

A Local Riemann Hypothesis, I

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PRELIMINARY VERSION

In Tate's thesis [30], Hecke L-functions are studied by means of the local integrals

$$\zeta(s, \nu, f) = \int_F f(x) \nu(x) |x|^s d^\times x,$$

where f is an element of the Schwartz space $S(F)$ on a local field F , and ν is a character of F^\times . Weil [35] defined a representation $\omega = \omega_\psi$ of the metaplectic group $\widetilde{SL}(2, F)$ on $S(F)$. We consider the restriction of ω to the special orthogonal group $SO(2)$ of $\widetilde{SL}(2, F)$, corresponding to the quadratic form $x^2 + y^2$. If -1 is not a square in F , this representation is multiplicity free, and $S(F)$ decomposes into a direct sum of one-dimensional invariant subspaces. The *Local Riemann Hypothesis* is the assertion that if f lies in one of these spaces, then the zeros of the local integral $\zeta(s, \nu, f)$ lie on the line $\operatorname{re}(s) = \frac{1}{2}$. (We refer to the text for the correct statement if -1 is a square.) This is proved in a substantial number of cases, in this paper and its companion piece by Kurlberg [19].

If $F = \mathbb{R}$, we will prove an extension of this result to the harmonic oscillator in n -dimensions. This result may be formulated in a way that makes sense over a p -adic field, though we have not investigated this yet. In this connection, we also have a *reciprocity law* for the values at negative integers of the Laguerre polynomials, and a geometrical interpretation of these values.

We will also state a certain conjecture, that if the spherical Whittaker function of a spherical representation of $GL(n, \mathbb{R})$ which is a functorial lift from $GL(2, \mathbb{R})$ vanishes anywhere on the group, then the representation is tempered. This generalizes a theorem of Pólya on the zeros of Bessel functions.

Up-to-date information on this topic may be found on the world-wide-web at:

<http://match.stanford.edu/rh/> .

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1. The zeros of the Mellin transforms of Hermite polynomials. For the quantum mechanical harmonic oscillator see Weyl [36], and Cartier [7].

We recall the result of Bump and Ng [5], showing that the Mellin transforms of the Hermite functions have their zeros on the line $\operatorname{re}(s) = \frac{1}{2}$. (At first Bump and Ng considered the case of H_n with n even, and Vaaler pointed out that the case n odd could be added.)

Our normalizations will be different than in [5]. Let

$$f_n(x) = 2^{-n/2} H_n(\sqrt{2\pi} x) e^{-\pi x^2},$$

where the Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The f_n are the eigenfunctions of the Hamiltonian $x^2 - \frac{1}{4\pi^2} \frac{d^2}{dx^2}$ of the quantum mechanical harmonic oscillator. That is, they satisfy the Schrödinger equation

$$\left(x^2 - \frac{1}{4\pi^2} \frac{d^2}{dx^2} \right) f_n = \frac{2n+1}{2\pi} f_n.$$

Define polynomials p_n by

$$M_n(s) = \begin{cases} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) p_n(s) & \text{if } n \text{ is even;} \\ \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) \sqrt{2\pi} p_n(s) & \text{if } n \text{ is odd.} \end{cases}$$

where the Mellin transform

$$M_n(s) = \int_0^\infty f_n(x) x^s \frac{dx}{x}.$$

We have

$$f_{n+1}(x) = \left(\sqrt{2\pi} x - \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \right) f_n(x),$$

and consequently, integrating by parts, we have

$$M_{n+1}(s) = \sqrt{2\pi} M_n(s+1) + \frac{s-1}{\sqrt{2\pi}} M_n(s-1).$$

This implies that

$$p_{n+1}(s) = \begin{cases} p_n(s+1) + p_n(s-1) & \text{if } n \text{ is even;} \\ s p_n(s+1) + (s-1) p_n(s-1) & \text{if } n \text{ is odd.} \end{cases}$$

The polynomials p_n have certain properties in common with the Riemann zeta function. We have the functional equation

$$p_n(1-s) = \begin{cases} p_n(s) & \text{if } n \equiv 0, 1 \pmod{4}; \\ -p_n(s) & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

Moreover

Theorem 1. *The zeros of p_n lie on the line $\operatorname{re}(s) = \frac{1}{2}$.*

We give two proofs of this. Another proof may be found in Bump and Ng [5].

FIRST PROOF. We recall a familiar classical fact, that *orthogonal polynomials have real zeros*. More precisely, let μ be a positive Borel measure on \mathbb{R} , and assume that μ is not supported on any finite set. We may apply Gram-Schmidt process to the sequence $\{1, x, x^2, \dots\}$ and obtain a sequence of polynomials P_0, P_1, P_2, \dots such that the degree of P_n is n , which are orthogonal with respect to μ . The zeros of these are real and simple. Indeed, after multiplying the polynomials P_n by suitable constants, they'll have real coefficients. If r_1, \dots, r_k are the zeros of P_n which have odd multiplicity, if $k < n$ we could expand $Q(x) = \prod_i (x - r_i)$ in terms of P_i with $i < n$, so Q would be orthogonal to P_n ; but patently $QP_n \geq 0$, so this is a contradiction.

