)} < \eta. \tag{3.20}$$

Alternatively to working with $g(\cdot)$, one could, in principle, work directly with $g^n(\cdot)$, by proving an analogue of Lemma 3.5 below, applicable for $g^n(\cdot)$ with n large, a direction we abandon. Corollary 3.4 naturally gives rise to the comparison to the defect variance of the random waves $g(\cdot)$. Note that, for our purposes of comparing the defect of the toral eigenfunctions to that of the random g_R, the C^1-estimate in (3.20) is too strong, and we could easily settle for an L^∞-estimate. Recall that $H(\cdot)$ is the sign function (1.6), and let

$$X_R = X_{\omega,R} := \frac{1}{\pi R^2} \int\limits_{B(R)} H(g(x))dx \tag{3.21}$$

be the (random) defect of $g(\cdot)$ restricted to the ball $B(R) \subseteq \mathbb{R}^2$. It is obvious that the expectation $\mathbb{E}[X_R] = 0$ vanishes, whereas the following easy, most likely sub-optimal, result asserts that so does its variance, asymptotically as $R \to \infty$.

Lemma 3.5. *As $R \to \infty$, the defect variance of $g(\cdot)$ restricted to $B(R)$ is vanishing:*

$$\mathrm{Var}(X_R) = O\left(\frac{1}{R^{1/2}}\right). \tag{3.22}$$

Proof. It is a well-known fact (see, e.g. [24, 25]) that every bivariate centred Gaussian random vector (X, Y) with covariance matrix $\Sigma = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$ with some $|r| \le 1$ satisfies

$$\mathbb{E}\left[H(X) \cdot H(Y)\right] = \frac{2}{\pi} \arcsin(r).$$

Hence, by setting $X = g(x), Y = g(y)$ of covariance $r = J_0(\|x - y\|)$, it follows that

$$\mathbb{E}[H(g(x)) \cdot H(g(y))] = \frac{2}{\pi} \arcsin(J_0(\|x - y\|), \tag{3.23}$$

analogous to the identity (2.7). We now use the definition (3.21) of the defect, and invert the integration order to write

$$\mathrm{Var}(X_R) = \frac{2}{\pi^3 R^4} \int\limits_{B(R)\times B(R)} \arcsin(J_0(\|x - y\|))dxdy, \tag{3.24}$$

with the use of (3.23). Now, for each $x \in B(R)$ fixed we separate the range of integration in (3.24) into $\|x - y\| < 1$ and $\|x - y\| > 1$ (say), so that

(3.25)

$$\begin{aligned}\mathrm{Var}(X_R) &= \frac{2}{\pi^3 R^4} \cdot \left(\int\limits_{\substack{x,y\in B(R)\\ \|x-y\|<1}} \arcsin(J_0(\|x - y\|))dxdy + \int\limits_{\substack{x,y\in B(R)\\ \|x-y\|>1}} \arcsin(J_0(\|x - y\|))dxdy \right) \\ &=: \frac{2}{\pi^3 R^4} \cdot (I_1 + I_2).\end{aligned}$$

We bound the contribution of the former range trivially as

$$|I_1| = \left| \int\limits_{\substack{x,y\in B(R)\\ \|x-y\|<1}} \arcsin(J_0(\|x - y\|))dxdy \right| = O(R^2), \tag{3.26}$$

whereas we use the standard asymptotics [1, formula (9.2.1)] for the Bessel J_0 function for $\|x-y\| > 1$:

$$|\arcsin(J_0(t))| \ll |J_0(t)| \ll \frac{1}{\sqrt{t}}$$

to bound the contribution of the latter range as

$$
\begin{aligned}
I_2 = \int\limits_{\substack{x,y\in B(R)\\ \|x-y\|>1}} \arcsin(J_0(\|x-y\|))dxdy \ll \int\limits_{B(R)} dx \int\limits_{y\in B(R):\, \|x-y\|>1} \frac{dy}{\|x-y\|^{1/2}} \\
\le R^2 \int\limits_1^R \frac{tdt}{\sqrt{t}} \ll R^{7/2}.
\end{aligned}
\tag{3.27}
$$

The statement of Lemma 3.5 finally follows upon substituting (3.26) and (3.27) into (3.25).

□

We will require the following notion, inspired by [33, 21], that will allow us to control the defect stability under small L^∞-perturbations.

Definition 3.6 (Stable event)**.** For $R > 0$, $\eta > 0$ and $\delta > 0$ we let the "$(R;\eta,\delta)$-unstable" event $\Omega_1(R;\eta,\delta) \subseteq \Omega$ be defined as

$$\Omega_1(R;\eta,\delta) := \left\{\omega \in \Omega : \frac{1}{\pi R^2} \cdot \text{meas}\{x \in B(R) : |g_\omega(x)| < \eta\} > \delta\right\} \tag{3.28}$$

the event that the proportion of $x \in B(R)$ so that $|g(x)|$ is small, is not negligible.

Lemma 3.7 (Stability estimate)**.** *For every $\delta, \epsilon > 0$, there exists an $\eta > 0$ sufficiently small, so that for every $R > 0$,*

$$\mathcal{P}r(\Omega_1(R;\eta,\delta)) < \epsilon.$$

Proof. Let $\mathcal{A}_{R;\eta} \subseteq B(R)$ be the (random) measure

$$\mathcal{A}_{R;\eta} := \text{meas}\{x \in B(R) : |g(x)| < \eta\}$$

of the set $g^{-1}([-\eta,\eta]) \cap B(R) \subseteq \mathbb{R}^2$. Clearly,

$$\mathcal{A}_{R;\eta} = \int\limits_{B(R)} \chi_{[-\eta,\eta]}(g(x))dx, \tag{3.29}$$

where $\chi_{[-\eta,\eta]}$ is the characteristic function of the interval $[-\eta,\eta] \subseteq \mathbb{R}$. Since, for every $x \in \mathbb{R}^2$, $g(x)$ is a standard Gaussian random variable, taking the expectation of both sides of (3.29) easily yields

$$\mathbb{E}[\mathcal{A}_{R;\eta}] = O(\eta R^2), \tag{3.30}$$

with the constant involved in the 'O'-notation absolute. Now, we have

$$\Omega_1(R;\eta,\delta) = \left\{\omega \in \Omega : \frac{1}{\pi R^2} \cdot \mathcal{A}_{R;\eta} > \delta\right\},$$

and, in light of (3.30), the conclusion of Lemma 3.7 follows from Markov's inequality.

□

After all the preparatory results of section 3.3, we are finally in a position to prove the principal de-randomization result.

3.4. **Spatial defect distribution: Proof of Proposition 3.1 via Bourgain's de-randomization.** We start with the following elementary lemma in probability theory, which is a criterion for the variance vanishing of *bounded* random variables, whose proof is thereupon conveniently omitted.

