Quadratic programming theory over the rational numbers

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Introduction

This presentation deals with existence of optimal solutions and duality theory for quadratic programming, in particular applied to a natural primal-dual pair of problems.

- All variables and input data are rational numbers. $(x_j \in \mathsf{Q})$
- Only elementary linear algebra in Q^n is used.
- "Real-number concepts" like square roots, infimum, derivatives, compact sets, limits, eigenvalues of matrices, etc. are avoided.
- But every single result holds, and every single proof is valid, also if Q is everywhere replaced by $\mathbb{R}$.

Quadratic programming problem on standard form

Consider first the following QP problem on standard form:

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{F}\mathbf{x} + \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{1}$$

where

$\mathbf{x} \in \mathbb{Q}^n$ is the variable vector,

$\mathbf{F} \in \mathbb{Q}^{n\times n}$ is symmetric and positive semidefinite,

$\mathbf{A} \in \mathbb{Q}^{m\times n}$, $\mathbf{b} \in \mathbb{Q}^m$ and $\mathbf{c} \in \mathbb{Q}^n$.

$\mathcal{F} = \{\, \mathbf{x} \in \mathbb{Q}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \,\}$, (the feasible region),

$\mathcal{D} = \{\, \mathbf{d} \in \mathbb{Q}^n \mid \mathbf{d} \geq \mathbf{0},\ \mathbf{A}\mathbf{d} = \mathbf{0},\ \mathbf{F}\mathbf{d} = \mathbf{0} \text{ and } \mathbf{c}^{\mathsf{T}}\mathbf{d} < 0 \,\}$.

First claim: There is at least one optimal solution to (1) if and only if the set $\mathcal{F}$ is non-empty while the set $\mathcal{D}$ is empty.

Partitioning of the feasible region

$\mathcal{F} = \{\, \mathbf{x} \in \mathsf{Q}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \,\} =$ the feasible region.

Assume that σ is a subset of the index set $\{1, \ldots, n\}$. Then let

$$\mathcal{F}_\sigma^+ = \{\mathbf{x} \in \mathcal{F} \mid x_j > 0 \text{ for } j \in \sigma, \text{ while } x_j = 0 \text{ for } j \notin \sigma\}.$$

There are 2^n different subsets σ of $\{1, \ldots, n\}$.
The corresponding 2^n regions $\mathcal{F}_\sigma^+$ are pair-wise disjunct,
and their union is equal to $\mathcal{F}$, since for each point $\mathbf{x} \in \mathcal{F}$ there is
a unique subset σ such that $\mathbf{x} \in \mathcal{F}_\sigma^+$, namely $\sigma = \{\, j \mid x_j > 0 \,\}$.

Note that $\mathcal{F}_\sigma^+$ may be empty for some subsets σ.

Interesting points

Consider the following QP problem without any sign constraints but with some variables fixed to zero:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = \tfrac{1}{2}\,\mathbf{x}^{\mathsf{T}}\mathbf{F}\mathbf{x} + \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & x_j = 0 \text{ for } j \notin \sigma. \end{aligned} \tag{2}$$

The subset σ is said to be an interesting subset if (2) has a unique optimal solution $\hat{\mathbf{x}}$, and $\hat{\mathbf{x}} \in \mathcal{F}_\sigma^+$, i.e. $x_j > 0$ for $j \in \sigma$. In this case, $\hat{\mathbf{x}}$ is said to be an interesting point.

Otherwise, if (2) does not have a unique optimal solution, or if (2) has a unique optimal solution $\hat{\mathbf{x}}$ but $\hat{\mathbf{x}} \notin \mathcal{F}_\sigma^+$, σ is said to be a boring subset, without any interesting point.

Most interesting point and optimal solution

If $\mathcal{F} \neq \emptyset$ then there is at least one and at most 2^n interesting points.