Let us show that the polynomials $p_{2n}(\frac{1}{2} + it)$ form an orthogonal family with respect to a suitable measure. Indeed, the even Hermite functions f_{2n} are eigenfunctions of a self-adjoint differential operator (the oscillator Hamiltonian), so they form an orthogonal family on the half-line \mathbb{R}^+ , which we parametrize exponentially. Thus, consider the functions $\phi_n(x) = f_{2n}(e^{2\pi x}) e^{\pi x}$. These are orthogonal with respect to Lebesgue measure on \mathbb{R} . The Fourier transform of ϕ_n is $2\pi M_{2n}(\frac{1}{2} + it)$, so by the Plancherel theorem these are orthogonal:

$$\int_{-\infty}^{\infty} M_{2n}(\frac{1}{2} + it) M_{2m}(\frac{1}{2} + it) dt = 0$$

if $m \neq n$. Thus the polynomials $p_{2n}(\frac{1}{2} + it)$ form an orthonormal family, with respect to the measure $|\Gamma(\frac{1}{4} + \frac{it}{2})|^2 dt$.

Similarly, the polynomials p_{2n+1} are orthogonal with respect to $|\Gamma(\frac{3}{4} + \frac{it}{2})|^2 dt$. They must therefore all have real zeros. ■

SECOND PROOF. Let f be an eigenfunction of the oscillator Hamiltonian. Thus, f satisfies the Schrödinger equation

$$\left(x^2 - \frac{1}{4\pi^2} \frac{d^2}{dx^2}\right) f = \frac{\lambda}{2\pi} f$$

for some value of λ . Define the Mellin transform

$$M(s) = \int_0^{\infty} f(x) x^s \frac{dx}{x}.$$

Integrating the above Schrödinger equation by parts gives

$$M(s+2) - \frac{1}{4\pi^2} (s-1)(s-2)M(s-2) = \frac{\lambda}{2\pi} M(s).$$

We have either

$$M(s) = \begin{cases} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) p(s) & \text{or} \\ \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right) \sqrt{2\pi} p(s), \end{cases}$$

with $p(s)$ a polynomial, according as $\hat{f} = \pm f$ or $\hat{f} = \pm if$ (i.e., according as $f = f_n$ with n even or n odd.) We have therefore either

$$\lambda p(s) = s p(s+2) - (s-1) p(s-2),$$

or

$$\lambda p(s) = (s+1) p(s+2) - (s-2) p(s-2).$$

The situation will be more symmetrical if we make the substitution $q(s) = p\left(s + \frac{1}{2}\right)$. Thus, we wish to show the zeros of q are purely imaginary, and we have

$$\lambda q(s) = (s+a) q(s+2) - (s-a) q(s-2),$$

with $a = \frac{1}{2}$ or $a = \frac{3}{2}$. The theorem now follows from the following

Lemma. *Let $q(s)$ be a polynomial, and assume that the zeros of $q(s)$ lie in the closed strip $\{\operatorname{re}(s) \in [-c, c]\}$ with $c > 0$. Then if $a > 0$, the zeros of*

$$r(s) = (s+a) q(s+2) - (s-a) q(s-2)$$

lie in the open strip $\{\operatorname{re}(s) \in (-c, c)\}$.

To prove this, suppose that $\operatorname{re}(s) \geq c$, yet $r(s) = 0$. We will obtain a contradiction. (The case $\operatorname{re}(s) \leq -c$ may be handled similarly.) Let $q(s) = c \prod_{i=1}^n (s - r_i)$. If $r(s) = 0$, then

$$|(s+a) q(s+2)| = |(s-a) q(s-2)|,$$

so

$$|s+a| \prod |s+2-r_i| = |s-a| \prod |s-2-r_i|.$$

Now since $\operatorname{re}(s) > 0$, $a > 0$, we have $|s+a| > |s-a|$; moreover, since $|\operatorname{re}(r_i)| \leq c$, $\operatorname{re}(s) > c$, we have $\operatorname{re}(s-r_i) \geq 0$, and so $|s+2-r_i| > |s-2-r_i|$. Multiplying these inequalities together, we obtain a contradiction. ■

The preceding proof is similar to the original proof of Pólya of an interesting property of the K -Bessel functions, namely, his theorem that if $y > 0$ and $K_\nu(y) = 0$, then ν is purely imaginary. Pólya's proof [23] depends on the recurrence identity (Watson [34], 3.71)

$$2\nu K_\nu(x) = x (K_{\nu+1}(x) - K_{\nu-1}(x)).$$

The operator which takes an even function $q(\nu)$ and replaces it by $\nu^{-1}(q(\nu+1) - q(\nu-1))$ has the property (like the operator $q \mapsto r$ in the Lemma) of moving the zeros of a function closer to the imaginary axis, and so an eigenfunction of this operator should have its zeros on the imaginary axis. Since $\nu \mapsto K_\nu(x)$ is not a polynomial function, making this argument rigorous requires care. An easier (but arguably less insightful) proof may be found in Titchmarsh [31], Section 10.23.