Lemma 3.8. *Let $\{X_k\}_{k\ge1}$ be a sequence of random variables $X_k : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \Sigma, \mathcal{P}r)$ satisfying $|X| \le 1$ a.s. and $\mathbb{E}[X_k] = 0$ for every $k \ge 1$. Then we have $\text{Var}(X_k) \to 0$ as $k \to \infty$, if and only if for every $\delta > 0$, the probability $\mathcal{P}r(|X_k| > \delta) \to 0$ vanishes as $k \to \infty$.*

Proof of Proposition 3.1. We are going to use Lemma 3.8 as a criterion for the variance vanishing, upon both exploiting the defect variance for Berry's random waves (Lemma 3.5), and also when proving the same for the toral eigenfunctions; note that the prescribed rate (3.22) is "lost" during this process for the latter. Let $\epsilon, \delta > 0$ be given. First, we invoke Lemma 3.7 with $\delta/4$ in place of δ, and $\epsilon/2$ in place of ϵ, to obtain a number $\eta = \eta(\epsilon/2, \delta/4)$ sufficiently small so that for all $R > 0$,

$$\mathcal{P}r(\Omega_1(R, \eta, \delta/4)) < \epsilon/2. \tag{3.31}$$

Next, we apply on Lemma 3.5 (along with the "only if" statement of Lemma 3.8), to obtain a number $R_0 = R_0(\delta/2, \epsilon/4)$ sufficiently large, so that for all $R > R_0$, we have

$$\mathcal{P}r\left\{|X_R| > \frac{\delta}{2}\right\} < \frac{\epsilon}{4}.$$

Let $\Omega_2 \subseteq \Omega$ be the corresponding event, i.e.

$$\Omega_2 = \Omega_2(R; \delta/2) := \left\{|X_R| > \frac{\delta}{2}\right\}, \tag{3.32}$$

of probability

$$\mathcal{P}r(\Omega_2) < \frac{\epsilon}{4}. \tag{3.33}$$

Finally, we apply Corollary 3.4 to obtain a number $n_0 = n_0(R; \eta, \epsilon/4)$, so that for all $n > n_0$ and $f_n \in \mathscr{B}_n$ there exists an event $\Omega' = \Omega'(n; f_n, R; \eta, \epsilon/4)$ of probability

$$\mathcal{P}r(\Omega') > 1 - \epsilon/4, \tag{3.34}$$

and a measure preserving map $\tau : \Omega' \to \mathbb{T}^2$ so that

$$\|g_{\omega;R} - F_{\tau(\omega);R}\|_{C^1([-1,1]^2)} < \eta, \tag{3.35}$$

where $g_{\omega;R}$ are the (scaled) Berry's random waves (3.18), and $F_{\tau(\omega);R}$ is the scaled version of the given $f_n \in \mathscr{B}_n$, defined in (3.17).

Recall that $X_{\omega;R}$ is the defect (3.21) of Berry's random waves restricted to $B(R)$. In light of (3.35), for $y \in [-1, 1]^2$ we have

$$H(g_{\omega;R}(y)) = H(F_{\tau(\omega);R}(y)),$$

unless $|g_{\omega;R}(y)| < \eta$. Hence, by the definition (3.28) of the unstable event Ω_1, it is clear (the magnitude of change in the sign function is at most 2, and the measure of the set of x for which $|g_\omega(x)| < \eta$ is at most $\delta/4$) that for all $\omega \in \Omega' \setminus \Omega_1$, one has

$$|X_{\omega,R} - Y_{f_n,R/\sqrt{n}}(\tau(\omega))| < 2 \cdot \frac{\delta}{4} = \frac{\delta}{2}. \tag{3.36}$$

Now, by the definition of Ω_2, for every $\omega \notin \Omega_2$ one has

$$|X_{\omega;R}| < \frac{\delta}{2}. \tag{3.37}$$

Hence (3.37) together with (3.32) imply that for all $\omega \in \Omega'' := (\Omega' \setminus \Omega_1) \setminus \Omega_2$, one has

$$|Y_{f_n,R/\sqrt{n}}(\tau(\omega))| \leq |X_{\omega,R}| + \frac{\delta}{2} < \delta.$$

Equivalently,

$$|Y_{f_n,R/\sqrt{n}}(x)| < \delta \tag{3.38}$$

for all $x \in \tau(\Omega'')$ of measure

$$\text{meas}(\tau(\Omega'')) \geq \mathcal{P}r(\Omega') - \mathcal{P}r(\Omega_1) - \mathcal{P}r(\Omega_2) > (1 - \frac{\epsilon}{4}) - \frac{\epsilon}{2} - \frac{\epsilon}{4} = 1 - \epsilon, \tag{3.39}$$

R	$\int_{S(R)} H(g(x))\,dx$	$(1/R^2)\cdot\int_{S(R)} H(g(x))\,dx$
5	-5.10561833230128	-0.204224733292051
15	-43.5759827038652	-0.193671034239401
25	-116.854534058787	-0.186967254494059
35	-247.264843494327	-0.201848851832104

TABLE 1. Integral values. Here $S(R) \subset \mathbb{R}^2$ is the square $[0,R]\times[0,R]$.

thanks to (3.31), (3.33) and (3.34), and the measure preserving property of τ. Finally, (3.38), (3.39), and the "if" direction of Lemma 3.8 allow us to deduce the conclusion of Proposition 3.1.

□

4. EIGENFUNCTIONS WITH NON-VANISHING DEFECT VARIANCE: PROOF OF THEOREM 1.2

4.1. **Large negative defect on hexagonal lattices.** We begin by constructing a completely flat Laplace eigenfunction g on a certain *hexagonal* torus T, such that the *total* defect of g is non-vanishing. In what follows it will be convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$.

Define $L := \mathbb{Z}[1+i/\sqrt{3}, 2i/\sqrt{3}]$, and let $T := \mathbb{C}/L$. Further, let $\hat{L} \subset \mathbb{C} \simeq \mathbb{R}^2$ denote the dual lattice to L, generated by the sixth roots of unity (or just by $\{1, e(1/6)\}$, where $e(z) := e^{2\pi i z}$). The Laplace eigenvalues on T are then given by $4\pi^2|v|^2$ for $v \in \hat{L}$. Let $v_1,\dots,v_6 \in \mathbb{R}^2$ denote the six elements in $\hat{L}$ with length one, and for $x \in \mathbb{R}^2$ define $f(x) = \sum_{i=1}^6 e(v_i \cdot x)$; f is then well defined on T (as well as totally flat), and is a Laplace eigenfunction on T, with eigenvalue $4\pi^2$.

Further, let $w_1, w_2, w_3 \in \mathbb{R}^2$ denote elements corresponding to the three third roots of unity. Using that $e(t)+e(-t) = 2\cos(t)$, and pairing off antipodal points (i.e. $v_i = -v_j$) define the completely flat function

$$g(x) := \sum_{i=1}^{3} \cos(2\pi w_i \cdot x) = f(x)/2. \tag{4.1}$$

Further, $g_m(x) := g(mx)$ is a Laplace eigenfunction on T with eigenvalue $4\pi^2 m^2$ (also completely flat if m is chosen to be a prime that is inert in $\mathbb{Z}[e^{2\pi i/3}]$), and the following proposition asserts that the total defect of g does not vanish.

Proposition 4.1. *We have*

$$c := \int_T H(g(y))\,dy < 0. \tag{4.2}$$

Further, for any $x \in T$, and $s > 0$

$$\frac{1}{\pi s^2}\int_{B_x(s)} H(g_m(y))\,dy = c\cdot\frac{\sqrt{3}}{2} + O(1/(ms))$$

A plot of $H(g(x_1,x_2))$ is shown in Figure 1. Since g is invariant under translation by L, unless the integral over the fundamental domain of L is exactly zero, we will get growth, of order R^2 in either the positive or the negative direction, when integrating over squares, say centred at $(R/2, R/2)$ and with sides length R growing. The numerics in Table 1 indicates that there is negative growth. These numerics can be made rigorous by bounding the gradient from above: this way we can ensure that the function does not change sign in most small disks. The following lemma, whose proof is obvious, introduces a stability notion, related to the one in section 3.3.

Lemma 4.2. *For the function g in* (4.1) *define*

$$M := \max_{x\in T} |\nabla g(x)|, \tag{4.3}$$

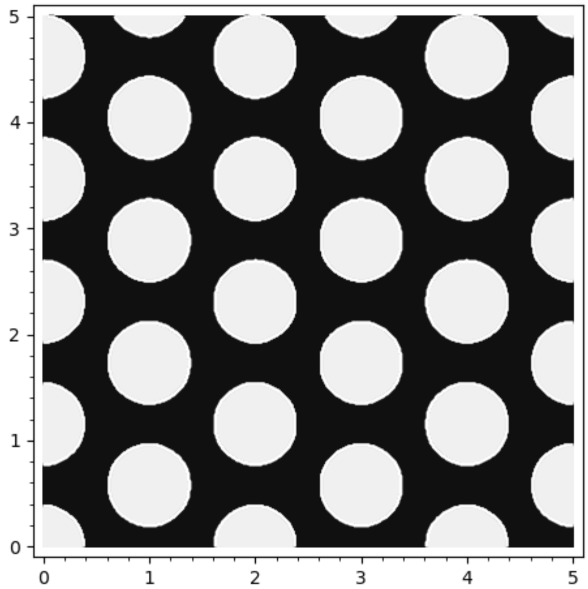

FIGURE 1. White regions denotes $g(x_1, x_2) > 0$, and black denotes $g(x_1, x_2) < 0$. Despite appearances, the white regions are *not* circles.

and let $D_x(r)$ denote a closed disk of radius $r > 0$ centred at x. Then $M \leq 2\pi \cdot 3$, and

$$\min_{y \in D_x(r)} |g(y)| \geq |g(x)| - r \cdot M.$$

Proof of Proposition 4.1. Recall that the lattice L is spanned by $u_1 = (1, 1/\sqrt{3})$ and $u_2 = (0, 2/\sqrt{3})$. The rhombus spanned by u_1, u_2 is a fundamental domain of L, as well as a fundamental domain for T. As it is more convenient to tile with rectangles rather than with rhombi we will prefer to evaluate the signed area on a rectangular fundamental domain, and show that the defect integral over the rectangle $\mathscr{R}$, having corners at $(0,0), (1,0), (0, 2/\sqrt{3}), (1, 2/\sqrt{3})$, easily seen to be a fundamental domain of T, is non-zero.

For some integer $N > 0$ we tile $\mathscr{R}$ by N^2 rectangles (modulo $\mathscr{R}$) centred at

$$h_{j,k} = \left(\frac{j}{N}, \frac{k}{N} \cdot \frac{2}{\sqrt{3}} \right)$$

for $0 \leq j, k < N$; each such rectangle can be covered with a disk of radius $r = \sqrt{7/12}/N$. If the inequality $|g(h_{j,k})| > 12\pi r > r \cdot M$, with M as in (4.3) is satisfied (using a factor of two safety margin), the corresponding rectangle centred at $h_{j,k}$ is said to be "stable", whence $g(\cdot)$ has constant sign on the whole rectangle by Lemma 4.2; otherwise it is said to be "unstable". Depending on the sign of $g(h_{j,k})$, we call the corresponding stable rectangle "positively stable" or "negatively stable".

For $N = 80$ one finds 2099 positively stable rectangles, 3299 negatively stable, and 1002 unstable ones. As $3299 - 2099 = 1200 > 1002$, we conclude that the defect (4.2) is nonzero (and in fact negative). Both assertions of Proposition 4.1 now follow: the first assertion follows from the presented numerical calculation, whereas the second one is an immediate consequence of the first assertion upon tiling $B_x(s)$ with $\pi(ms)^2/(2/\sqrt{3}) + O(ms)$ copies of fundamental domains associated with the lattice $\frac{1}{m}L$ (note that the boundary of $B_x(s)$ can be covered with $O(ms)$ tiles.) One can obtain more precise estimates on c in (4.2), by increasing N, and thus decreasing the mesh size: for example, for $N = 500$, the corresponding counts are respectively $96639, 147207$, and 6154. □

4.2. **Defect stability w.r.t. perturbations of g.** For later use we show that a small perturbation of g only changes the defect by a small amount. For convenience we work in the rescaled region where the eigenvalues are normalized to $4\pi^2$, hence we should consider the defect over balls of radius R (or squares of sides R) with R growing. We start by showing that simultaneous vanishing of both g and its gradient ∇g is impossible.

Lemma 4.3. *Let $Z_1 := \{x \in T : g(x) = 0\}$ and let $Z_2 := \{x \in T : \nabla g(x) = (0,0)\}$. Then $Z_1 \cap Z_2 = \emptyset$.*

Proof. The linear map $\mathbb{R}^3 \to \mathbb{R}^2$, given by $(a_1, a_2, a_3) \to \sum_{i=1}^3 a_i w_i$ with w_i as in (4.1), clearly has full range, hence a one dimensional kernel, spanned by $(1, 1, 1)$. In particular, if $\sum_{i=1}^3 a_i w_i = 0$, then $a_1 = a_2 = a_3 = C$ for some C. Therefore, $\nabla g(x) = 0$ implies that $\cos(2\pi w_1 \cdot x) = \cos(2\pi w_2 \cdot x) = \cos(2\pi w_3 \cdot x) = C$ for some C. Further, $g(x) = 0$ implies that $0 = \sum_{i=1}^3 \cos(2\pi w_i \cdot x) = 3C$, and thus $C = 0$ for any point where g and ∇g both vanish. In particular, we find that $2\pi w_i \cdot x = \pm\pi/2 + 2\pi k_i$ for $k_i \in \mathbb{Z}$. On the other hand, as $\sum_{i=1}^3 w_i = 0$, we find, on multiplying by $2/\pi$ that

$$0 \equiv \pm 1 + \pm 1 + \pm 1 \mod 4$$

which is impossible since the right hand side is odd no matter what signs are chosen. □

In light of Lemma 4.3 and the compactness of T, it follows that the gradient of g is uniformly bounded below on the zero set of $g(\cdot)$:

Corollary 4.4. *There exist $C > 0$ such that $|\nabla g(x)| \geq C$ for all $x \in Z_1 = g^{-1}(0)$.*