Pólya connects his result with the Riemann hypothesis by arguing that

$$\pi^2 (K_{\frac{q}{4} + \frac{it}{2}}(2\pi) + K_{\frac{q}{4} - \frac{it}{2}}(2\pi))$$

has analytic properties similar to $\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, with $s = \frac{1}{2} + it$. (Actually this value, taken from Titchmarsh [31], seems to us to be off by a constant, but this is unimportant.) This function also has its zeros on the line $\operatorname{re}(s) = \frac{1}{2}$.

It is worth pointing out that there is another more “philosophical” way of connecting Pólya's result on the Bessel functions with the Riemann hypothesis. We begin by noting that it implies a Riemann hypothesis for the Fourier coefficients of Eisenstein series. Consider the classical $SL(2, \mathbb{Z})$ Eisenstein series

$$E(z, s) = \frac{1}{2} \pi^{-s} \Gamma(s) \sum \frac{y^s}{|cz + d|^{2s}},$$

where the summation is over nonzero pairs of integers (c, d) . It is well known that if $n \neq 0$, then the n -th Fourier coefficient

$$\int_0^1 E(x + iy) e^{2\pi i n x} dx = 2 |n|^{s-1/2} \sigma_{1-2s}(|n|) \sqrt{y} K_{s-1/2}(2\pi |n| y).$$

(See Bump [2] Section I.6.) Both the divisor function $\sigma_{1-2s}(|n|)$ and the K -Bessel function $K_{s-1/2}$ have their zeros on the line $\operatorname{re}(s) = \frac{1}{2}$. Now if, on the other hand, we consider the Eisenstein series of half-integral weight (see Maass [20], Shimura [28] and Goldfeld and Hoffstein [13]), the Fourier coefficients are quadratic L-functions. So the analogous assertion—that the Fourier coefficients of the Eisenstein series satisfy a Riemann hypothesis—in the case of the Eisenstein series of half-integral weight, should reduce to the classical Riemann hypothesis.

One may be a bit more careful here. Actually the Fourier coefficients of these Eisenstein series are the products of quadratic L-functions with certain finite Dirichlet polynomials, and one would like to assert that these polynomials themselves have their zeros

on the line $\operatorname{re}(s) = 1/2$. David Cardon has looked at the case of Eisenstein series on the double cover of $GL(2)$ over a rational function field, and his work suggests that the correct formulation is that *the Whittaker coefficients in the modified sense of Gelbart, Howe and Piatetski-Shapiro [11] should satisfy the Riemann hypothesis.*

We propose here a conjectural generalization of Pólya’s result on the zeros of the Bessel function K_ν . Let π be a spherical principal series representation of $PGL(2, \mathbb{R})$, and let W be the $SO(2)$ -fixed vector (determined up to constant multiple) in its Whittaker model with respect to the additive character $\psi(x) = e^{2\pi ix}$ of \mathbb{R} . Then

$$W\left(\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} k\right) = \sqrt{y} K_\nu(2\pi y) e^{2\pi ix},$$

when $k \in SO(2)$, for some complex number ν . So Pólya’s result may be formulated as saying that *if the $SO(2)$ -fixed Whittaker vector in a spherical principal series representation vanishes anywhere on $PGL(2, \mathbb{R})$, then the representation is tempered.*

More generally, let π be a spherical principal series representation of $PGL(n, \mathbb{R})$, and assume that π is a symmetric $n - 1$ -st power lifting of a spherical principal series representation of $PGL(2, \mathbb{R})$. This means that there is a quasicharacter χ of $\mathbb{R}^\times / \{\pm 1\}$ such that π is obtained by normalized parabolic induction from the character

$$\begin{pmatrix} y_1 & * & \cdots & * \\ & y_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & y_n \end{pmatrix} \mapsto \chi(y_1)^{n-1} \chi(y_2)^{n-3} \cdots \chi(y_n)^{1-n}.$$

Let W be the $SO(n)$ -fixed vector in the Whittaker model of π , determined up to constant multiple.

Conjecture. *In this setting, if W vanishes anywhere on $GL(n, \mathbb{R})$, then π is tempered (i.e. χ is unitary).*

We will offer three pieces of evidence for this statement.

Firstly, it is true when $n = 2$ by Pólya’s result.

Secondly, for one particular nontempered spherical Whittaker function (which is a symmetric square lift from $GL(2)$) on $GL(3, \mathbb{R})$ we can verify this claim—we recall that the spherical Whittaker functions on $GL(3, \mathbb{R})$ and $GL(3, \mathbb{C})$ are the same, and that for one particular principal series representation, corresponding to the cubic theta function on $GL(3, \mathbb{C})$, the Whittaker function can be expressed in terms of the Bessel function $K_{1/3}$, so the asserted nonvanishing follows from Pólya’s result. See Bump and Friedberg [3] and Bump and Huntley [4].