It is now straightforward to prove stability of the defect of g w.r.t. perturbations. Given $R \geq 1$ and a continuous function $f \in C(\mathbb{R}^2)$, define

$$Y_{f,R}(x) := \frac{1}{\pi R^2} \int_{B_x(R)} H(f(y))\, dy,$$

Lemma 4.5. *Let g be the function (4.1), and $R \geq 1$. Then for all $\epsilon > 0$ sufficiently small, if $f \in C(\mathbb{R}^2)$ is such that $|g(y) - f(y)| < \epsilon$ holds for all $y \in B_x(R)$, one has*

$$Y_{f,R}(x) = Y_{g,R}(x) + O(\epsilon).$$

Proof. It is sufficient to show that the measure of the set

$$\{x \in T : |g(x)| \leq \epsilon\}$$

is $O(\epsilon)$, for all sufficiently small ϵ, as we can then tile $B_x(R)$ with $\asymp R^2$ copies of the fundamental domain. Now, there exist some open neighborhood of $Z_1 = g^{-1}(0)$, outside of which $|g(x)|$ is uniformly bounded away from zero (say, using compactness of the closed complement). In other words, if $|g(x)|$ is small then we must have $d(x, Z_1)$ small, where $d(x, Z_1)$ denotes the distance between x and the zero set Z_1. Further, all x for which $d(x, Z_1)$ is sufficently small is contained in some small tubular neighbourhood of Z_1. The lower bound on the gradient of Corollary 4.4 implies that $|g(x)| \gg d(x, Z_1) + O(d(x, Z_1)^2)$, and hence the measure of the set of x for which $|g(x)| < \epsilon$ is $\ll \epsilon$. □

4.3. **Approximating g on the standard torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$: proof of Theorem 1.2.** We next show that a perturbed variant of the hexagonal lattice construction can be translated to the square torus. We begin by showing that the set of Gaussian integers, scaled to have norm one, can very well approximate third roots of unity.

Proposition 4.6. *The Pell equation*

$$b^2 - 3a^2 = 1 \tag{4.4}$$

admits infinitely many solutions. Further, let

$$S'' = \{n = a^2 + b^2\} \tag{4.5}$$

be the infinite sequence of integers of the form a^2+b^2 *with* (a,b) *as in* (4.4)*, and for* $n\in S''$ *we define the Gaussian integers* $z_1=z_{n,1}, z_2=z_{n,2}, z_3=z_{n,3}$ *as*

$$z_1:=-a+bi,\quad z_2:=-a-bi,\quad z_3:=2a+i. \tag{4.6}$$

Then, as $n\to\infty$ *along* S''*, we have*

$$z_1/|z_1|=e^{2\pi i/3}+O\left(n^{-1/2}\right)\quad z_2/|z_2|=e^{-2\pi i/3}+O\left(n^{-1/2}\right),\quad z_3/|z_3|=1+O\left(n^{-1/2}\right). \tag{4.7}$$

Proof. Since the Pell equation $b^2-3a^2=1$ has the solution $a=1, b=2$, it has infinitely many integer solutions. Moreover, we find that $|z_1|^2=|z_2|^2=|z_3|^2=4a^2+1$, and

$$\frac{z_1}{|z_1|}=\frac{-1+i\sqrt{3}}{2}+O(\frac{1}{a}),\quad \frac{z_2}{|z_2|}=\frac{-1-i\sqrt{3}}{2}+O(\frac{1}{a}),\quad \frac{z_3}{|z_3|}=1+O(\frac{1}{a}). \tag{4.8}$$

Thus, taking $n=a^2+b^2=4a^2+1$ we have $1/a=O(n^{-1/2})$, and the proof of Proposition 4.6 is concluded. □

Proof of Theorem 1.2. We claim that the statement of Theorem 1.2 holds, with S'' prescribed by (4.5), satisfying, in particular, the statement (4.7) of Proposition 4.6. To construct eigenfunctions on $\mathbb{T}=\mathbb{R}^2/\mathbb{Z}^2$ having large defect it is convenient to rescale $\mathbb{T}$ so that the eigenvalue equals $4\pi^2$, and correspondingly the torus must be rescaled so that the fundamental domain is a square with sides $n^{1/2}$ (where $\lambda=4\pi^2 n$ denotes the unscaled eigenvalue.) Given $n=a^2+b^2\in S''$ with $b^2-3a^2=1$ define the unit vectors $\widetilde{w}_i:=\frac{z_i}{|z_i|}\in\mathbb{R}^2$, $i=1,2,3$, with z_i as in (4.6), and the Laplace eigenfunction G, on the re-scaled torus $\mathbb{R}^2/(\sqrt{n}\mathbb{Z}^2)$, by

$$G(x):=\sum_{i=1}^{3}\cos(2\pi\widetilde{w}_i\cdot x)$$

A simple calculation shows that G is a Laplace eigenfunction, with eigenvalue $4\pi^2$, and that, with w_i as in (4.1), the asymptotic approximation (4.8) reads

$$|w_i-\widetilde{w}_i|=O(1/a)=O(1/n^{1/2}).$$

Hence, for any $x\in\mathbb{R}^2$, we have

$$|g(x)-G(x)|\ll|x|/n^{1/2}.$$

In particular, for $|x|=o(n^{1/2})$, we have $G(x)=g(x)+o(1)$, and thus, if $R=o(n^{1/2})$ grows with n we find, thanks to Lemma 4.5, that

$$Y_{G,R}(x)=Y_{g,R}(x)+o(1)=C+o(1)$$

for $C:=c\cdot\sqrt{3}/2<0$. In the macroscopic regime, i.e. when R is of size $n^{1/2}$, we similarly find that for $|x|\ll\epsilon n^{1/2}$,

$$Y_{G,R}(x)=Y_{g,R}(x)+O(\epsilon)=C+O(\epsilon).$$

Thus, if for $n\in S''$ we construct G as described above and define $f_n(x):=G(\sqrt{n}x)$, we obtain an eigenfunction on $\mathbb{T}^2$, with eigenvalue $4\pi^2 n$, and find that the defect integral over $B_x(s)$ (keeping in mind that $s=R/\sqrt{n}$ when we undo the scaling) is bounded away from zero for $|x|<\epsilon$; hence the variance is bounded from below, and the proof is concluded.

□

5. The defect of Arithmetic Random Waves: proof of theorems 1.3-1.5

5.1. **Preliminary lemmas.** Let $f_n(\cdot)$ be the Arithmetic Random Wave corresponding to (1.1), so that $f_n(\cdot)$ is a unit variance stationary Gaussian random field with covariance function (1.5). We first establish the precise expression (2.8) for the variance of the defect $\mathcal{D}_{n;s}$.

Lemma 5.1. *We have*

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right) = \frac{2}{\pi^3 s^4} \int\limits_{B(s)\times B(s)} \arcsin\left(r_n\left(x-y\right)\right) dxdy.$$

Proof. By the vanishing of the defect expectation (1.11), we have

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right) = \mathbb{E}\left[\mathcal{D}_{n;s}^2\right] = \frac{1}{\left(\pi s^2\right)^2}\mathbb{E}\left[\int_{B(s)\times B(s)} H\left(f_n\left(x\right)\right)\cdot H\left(f_n\left(y\right)\right) dxdy\right]. \tag{5.1}$$

Changing the order of expectation and integration in (5.1) together with the identity (2.7) (following along the same lines as the ones leading to (3.23)), gives the desired formula for the defect variance. □

As we will see below, the defect variance $\mathrm{Var}\left(\mathcal{D}_{n;s}\right)$ is intimately related to the (restricted) moments of the covariance function $r_n(\cdot)$. The following lemma gives a useful arithmetic formula for these moments.

Lemma 5.2. *Let* $l \geq 1$. *We have*

$$\int\limits_{B(s)\times B(s)} r_n\left(x-y\right)^l dxdy = \left(\pi s^2\right)^2 \frac{\#\mathscr{P}_n\left(l\right)}{N_n^l} + \frac{s^2}{N_n^l} \sum_{\left(\lambda^1,\dots,\lambda^l\right)\notin\mathscr{P}_n(l)} \frac{J_1\left(2\pi s\left\|\lambda^1+\dots+\lambda^l\right\|\right)^2}{\left\|\lambda^1+\dots+\lambda^l\right\|^2}. \tag{5.2}$$

Moreover, if $l = 2k+1$, *then by* (2.3) *we have* $\#\mathscr{P}_n(l) = 0$, *so that* (5.2) *reads* (2.11).

Proof. Expanding the covariance function (1.5), and recalling the definition (2.2) of $\mathscr{P}_n\left(l\right)$, we obtain

$$\begin{aligned}\int\limits_{B(s)\times B(s)} r_n\left(x-y\right)^l dxdy &= \frac{1}{N_n^l} \int\limits_{B(s)\times B(s)} \sum_{\lambda^1,\dots,\lambda^l\in\mathcal{E}_n} e\left(\left\langle\lambda^1+\dots+\lambda^l, x-y\right\rangle\right) dxdy \\ &= \left(\pi s^2\right)^2 \frac{\#\mathscr{P}_n\left(l\right)}{N_n^l} + \frac{1}{N_n^l} \sum_{\left(\lambda^1,\dots,\lambda^l\right)\notin\mathscr{P}_n(l)} \left|\int_{B(s)} e\left(\left\langle\lambda^1+\dots+\lambda^l, x\right\rangle\right) dx\right|^2.\end{aligned}$$

Formula (5.2) now follows from the identity

$$\int\limits_{B(s)} e\left(\left\langle v, x\right\rangle\right) dx = \frac{sJ_1\left(2\pi s\left\|v\right\|\right)}{\left\|v\right\|}.$$

□

5.2. **Upper bounds.** We now turn to prove the upper bounds for $\mathrm{Var}\left(\mathcal{D}_{n;s}\right)$. We begin with the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.1 and the elementary bound $\arcsin x = x + O(x^2)$ we have

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right) = \frac{2}{\pi^3 s^4}\left(\int\limits_{B(s)\times B(s)} r_n\left(x-y\right) dxdy + O\left(\int\limits_{B(s)\times B(s)} r_n\left(x-y\right)^2 dxdy\right)\right) \tag{5.3}$$

By Lemma 5.2,

$$\int_{B(s)\times B(s)} r_n(x-y)\,dxdy = \frac{s^2}{N_n}\sum_{\lambda\in\mathcal{E}_n}\frac{J_1(2\pi s\|\lambda\|)^2}{\|\lambda\|^2}$$

which, using the bound,

$$J_1(x) \ll \min\left\{x^{-1/2}, x\right\} \tag{5.4}$$

(see formulas (9.1.7) and (9.2.1) in [1]) is

$$\ll \frac{s^2}{N_n}\frac{N_n}{s\|\lambda\|^3} = \frac{s^4}{(sn^{1/2})^3} \le s^4 n^{-3\epsilon}$$

for all $s > n^{-1/2+\epsilon}$. We find that the contribution from the first integral on the r.h.s. of (5.3) is $\ll n^{-3\epsilon}$.

We next evaluate the second integral on the r.h.s. of (5.3). By Lemma 5.2, we have

$$\int_{B(s)\times B(s)} r_n(x-y)^2\,dxdy = \frac{\pi^2 s^4}{N_n} + \frac{s^2}{N_n^2}\sum_{\lambda^1\neq\lambda^2\in\mathcal{E}_n}\frac{J_1(2\pi s\|\lambda^1-\lambda^2\|)^2}{\|\lambda^1-\lambda^2\|^2}, \tag{5.5}$$

where we used the fact that $(\lambda^1,\lambda^2)\in\mathscr{P}_n(2)$ if and only if $\lambda^1 = -\lambda^2$, and in particular

$$\#\mathscr{P}_n(2) = N_n,$$

and the symmetry $\lambda\in\mathcal{E}_n \iff -\lambda\in\mathcal{E}_n$. Again using the bound (5.4) we have

$$\sum_{\lambda^1\neq\lambda^2\in\mathcal{E}_n}\frac{J_1(2\pi s\|\lambda^1-\lambda^2\|)^2}{\|\lambda^1-\lambda^2\|^2} \ll \sum_{\lambda^1\neq\lambda^2\in\mathcal{E}_n}\min\left\{\frac{1}{s\cdot\|\lambda^1-\lambda^2\|^3}, s^2\right\},$$

and therefore, for any $0<\eta<1/2$, we have

$$\sum_{\lambda^1\neq\lambda^2\in\mathcal{E}_n}\frac{J_1(2\pi s\|\lambda^1-\lambda^2\|)^2}{\|\lambda^1-\lambda^2\|^2} \ll s^2\sum_{\substack{\lambda^1,\lambda^2\in\mathcal{E}_n\\ 0<\|\lambda^1-\lambda^2\|<n^{1/2-\eta}}} 1 + s^{-1}\sum_{\substack{\lambda^1,\lambda^2\in\mathcal{E}_n\\ \|\lambda^1-\lambda^2\|\ge n^{1/2-\eta}}}\frac{1}{\|\lambda^1-\lambda^2\|^3}. \tag{5.6}$$

We estimate the sums on the r.h.s. of (5.6) separately. The second sum on the r.h.s. of (5.6) can be bounded trivially:

$$s^{-1}\sum_{\substack{\lambda^1,\lambda^2\in\mathcal{E}_n\\ \|\lambda^1-\lambda^2\|\ge n^{1/2-\eta}}}\frac{1}{\|\lambda^1-\lambda^2\|^3} \le N_n^2 s^{-1}\left(n^{1/2-\eta}\right)^{-3}, \tag{5.7}$$

whereas the first sum on the r.h.s. of (5.7) is the number of "close-by pairs", bounded in [12] (see Theorem 1.8 there and the remark following it) by

$$\sum_{\substack{\lambda^1,\lambda^2\in\mathcal{E}_n\\ 0<\|\lambda^1-\lambda^2\|<n^{1/2-\eta}}} 1 \ll N_n^{2-\tau\eta} \tag{5.8}$$

for any $\tau<4$ and $\eta>0$ sufficiently small.

Substituting the bounds (5.7) and (5.8) into (5.6), and then back into (5.5), we obtain the bound

$$\int_{B(s)\times B(s)} r_n(x-y)^2\,dxdy \ll s^4N_n^{-1}+sn^{-3/2+3\eta}+s^4N_n^{-\tau\eta} = s^4(N_n^{-1}+n^{3\eta}/(sn^{1/2})^3+N_n^{-\tau\eta}). \tag{5.9}$$

Let $0<\delta<4\epsilon$, and write $\delta=\tau\eta$ where $\tau<4$ and $\eta<\epsilon$. Then (5.9), together with (5.3), the bound (1.4), and the previous bound on the first integral on the r.h.s. of (5.3), gives $\mathrm{Var}(\mathcal{D}_{n;s}) \ll N_n^{-\delta}$ uniformly for all $s>n^{-1/2+\epsilon}$, completing the proof of Theorem 1.3.