And thirdly, an analogous statement is true for spherical Whittaker functions on $PGL(n, F)$, when F is a nonarchimedean local field. Let π be a spherical principal series

representation with Satake parameters $\alpha_1, \dots, \alpha_n$. Let

$$h = \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{pmatrix}$$

be a dominant element of the diagonal subgroup, so that if λ_i is the valuation of y_i , we have $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Let s_λ be the Schur polynomial corresponding to the partition $\lambda = (\lambda_1, \dots, \lambda_n)$, a symmetric polynomial in n variables (Macdonald [21]). According to Shintani [29] and Casselman and Shalika [8], the value $W(h)$ of the normalized Whittaker function with respect to an additive character ψ whose conductor is the ring \mathfrak{o} of integers in F equals $\delta(h)^{1/2} s_\lambda(\alpha_1, \dots, \alpha_n)$, where δ is the modular quasicharacter of the Borel subgroup of $GL(n, F)$. Now suppose that π is a symmetric $n - 1$ -st power lift from $GL(2)$. Thus we assume that there exists a complex number α such that

$$(\alpha_1, \dots, \alpha_n) = (\alpha^{n-1}, \alpha^{n-3}, \dots, \alpha^{1-n}).$$

Proposition. *In this situation, if $W(h) = 0$ for h dominant, then π is tempered.*

PROOF. We have $s_\lambda(\alpha^n, \alpha^{n-2}, \dots, \alpha^{-n}) = 0$, and we will show that $|\alpha| = 1$. Indeed, by homogeneity of the Schur polynomial, we have $s_\lambda(\alpha^{2n-2}, \alpha^{2n-4}, \dots, 1) = 0$. We recall that

$$s_\lambda(\alpha_1, \dots, \alpha_n) = \frac{\begin{vmatrix} \alpha_1^{\lambda_1+n-1} & \alpha_2^{\lambda_1+n-1} & \dots & \alpha_n^{\lambda_1+n-1} \\ \alpha_1^{\lambda_2+n-2} & \alpha_2^{\lambda_2+n-2} & \dots & \alpha_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{\lambda_n} & \alpha_2^{\lambda_n} & \dots & \alpha_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \dots & \alpha_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix}}.$$

Substituting $(\alpha^{2n-2}, \alpha^{2n-4}, \dots, 1)$ for $(\alpha_1, \dots, \alpha_n)$, the numerator here becomes

$$\begin{vmatrix} \beta_1^{n-1} & \beta_1^{n-2} & \dots & 1 \\ \beta_2^{n-1} & \beta_2^{n-2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n^{n-1} & \beta_n^{n-2} & \dots & 1 \end{vmatrix} = \prod_{i < j} (\beta_i - \beta_j),$$

where $\beta_i = \alpha^{2(\lambda_i+n-i)}$. If this is zero, then some $\beta_i = \beta_j$, which implies that α is a root of unity. Thus $|\alpha| = 1$, so π is tempered. ■

2. The metaplectic representation. Witten, Brekke, Freund and Olsen in [1], [10] and [9] considered p -adic analogs of bosonic string theory. This led Ruelle, Thiran, Versteegen and Weyers [27] to consider the p -adic harmonic oscillator, also studied in the recent book of Vladimirov, Volovich and Zelenov [32]. The p -adic harmonic oscillator may be understood in terms of the restriction of the metaplectic representation of the double cover of $SL(2, \mathbb{R})$ on $L^2(\mathbb{R})$ to the group $SO(2)$ of symmetries of the Hamiltonian of a single particle moving in a quadratic potential field. In this formulation, there is no obstacle to replacing \mathbb{R} by an arbitrary local field, and this is the point of view we will take.

Let F be a local field of characteristic not equal to 2. Let $(\ , \)$ denote the Hilbert symbol of F . Let ψ denote a nontrivial additive character of F . Let dx denote the measure on F which is self-dual with respect to the Fourier transform; thus if

$$\hat{f}(x) = \int_F f(y) \psi(2xy) dy,$$

dx is self-dual if $\hat{\hat{f}}(x) = f(-x)$. If $t \in F^\times$, let

$$\gamma(t) = |t|^{1/2} \int_F \psi(tx^2) dx.$$

This oscillatory integral is conditionally convergent in an obvious sense. The absolute value of γ equals 1—indeed it is an eight-th root of unity—and

$$\gamma(a) \gamma(b) = (a, b) \gamma(ab) \gamma(1).$$

Furthermore, we have

$$\gamma(b^2 a) = \gamma(a), \quad \gamma(-a) = \gamma(a)^{-1}.$$

Let $G = SL(2, F)$, and let \tilde{G} be the metaplectic double cover of $SL(2, F)$ defined by Kubota's cocycle $\sigma : G \times G \rightarrow \mu_2 = \{\pm 1\}$. Thus in terms of the Hilbert symbol,

$$\sigma(g_1, g_2) = \left(\frac{X(g_1)}{X(g_1 g_2)}, \frac{X(g_2)}{X(g_1 g_2)} \right),$$

where

$$X \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0; \\ d & \text{otherwise.} \end{cases}$$

Let $\mathbf{s} : G \rightarrow \tilde{G}$ be the standard section, so that

$$\mathbf{s}(g_1) \mathbf{s}(g_2) = \sigma(g_1, g_2) \mathbf{s}(g_1 g_2).$$

We will also use the notation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{s}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \tilde{G}.$$

The metaplectic representation $\omega = \omega_\psi$ is an action of \tilde{G} on the Schwartz space $S(F)$. It is given on generators by

$$\begin{aligned} \left(\omega \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} f\right)(x) &= \psi(tx^2) f(x), \\ \left(\omega \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} f\right)(x) &= \gamma(1) \hat{f}(x), \\ \left(\omega \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} f\right)(x) &= |a|^{1/2} \frac{\gamma(1)}{\gamma(a)} f(ax). \end{aligned}$$

See Weil [35] and Gelbart and Piatetski-Shapiro [12].

Let

$$H = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in F, a^2 + b^2 = 1 \right\},$$

and let \tilde{H} be the preimage of H in \tilde{G} . Let H' be the unique maximal compact subgroup of H , \tilde{H}' its preimage in \tilde{H} . If -1 is not a square in F , then H is compact, so actually $H' = H$ and $\tilde{H}' = \tilde{H}$. On the other hand, if -1 is a square, then $H \cong F^\times$, so H' is a proper subgroup. The action of \tilde{H} on the Schwartz space by means of the metaplectic representation is given by the following formula:

$$\left(\omega \begin{bmatrix} a & -b \\ b & a \end{bmatrix} f\right)(x) = |b|^{-1/2} \gamma(b)^{-1} \int_F \psi\left(\frac{1}{b}(ax^2 - 2xy + ay^2)\right) f(y) dy.$$

If -1 is not a square, so that \tilde{H} is compact, then the restriction of ω to \tilde{H} is multiplicity-free. If $F = \mathbb{R}$, this follows from our proof of Theorem 2 below (though it was known long before by Howe). If F is p -adic, this follows from the Howe duality principle for the dual pair $U(1) \times U(1)$ in $SL(2)$. (Our group $SO(2)$ is the same as $U(1)$.) See Howe [16] and Waldspurger [33] for Howe duality, which is a theorem except in residual characteristic two. Other papers concerned specifically with the character of the metaplectic representation restricted to $SO(2)$ in the case of odd residual characteristic are Moen [22] and Prasad [24]. Tonghai Yang [37] has formulas for the actual eigenfunctions of $U(1)$ acting on the Schwartz space.

In the case of residue characteristic two, the fact that the restriction of the metaplectic representation to compact $SO(2)$ is multiplicity-free is still known. This is implicit in the

work of Rogawski [26], which uses global to local methods, and a purely local proof may be found in Harris, Kudla and Sweet [14]. Also P. Ruelle, E. Thiran, D. Versteegen and J. Weyers [27] have calculated the character of the restriction of the metaplectic representation to tori in the fields \mathbb{Q}_p , including \mathbb{Q}_2 , and their result implies this multiplicity one statement for \mathbb{Q}_2 .

On the other hand if -1 is a square in F , the restriction of ω to \tilde{H} does not decompose into a direct sum of constituents (though its dual space of distributions does so decompose). Instead we will consider the group \tilde{H}' . The restriction of ω to this group is not multiplicity free.

The metaplectic cover splits over \tilde{H} . Indeed, if -1 is not a square, \tilde{H} is contained in $SL(2, \mathfrak{o})$, and an explicit splitting over this maximal compact subgroup was given by Kubota [18]. If we define

$$\kappa \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{cases} -1 & \text{if } v(b) \text{ is odd and } a \equiv -1 \text{ modulo } \mathfrak{p}; \\ 1 & \text{otherwise,} \end{cases}$$

then

$$\sigma(g_1, g_2) = \frac{\kappa(g_1) \kappa(g_2)}{\kappa(g_1 g_2)}$$

when $g_1, g_2 \in H$. (It is worth mentioning that if the valuation $v(b) > 0$, then $a \equiv \pm 1$ modulo \mathfrak{p} since $a^2 + b^2 = 1$.) We may therefore define a representation of the abelian group H by

$$\left(\omega \begin{pmatrix} a & -b \\ b & a \end{pmatrix} f \right) (x) = \kappa \begin{pmatrix} a & -b \\ b & a \end{pmatrix} |b|^{-1/2} \gamma(b)^{-1} \int_F \psi \left(\frac{1}{b} (ax^2 - 2xy + ay^2) \right) f(y) dy.$$

On the other hand, if -1 is a square in F , then H is conjugate to the diagonal torus in $SL(2)$, and it is well known (and easy to prove from Kubota's cocycle formula) that the metaplectic cover splits over this subgroup. Since the cover splits over H' , we may regard ω as giving a representation of this group.