□

We now prove Theorem 2.6 which, as argued above, immediately implies Theorem 1.4.

Proof of Theorem 2.6. Recall that the Taylor series of $\arcsin(t)$ is given by (2.9) where

$$a_k = \frac{1}{2^{2k}}\binom{2k}{k}\frac{1}{2k+1}, \tag{5.10}$$

so that by Stirling's approximation $a_k \sim \frac{1}{2\sqrt{\pi}}k^{-3/2}$, and the convergence is uniform on $[-1,1]$. In particular for $K \geq 0$, the Taylor polynomial of $\arcsin(t)$ is given by

$$\arcsin(t) = \sum_{k=0}^{K} a_k t^{2k+1} + O\left(|t|^{2K+3}\right). \tag{5.11}$$

Substituting (5.11) into (2.8) yields

$$\text{Var}(\mathcal{D}_{n;s}) = \frac{2}{\pi^3 s^4}\sum_{k=0}^{K} a_k \int\limits_{B(s)\times B(s)} r_n(x-y)^{2k+1}\,dxdy + O\left(\frac{1}{s^4}\int\limits_{B(s)\times B(s)} |r_n(x-y)|^{2K+3}\,dxdy\right). \tag{5.12}$$

Let $0 \leq k \leq K$. Recall the identity (2.11), and that $n \in S'''$ where the sequence $S''' \subseteq S$ satisfies the axiom $\mathcal{A}(\delta)$ as in Definition 2.4, so that the condition $\left(\lambda^1,\ldots,\lambda^{2k+1}\right) \notin \mathscr{P}_n(2k+1)$ in (2.11) implies that

$$\left\|\lambda^1+\cdots+\lambda^{2k+1}\right\| \gg_K n^{1/2-\delta}. \tag{5.13}$$

Substituting the bound (5.13) together with the bound (5.4) into (2.11), we get that

$$\begin{aligned}\frac{1}{s^4}\int\limits_{B(s)\times B(s)} r_n(x-y)^{2k+1} \ll &\frac{1}{s^3 N_n^{2k+1}}\sum_{\left(\lambda^1,\ldots,\lambda^{2k+1}\right)\notin\mathscr{P}_n(2k+1)}\frac{1}{\|\lambda^1+\cdots+\lambda^{2k+1}\|^3} \ll_K s^{-3}n^{-3/2+3\delta}\\ \leq &n^{-3(\epsilon-\delta)}\end{aligned} \tag{5.14}$$

uniformly for $s > n^{-1/2+\epsilon}$. We can now use (5.14) to bound the summation in the variance formula (5.12), which gives

$$\text{Var}(\mathcal{D}_{n;s}) \ll_K n^{-3(\epsilon-\delta)} + \frac{1}{s^4}\int\limits_{B(s)\times B(s)} |r_n(x-y)|^{2K+3}\,dxdy. \tag{5.15}$$

To control the $(2K+3)$'th moment of the absolute value of $r_n(\cdot)$, we use the Cauchy–Schwarz inequality to discard the absolute value:

$$\int\limits_{B(s)\times B(s)} |r_n(x-y)|^{2K+3}\,dxdy \leq \pi s^2\left(\int_{B(s)\times B(s)} r_n(x-y)^{4K+6}\,dxdy\right)^{1/2}. \tag{5.16}$$

By Lemma 5.2, we have

$$\begin{aligned}\frac{1}{s^4}\int\limits_{B(s)\times B(s)} r_n(x-y)^{4K+6}\,dxdy =&\pi^2\frac{\#\mathscr{P}_n(4K+6)}{N_n^{4K+6}}\\ &+\frac{1}{s^2 N_n^{4K+6}}\sum_{(\lambda^1,\ldots,\lambda^{4K+6})\notin\mathscr{P}_n(4K+6)}\frac{J_1\left(2\pi s\left\|\lambda^1+\cdots+\lambda^{4K+6}\right\|\right)^2}{\|\lambda^1+\cdots+\lambda^{4K+6}\|^2}.\end{aligned} \tag{5.17}$$

Since S''' is correlation-tame (Definition 2.2), we have $\#\mathscr{P}_n(4K+6) \ll_K N_n^{2K+3}$. This, together with (5.17) and the estimate (5.4), yields

(5.18)
$$\frac{1}{s^4}\int\limits_{B(s)\times B(s)} r_n(x-y)^{4K+6}\,dxdy \ll_K \frac{1}{N_n^{2K+3}}+\frac{1}{s^3N_n^{4K+6}}\sum_{(\lambda^1,\dots,\lambda^{4K+6})\notin\mathscr{P}_n(4K+6)}\frac{1}{\|\lambda^1+\cdots+\lambda^{4K+6}\|^3}.$$

By the lower bound (5.13), we have

$$\frac{1}{s^3N_n^{4K+6}}\sum_{(\lambda^1,\dots,\lambda^{4K+6})\notin\mathscr{P}_n(4K+6)}\frac{1}{\|\lambda^1+\cdots+\lambda^{4K+6}\|^3} \ll_K s^{-3}n^{-3/2+3\delta} \le n^{-3(\epsilon-\delta)} \tag{5.19}$$

uniformly for $s > n^{-1/2+\epsilon}$. Substituting the bound (5.19) into (5.18) and bearing in mind (1.4) gives

$$\frac{1}{s^4}\int\limits_{B(s)\times B(s)} r_n(x-y)^{4K+6}\,dxdy \ll_K \frac{1}{N_n^{2K+3}}. \tag{5.20}$$

Finally, we substitute the bound (5.20) into (5.16), and then into (5.15). Using again (1.4), we get that

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right) \ll_K \frac{1}{N_n^{K+3/2}}.$$

This completes the proof of Theorem 2.6, since K can be taken arbitrarily large. □

5.3. **Lower bound.** In order to prove the lower bound for $\mathrm{Var}\left(\mathcal{D}_{n;s}\right)$ stated in Theorem 1.5, we will require a result on Diophantine approximation by multiples of square roots of prime numbers. For $t \in \mathbb{R}$, we denote $\langle t\rangle$ to be the distance of t to the nearest integer number, and let

$$P_K := \{p \text{ prime}:\ p \equiv 1 \pmod 4,\ p \le K\} \tag{5.21}$$

denote the set of primes $p \le K$ congruent to 1 modulo 4.

Lemma 5.3. *Let $K > 1$ be an integer, and let $\epsilon > 0$. For every integer $q \ge 1$, we have*

$$\max_{p\in P_K}\langle q\sqrt{p}\rangle \gg_{K,\epsilon} q^{-\frac{2\log K}{K}-\epsilon}. \tag{5.22}$$

The proof of Lemma 5.3 will invoke two classical results from the theory of Diophantine approximation: Besicovitch's theorem on the linear independence over $\mathbb{Q}$ of the square roots of distinct square-free positive integers, and Schmidt's theorem on simultaneous Diophantine approximation, that, for the reader's convenience, we cite next, in the form used subsequently.