Local Riemann Hypothesis. *Suppose that F is a local field. Assume that F is not complex, and that the characteristic of F is not equal to 2. Let $f \in S(F)$ be an eigenfunction of this action of $H \cap K$, and let ν be a character of F^\times . Then the Mellin transform*

$$\int_F f(x) \nu(x) |x|^s d^\times x,$$

if not identically zero, has its only zeros on the line $\operatorname{re}(s) = \frac{1}{2}$.

This assertion is largely proved, in this paper and its companion piece, Kurlberg [19].

Let us study what happens when we change the additive character. If $\lambda \in F^\times$, let ψ_λ be the character $x \mapsto \psi(\lambda x)$. Let $d_\psi x$ denote the additive Haar measure which is self-dual with respect to ψ . Then $d_{\psi_\lambda} x = |\lambda|^{1/2} d_\psi x$. Let ω_ψ denote the metaplectic representation parametrized by ψ . If $f \in S(F)$, let $f_\lambda(x) = f(\lambda x)$. Then it is easy to see that

$$\omega_{\psi_{\lambda^2}} \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} f_\lambda \right) (x) = \omega_\psi \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} f \right) (\lambda x).$$

Thus if f is an eigenfunction of \tilde{H} under the representation ω_ψ , then f_λ is an eigenfunction of \tilde{H} under $\omega_{\psi_{\lambda^2}}$. The zeros of $\zeta(s, \nu, f)$ and $\zeta(s, \nu, f_\lambda)$ are at the same places, so we have the freedom to change ψ to ψ_{λ^2} for any square λ^2 .

Theorem 2. *The Local Riemann Hypothesis is true if $F = \mathbb{R}$.*

PROOF. We reduce this to Theorem 1. Since we have the freedom to change ψ by a square, we may assume that $\psi(x) = e^{\pm i\pi x}$. We will assume that $\psi(x) = e^{i\pi x}$; the other case is obtained by replacing i by $-i$ throughout the following discussion.

In this case, the self-dual measure on \mathbb{R} coincides with Lebesgue measure, and

$$\gamma(1) = \int_{-\infty}^{\infty} e^{i\pi x^2} dx = \lim_{t \rightarrow 0^+} \int_{-\infty}^{\infty} e^{-\pi(t-i)x^2} dx = \lim_{t \rightarrow 0^+} (t-i)^{-1/2} = \frac{1}{\sqrt{2}}(1-i).$$

Let \mathfrak{g} be the Lie algebra of $SL(2, \mathbb{R})$. The exponential map $\mathfrak{g} \rightarrow SL(2, \mathbb{R})$ lifts to a map $\widetilde{\exp} : \mathfrak{g} \rightarrow \tilde{G}$. We then have a representation $d\omega$ of \mathfrak{g} on $S(\mathbb{R})$ by

$$((d\omega X)(f))(x) = \frac{d}{dt} (\widetilde{\exp}(tX) f)(x)|_{t=0}.$$

Let $\mathfrak{F} : S(\mathbb{R}) \rightarrow S(\mathbb{R})$ denote the Fourier transform $\mathfrak{F}f = \hat{F}$, and let \mathfrak{F}^{-1} be its inverse:

$$(\mathfrak{F}^{-1}f)(x) = \int_{-\infty}^{\infty} f(y) e^{-2\pi ixy} dy.$$

Define ‘‘momentum’’ and ‘‘position’’ operators P and Q on the Schwartz space by

$$(Pf)(x) = \frac{1}{2\pi i} \frac{df}{dx}(x), \quad (Qf)(x) = x f(x).$$

We have

$$\mathfrak{F}^{-1} Q^2 \mathfrak{F} = P^2.$$

Indeed, $(\mathfrak{F}^{-1} Q^2 \mathfrak{F} f)(x)$ equals

$$\int_{-\infty}^{\infty} y^2 \hat{f}(y) e^{-2\pi ixy} dy = -\frac{1}{4\pi^2} \frac{d^2}{dx^2} \int_{-\infty}^{\infty} \hat{f}(y) e^{-2\pi ixy} dy = -\frac{1}{4\pi^2} \frac{d^2 f}{dx^2}(x).$$

We now prove that

$$d\omega \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = i\pi Q^2, \quad d\omega \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = i\pi P^2.$$

The first identity follows directly from the definitions:

$$\left(d\omega \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f \right) (x) = \frac{d}{dt} \left(\omega \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} f \right) (x) \Big|_{t=0} = \frac{d}{dt} e^{i\pi x^2 t} f(x) \Big|_{t=0} = i\pi x^2 f(x).$$

Since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

we have

$$d\omega \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \left(\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \left(d\omega \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \left(\omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) = i\pi \mathfrak{F}^{-1} Q^2 \mathfrak{F},$$

and so the second identity follows from the first.

Now suppose that f is an eigenfunction of \tilde{H} . Since

$$\tilde{H} = \widetilde{\exp} \left(\mathbb{R} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right),$$

f is also an eigenfunction of

$$d\omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = d\omega \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + d\omega \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = i\pi(P^2 + Q^2),$$

which is (up to constant) the oscillator Hamiltonian. Hence f is one of the functions f_n .