Theorem 5.4 (Besicovitch [3])**.** *Let $q_1,\dots,q_m$ be distinct squarefree positive integers. The numbers $\sqrt{q_1},\dots,\sqrt{q_m}$ are linearly independent over $\mathbb{Q}$.*

Theorem 5.5 (Schmidt [32])**.** *Let $\alpha_1,\dots,\alpha_m$ be real algebraic numbers so that $1,\alpha_1,\dots,\alpha_m$ are linearly independent over the rationals. Then for every $\epsilon > 0$ and for every integer $q \ge 1$, we have*

$$\max_{1\le i\le m}\langle q\alpha_i\rangle \gg q^{-1/m-\epsilon}$$

where the implied constant depends on ϵ and on $\alpha_1,\dots,\alpha_m$.

Proof of Lemma 5.3. By Theorem 5.4, the elements of the set $\{1\}\cup\left\{\sqrt{p} : p \in P_K\right\}$ are linearly independent over the rationals. Since $\#P_k \sim \frac{K}{2\log K}$ as $K \to \infty$, the bound (5.22) follows from Theorem 5.5. □

We are finally in a position to prove Theorem 1.5.

Proof of Theorem 1.5. Recall that substituting the Taylor series of the arcsine function (2.9) in (2.8) gives formula (2.10):

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right)=\frac{2}{\pi^3 s^4}\sum_{k=0}^{\infty}a_k\int\limits_{B(s)\times B(s)}r_n\left(x-y\right)^{2k+1}dxdy,$$

where a_k are given by (5.10), and in particular $a_k>0$, $a_0=1$, and $a_k\sim\frac{1}{2\sqrt{\pi}}k^{-3/2}$. Hence, Lemma 5.2 yields

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right)=\frac{2}{\pi^3 s^2}\sum_{k=0}^{\infty}\frac{a_k}{N_n^{2k+1}}\sum_{\left(\lambda^1,\dots,\lambda^{2k+1}\right)\notin\mathscr{P}_n(2k+1)}\frac{J_1\left(2\pi s\left\|\lambda^1+\cdots+\lambda^{2k+1}\right\|\right)^2}{\left\|\lambda^1+\cdots+\lambda^{2k+1}\right\|^2}. \tag{5.23}$$

By the positivity of the coefficients a_k, we may obtain a lower bound by discarding all terms in (5.23) but one with $k=0$:

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right)\geq\frac{2}{\pi^3}\frac{J_1\left(2\pi T\right)^2}{T^2}, \tag{5.24}$$

with $T=s\cdot\sqrt{n}$, as in Theorem 1.5.

Recall that for large z, we have [1, formula (9.2.1)]

$$J_1\left(z\right)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{3}{4}\pi\right)+O\left(\frac{1}{z^{3/2}}\right), \tag{5.25}$$

so that

$$J_1\left(2\pi T\right)=\pi^{-1}T^{-1/2}\cos\left(\left(2T-\frac{3}{4}\right)\pi\right)+O\left(T^{-3/2}\right). \tag{5.26}$$

We write

$$2T-\frac{1}{4}=t+\rho \tag{5.27}$$

where $t=t\left(T\right)\in\mathbb{Z}$ and $|\rho|\leq 1/2$, so that

$$\left|\cos\left(\left(2T-\frac{3}{4}\right)\pi\right)\right|=\left|\sin\left(\rho\pi\right)\right|\gg\left|\rho\right|. \tag{5.28}$$

The tth zero $j_{1,t}$ of J_1 satisfies

$$j_{1,t}=\left(t+\frac{1}{4}\right)\pi+O(1/t)$$

(see, e.g. [1, formula (9.5.12)]), so that

$$|2\pi T-j_{1,t}|=\pi|\rho|+O(1/T). \tag{5.29}$$

In particular, if $2\pi T$ is bounded away from $j_{1,t}$, then (5.29) yields $\rho\gg 1$, so that (5.26) and (5.28) give $J_1\left(2\pi T\right)^2\gg T^{-1}$, which together with (5.24) yields

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right)\gg T^{-3}.$$

Given $\delta>0$, we consider two cases, whether $|\rho|\geq T^{-\delta/2}$ or $|\rho|<T^{-\delta/2}$, aiming at proving (1.13) with the same δ. If $|\rho|\geq T^{-\delta/2}$, then by (5.26) and (5.28) it follows that $J_1\left(2\pi T\right)^2\gg T^{-1-\delta}$ so that (5.24) gives

$$\mathrm{Var}\left(\mathcal{D}_{n;s}\right)\gg T^{-3-\delta}, \tag{5.30}$$

stronger than (1.13) with $A>0$ arbitrary. Assume otherwise that $|\rho|<T^{-\delta/2}$, and observe that all odd numbers $m\in S$ are expressible as

$$m=a^2+\left(2k+1-a\right)^2 \tag{5.31}$$

for some $k \geq 0$ and $1 \leq a \leq 2k+1$. Consider all tuples of the form

$$\left(\lambda^1, \ldots, \lambda^{2k+1}\right) = \left(\overbrace{\lambda, \ldots, \lambda}^{a \text{ times}}, \overbrace{i\lambda, \ldots, i\lambda}^{2k+1-a \text{ times}}\right). \tag{5.32}$$

The number of such tuples is precisely N_n, and they satisfy

$$\left\|\lambda^1 + \cdots + \lambda^{2k+1}\right\| = \sqrt{nm}. \tag{5.33}$$

By the inequality $\alpha^2 + \beta^2 \geq \frac{(\alpha+\beta)^2}{2}$ applied to (5.31), we get that $m \geq \frac{(2k+1)^2}{2} \geq 2k^2$ so that

$$k \leq \sqrt{m/2}. \tag{5.34}$$

By the positivity of all the terms in (5.23), we can bound $\text{Var}\left(\mathcal{D}_{n;s}\right)$ from below by restricting the inner summation in (5.23) to tuples of the form (5.32). This together with (5.33) and (5.34) (note that $a_k \gg m^{-3/4}$) gives the lower bound

$$\text{Var}\left(\mathcal{D}_{n;s}\right) \gg \frac{1}{T^2} \sum_{\substack{m \in S \\ m \text{ odd}}} \frac{1}{N_n^{\sqrt{2m}}} \frac{J_1\left(2\pi T\sqrt{m}\right)^2}{m^{7/4}}. \tag{5.35}$$

Let $K > 1$ be a sufficiently large parameter to be chosen later, and restrict the summation in (5.35) to primes $p \in P_K$ in (5.21) (these are the primes $p \equiv 1 \pmod 4$ which are less or equal to K). Then

$$\text{Var}\left(\mathcal{D}_{n;s}\right) \gg_K \frac{1}{N_n^{\sqrt{2K}} T^2} \sum_{p \in P_K} J_1\left(2\pi T\sqrt{p}\right)^2. \tag{5.36}$$

By (5.25), we have

$$J_1\left(2\pi T\sqrt{p}\right) = \pi^{-1} T^{-1/2} p^{-1/4} \cos\left(\left(2T\sqrt{p} - \frac{3}{4}\right)\pi\right) + O\left(T^{-3/2}\right). \tag{5.37}$$

We write

$$2T\sqrt{p} - \frac{1}{4} = l + \eta$$

where $l = l\left(T, p\right) \in \mathbb{Z}$ and $|\eta| \leq 1/2$. Then by (5.27),

$$\left|\cos\left(\left(2T\sqrt{p} - \frac{3}{4}\right)\pi\right)\right| = \left|\sin\left(\eta\pi\right)\right| \gg |\eta| = \left|2T\sqrt{p} - l - \frac{1}{4}\right| = \left|\left(t + \frac{1}{4} + \rho\right)\sqrt{p} - l - \frac{1}{4}\right|$$

$$\gg \left|\left(4t+1\right)\sqrt{p} - \left(4l+1\right)\right| - 4\left|\rho\right|\sqrt{p}. \tag{5.38}$$

By Lemma 5.3, there exists $p_0 \in P_K$ such that

$$\left|\left(4t+1\right)\sqrt{p_0} - \left(4l+1\right)\right| \gg_{K,\epsilon} t^{-2\log K/K - \epsilon}. \tag{5.39}$$

Since $|\rho| < T^{-\delta/2}$, by choosing $K = K\left(\delta\right)$ sufficiently large so that $2\log K/K < \delta/4$ (keeping in mind that $t = 2T + O(1)$), we conclude upon substituting the bound (5.39) in (5.38) that

$$\left|\cos\left(\left(2T\sqrt{p_0} - \frac{3}{4}\right)\pi\right)\right| \gg_\delta T^{-\delta/4},$$

which by (5.37) implies

$$J_1\left(2\pi T\sqrt{p_0}\right)^2 \gg_\delta T^{-1-\delta/2}.$$

This, together with (5.36) gives

$$\text{Var}\left(\mathcal{D}_{n;s}\right) \gg_\delta \frac{1}{N_n^{\sqrt{2K}} T^2} J_1\left(2\pi T\sqrt{p_0}\right)^2 \gg_\delta \frac{1}{N_n^{\sqrt{2K}} T^{3+\delta/2}}. \tag{5.40}$$

To summarize, the bounds (5.30) and (5.40) imply that, in either case, (1.13) holds with $A = \sqrt{2K}$, which is the statement of Theorem 1.5. □

REFERENCES

[1] Abramowitz, M. and Stegun, I. A. Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards Applied Mathematics Series 55, Tenth Prrinting, Washington, D.C. (1972).

[2] Berry, M.V., 2002. Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. Journal of Physics A: Mathematical and General, 35(13), p.3025.

[3] Besicovitch, A.S., On the linear independence of fractional powers of integers, J. London Math. Soc. 15, 3–6 (1940).

[4] Blum, G., Gnutzmann, S. and Smilansky, U. Nodal domains statistics: A criterion for quantum chaos. Physical Review Letters, 88(11), p.114101 (2002).

[5] Benatar, J., Marinucci, D. and Wigman, I. Planck-scale distribution of nodal length of arithmetic random waves. Journal d'Analyse Mathématique, 141(2), pp.707–749 (2020).

[6] Bombieri, E., Bourgain, J. A problem on sums of two squares. International Mathematics Research Notices, 2015(11), pp.3343–3407.

[7] Bourgain, J. On toral eigenfunctions and the random wave model. Israel Journal of Mathematics, 201(2), pp.611–630 (2014).

[8] Buckley, J. and Wigman, I. On the number of nodal domains of toral eigenfunctions. In Annales Henri Poincaré, Vol. 17, No. 11, pp. 3027–3062, Springer International Publishing (2016).

[9] Cilleruelo, J. The distribution of the lattice points on circles. Journal of Number theory, 43(2), pp.198–202 (1993).

[10] Erdös, P. and Hall, R.R. On the angular distribution of Gaussian integers with fixed norm. Discrete mathematics, 200(1-3), pp.87–94 (1999).

[11] Fainsilber, L., Kurlberg, P. and Wennberg, B. Lattice points on circles and discrete velocity models for the Boltzmann equation. SIAM journal on mathematical analysis, 37(6), pp.1903–1922 (2006).

[12] Granville, A. and Wigman, I., Planck-scale mass equidistribution of toral Laplace eigenfunctions. Comm. Math. Phys. 355, no. 2, 767–802 (2017).

[13] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford, at the Clarendon Press, 1975. 4th ed.

[14] Humphries, P. Equidistribution in shrinking sets and L^4-norm bounds for automorphic forms. Mathematische Annalen, 371(3-4), pp.1497–1543 (2018).

[15] Kátai, I. and Környei, I. On the distribution of lattice points on circles. Ann. Univ. Sci. Budapest. Eotvos Sect. Math, 19(87–91), (1977).

[16] Krishnapur, M., Kurlberg, P. and Wigman, I. Nodal length fluctuations for arithmetic random waves. Annals of Mathematics, pp.699–737 (2013).

[17] Kurlberg, P. and Wigman, I. On probability measures arising from lattice points on circles. Mathematische Annalen, 367(3-4), pp.1057–1098 (2017).

[18] Landau, E., 1909. Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate.

[19] Marinucci, D. and Wigman, I. The defect variance of random spherical harmonics. Journal of Physics A: Mathematical and Theoretical, 44(35), p.355206 (2011)

[20] Marinucci, D. and Wigman, I On nonlinear functionals of random spherical eigenfunctions. Communications in Mathematical Physics, 327(3), pp.849–872 (2014).

[21] Nazarov, F. and Sodin, M. Asymptotic Laws for the Spatial Distribution and the Number of Connected Components of Zero Sets of Gaussian Random Functions. J. Math. Phys., Anal., Geom. (2016)

[22] Nazarov, F., Polterovich, L. and Sodin, M. Sign and area in nodal geometry of Laplace eigenfunctions. American Journal of Mathematics, 127(4), pp.879–910 (2005).

[23] Oravecz, F., Rudnick, Z. and Wigman, I. The Leray measure of nodal sets for random eigenfunctions on the torus. Annales de l'institut Fourier, Vol. 58, No. 1, pp. 299–335 (2008).

[24] Rice, S.O., Mathematical analysis of random noise, Bell Syst. Tech. J., 23, 282 (1944).

[25] Rice, S.O., Mathematical analysis of random noise – conclusion, Bell Syst. Tech. J., 24, 46 (1945).

[26] Rudnick, Z. and Wigman, I. On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. Annales Henri Poincare, Vol. 9, No. 1, pp. 109–130 (2008).

[27] Sarnak, P. Variance sums on symmetric spaces. Private communication.

[28] Sarnak, P., Wigman, I. Topologies of Nodal Sets of Random Band-Limited Functions. Communications on pure and applied mathematics, 72(2), pp.275–342 (2019).

[29] Sartori, A. On the fractal structure of attainable probability measures. Bulletin Polish Acad. Sci. Math. 66, 123–133 (2018).

[30] Sartori, A. Mass distribution for toral eigenfunctions via Bourgain's de-randomization. The Quarterly Journal of Mathematics, 71(1), pp.1–20 (2020).

[31] Sartori, A. Spectral quasi correlations and phase-transitions for the nodal length of Arithmetic Random Waves, Int. Math. Res. Not. (IMRN), to appear, available online (2020) `https://arxiv.org/abs/2005.04698`.

[32] Schmidt, W.M. Simultaneous approximation to algebraic numbers by rationals. Acta Mathematica, 125(1), pp.189–201 (1970).

[33] Sodin, M. Lectures on random nodal portraits. Probability and statistical physics in St. Petersburg, 91, pp.395–422 (2016).

[34] Wigman, I. and Yesha, N. Central limit theorem for Planck-scale mass distribution of toral Laplace eigenfunctions. Mathematika, 65(3), pp.643–676.