There are two possibilities for ν : $\nu(x) = \text{sgn}(x)^\delta$, where $\delta = 0$ or 1 . Depending on whether f is even or odd, exactly one of the integrals $\int f(x) \nu(x) |x|^s dx/x$ will be nonzero, and this one will be just twice the Mellin transform of f . Consequently, Theorem 2 follows from Theorem 1. ■

We turn now to the case of a p -adic field F . In this case, following some preliminary investigation by Bump and Hoffstein, Kurlberg [19] has proved:

Theorem 3. *The Local Riemann Hypothesis is true if F is a nonarchimedean local field of odd residue characteristic.*

On the other hand, Kurlberg has also shown that the Local Riemann Hypothesis is false if $F = \mathbb{C}$.

3. Laguerre polynomials, the n -dimensional harmonic oscillator and a reciprocity law. The *Laguerre polynomials* (cf. Rainville [25]) are defined by:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \sum_{k=0}^n \frac{(1+\alpha)_n (-x)^k}{k! (n-k)! (1+\alpha)_k},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. They satisfy the differential equation

$$x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (1+\alpha-x) \frac{d}{dx} L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0,$$

and the orthogonality relation:

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{\Gamma(1+\alpha+n)}{n!} & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}_n^{(\alpha)}(x) = x^{\alpha/2} e^{-x/2} L_n^{(\alpha)}(x)$. Then the *Laguerre functions* $\mathcal{L}_n^{(\alpha)}$ are orthogonal with respect to Lebesgue measure on $[0, \infty)$. Their Mellin transforms

$$\mathcal{M}_n^{(\alpha)}(s) = \int_0^\infty \mathcal{L}_n^{(\alpha)}(x) x^{s-1} dx = 2^{s+\frac{\alpha}{2}} \Gamma\left(s + \frac{\alpha}{2}\right) P_n^{(\alpha)}(s),$$

where

$$P_n^{(\alpha)}(s) = \sum_{k=0}^n 2^k \binom{n+\alpha}{n-k} \binom{-s-\frac{\alpha}{2}}{k}.$$

Theorem 4. *The zeros of $P_n^{(\alpha)}(s)$ lie on the line $\operatorname{re}(s) = \frac{1}{2}$.*

PROOF. The first proof of Theorem 1 is easily adapted. Using the orthogonality of the Laguerre functions, we see that the polynomials $P_n^{(\alpha)}\left(\frac{1}{2} + it\right)$ are orthogonal with respect to the measure $2^{1+\alpha} |\Gamma\left(\frac{1}{2} + \frac{\alpha}{2} + it\right)|^2 dt$, and their zeros are therefore real. ■

The polynomials $P_n^{(\alpha)}(s)$ satisfy a functional equation:

$$P_n^{(\alpha)}(s) = (-1)^n P_n^{(\alpha)}(1-s).$$

We may prove this as follows. We start with the generating function for the Laguerre polynomials (Rainville [25], p. 202):

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n = (1-t)^{-1-\alpha} e^{-xt/(1-t)}.$$

Taking the Mellin transform in this identity yields

$$\sum_{n=0}^{\infty} P_n^{(\alpha)}(s) t^n = (1-t)^{s-1-\alpha/2} (1+t)^{-s-\alpha/2},$$

whence the functional equation.

Now let us investigate the harmonic oscillator in n -dimensions. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $r = |x| = \sqrt{\sum_i x_i^2}$ be the radial distance from the origin, and let Δ be the n -dimensional Laplacian $\sum_i \partial^2/\partial x_i^2$. Then consider the Schrödinger equation corresponding to a quadratic potential $V(r) = r^2$:

$$(4) \quad (-\Delta + r^2)\phi = \epsilon\phi.$$

The eigenvalue ϵ is the energy level. The potential is rotationally symmetric and the Hamiltonian $-\Delta + r^2$ commutes with the orthogonal group. We may thus restrict ourselves to ϕ which lie in an irreducible subspace of $O(n)$.

Theorem 5. *Let ϕ be a solution to (4) lying in an irreducible subspace of $O(n)$. Let X be any radially symmetric function on \mathbb{R}^n , so that $X(tx) = X(x)$. Then the Mellin transform*

$$(5) \quad \int_{\mathbb{R}^n} \phi(x) X(x) |x|^{2s-\frac{n}{2}-1} dx$$

has its zeros on the line $\operatorname{re}(s) = 1/2$.

PROOF. We make use of spherical coordinates. Thus if $x \in \mathbb{R}^n$ is given, we take $r = |x| \in \mathbb{R}^+$ and $\xi = x/|x| \in S^{n-1}$ as basic coordinates. The group $O(n)$ acts on $L^2(S^{n-1})$, which decomposes as a direct sum of irreducible subspaces, each with multiplicity one. Because of this, our assumption that ϕ lies in an irreducible subspace of $O(n)$ implies that ϕ may be written in the form $\phi_0(r)\Phi(\xi)$, where Φ lies in one of these irreducible subspaces of $L^2(S^{n-1})$. Since $dx = r^{n-1} dr d\xi$, the integral equals

$$(6) \quad \int_0^{\infty} \phi_0(r) r^{2s+\frac{n}{2}-1} \frac{dr}{r}$$

times the inner product on S^{n-1} of X and Φ . In spherical coordinates, the Laplacian in n dimensions has the form:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Lambda,$$

where Λ is the Laplacian on S^{n-1} (Helgason, *Groups and Geometric Analysis* p.16). Moreover, the eigenvalue of Λ on an element of an irreducible subspace of S^{n-1} is equal to the eigenvalue of the Casimir operator on the corresponding irreducible representation, which Helgason shows has the form $-l(l+n-2)$, where $l \in \mathbb{Z}$. We thus have the differential equation (with eigenvalue λ for Λ):

$$\phi_0'' + \frac{n-1}{r} \phi_0' + \left(\frac{-l(l+n-2)}{r^2} - r^2 + \epsilon \right) \phi_0 = 0.$$

In order for $\phi_0 = e^{-r^2/2} r^l L(r^2)$ to satisfy this differential equation, we need

$$r L'' + \left(l + \frac{n}{2} - r \right) L' + \left(\frac{\epsilon}{4} - \frac{l}{2} - \frac{n}{4} \right) L = 0.$$

This differential equation has a regular singular point at the origin, and a solution that is well-behaved there must be a constant multiple of $L = L_k^{(l+\frac{n}{2}-1)}$, where k is an integer, and $\epsilon = 4k + 2l + n$. The result now follows from Theorem 4. ■

We note that this setup can be adapted to the metaplectic group by means of the Weil representation. The eigenfunctions at hand live in irreducible subspaces for the group $O(2) \times O(n)$, which is a maximal compact subgroup of the dual pair $SL(2, \mathbb{R}) \times O(n)$ in $Sp(2n, \mathbb{R})$, acting on $L^r(\mathbb{R}^n)$ via the standard polarization in the Weil representation. Expressed this way, the integrals of Theorem 5 have p -adic analogs, and though we haven't had a chance to investigate whether these satisfy a Riemann hypothesis, we hazard to conjecture that they do, at least in the case of anisotropic $O(n)$.

The polynomials $P_n^{(\alpha)}$ satisfy a *reciprocity law* relating their values at negative integers. We will show that

$$(7) \quad \binom{m+\alpha}{m} P_n^{(\alpha)}\left(-m - \frac{\alpha}{2}\right) = \binom{n+\alpha}{n} P_m\left(-n - \frac{\alpha}{2}\right).$$

Indeed, the left side equals

$$\sum_{k=0}^n 2^k \binom{m+\alpha}{m} \binom{n+\alpha}{n-k} \binom{m}{k},$$

and the reciprocity law follows from the identity

$$\binom{m+\alpha}{m} \binom{n+\alpha}{n-k} \binom{m}{k} = \binom{n+\alpha}{n} \binom{m+\alpha}{m-k} \binom{n}{k}.$$

We note the special case

$$(8) \quad P_n^{(0)}(-m) = P_m^{(0)}(-n).$$

This identity has an interesting *combinatorial interpretation*.

Theorem 6. $P_n^{(0)}(-m)$ equals the number of lattice points $(x_1, \dots, x_n) \in \mathbb{Z}^n$ such that $\sum |x_i| \leq m$.

PROOF. We can count the number of lattice points in \mathbb{Z}^n satisfying $\sum |x_i| \leq m$ as follows. The number of lattice points having exactly k nonzero entries is $2^k \binom{n}{k} \binom{m}{k}$ if $0 \leq k \leq \min(m, n)$, because there are $\binom{n}{k}$ choices for which coordinates shall be nonzero; and once this choice is fixed, there are 2^k possible distributions of signs, and $\binom{m}{k}$ possible distributions of absolute values. Hence the number of lattice points is

$$\sum_{k=0}^{\min(m,n)} 2^k \binom{n}{k} \binom{m}{k} = P_n^{(0)}(-m).$$

This completes the proof. ■

We derive a generating function for $P_n^{(0)}(-m)$. Let $a(m, n)$ be the number of lattice points satisfying the condition on the theorem. Then $a(m, n) - a(m, n-1)$ is the number of lattice points satisfying exactly $\sum |x_i| \leq m$ having a nonzero last component. If the last component is $\pm m - k$, with $0 \leq k \leq m-1$, then the number of possibilities for the first $n-1$ components is $a(k, n-1)$, and so we have

$$a(m, n) - a(m, n-1) = 2 \sum_{k=0}^{m-1} a(k, n-1).$$

Hence (assuming $m, n > 0$) we have

$$a(m, n) - a(m, n-1) - a(m-1, n) + a(m-1, n-1) = 2a(m-1, n-1),$$

which leads to the recursion

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a(m, n) x^m y^n = (1 - x - y - xy)^{-1}.$$

The reciprocity law (8) is reflected by the symmetry of the generating function.

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