An interesting point $\hat{\mathbf{x}}$ is said to be a most interesting point if $f(\hat{\mathbf{x}}) \leq f(\mathbf{x})$ for all interesting points $\mathbf{x}$.

If $\mathcal{F} \neq \emptyset$ then there is at least one most interesting point.

Theorem: If $\mathcal{F} \neq \emptyset$, $\mathcal{D} = \emptyset$ and $\hat{\mathbf{x}}$ is a most interesting point, then $\hat{\mathbf{x}}$ is an optimal solution to the considered problem (1).

Main step in proof: If $\mathcal{F} \neq \emptyset$, $\mathcal{D} = \emptyset$ and σ is a boring subset with $\mathcal{F}_\sigma^+ \neq \emptyset$ then, for each point $\mathbf{x} \in \mathcal{F}_\sigma^+$, there is a strict subset $\tilde{\sigma} \subset \sigma$ and a point $\mathbf{x} + t\,\mathbf{d} \in \mathcal{F}_{\tilde{\sigma}}^+$ such that $f(\mathbf{x} + t\,\mathbf{d}) \leq f(\mathbf{x})$.

Sign-constrained least squares problems

An important special case is the sign-constrained least squares problem:

$$\text{minimize } \tfrac{1}{2}\,\|\mathbf{B}\mathbf{x}-\mathbf{p}\|^2 \text{ subject to } \mathbf{x} \geq \mathbf{0}. \tag{3}$$

where $\mathbf{B} \in \mathsf{Q}^{k\times n}$ is a given matrix and $\mathbf{p} \in \mathsf{Q}^k$ is a given vector.

This is a problem of the form (1) with $\mathbf{F} = \mathbf{B}^{\mathsf{T}}\mathbf{B}$, $\mathbf{c} = -\mathbf{B}^{\mathsf{T}}\mathbf{p}$, and no equality constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then

$$\mathcal{F} = \{\, \mathbf{x} \in \mathsf{Q}^n \mid \mathbf{x} \geq \mathbf{0} \,\} \neq \emptyset \quad \text{and}$$

$$\mathcal{D} = \{\, \mathbf{d} \in \mathsf{Q}^n \mid \mathbf{d} \geq \mathbf{0},\ \mathbf{F}\mathbf{d} = \mathbf{0} \text{ and } \mathbf{c}^{\mathsf{T}}\mathbf{d} < 0 \,\} =$$

$$= \{\, \mathbf{d} \in \mathsf{Q}^n \mid \mathbf{d} \geq \mathbf{0},\ \mathbf{B}^{\mathsf{T}}\mathbf{B}\,\mathbf{d} = \mathbf{0} \text{ and } \mathbf{p}^{\mathsf{T}}\mathbf{B}\,\mathbf{d} > 0 \,\} = \emptyset.$$

Thus, there is always at least one optimal solution to (3).

Moreover, it is easy to deduce optimality conditions:

The point $\hat{\mathbf{x}}$ is an optimal solution to (3) if and only if
$\hat{\mathbf{x}} \geq \mathbf{0}$, $\mathbf{B}^{\mathsf{T}}(\mathbf{B}\hat{\mathbf{x}}-\mathbf{p}) \geq \mathbf{0}$ and $\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}(\mathbf{B}\hat{\mathbf{x}}-\mathbf{p}) = 0$.

An elementary proof of Farkas lemma

Farkas lemma: Assume that $\mathbf{B} \in \mathrm{Q}^{k \times n}$ and $\mathbf{p} \in \mathrm{Q}^{k}$ are given.
Then exactly one of the following two systems has a solution.

System 1: $\mathbf{B}\mathbf{x} = \mathbf{p}$ and $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{x} \in \mathrm{Q}^{n}$

System 2: $\mathbf{B}^{\mathsf{T}}\mathbf{y} \geq \mathbf{0}$ and $\mathbf{p}^{\mathsf{T}}\mathbf{y} < 0$, where $\mathbf{y} \in \mathrm{Q}^{k}$.

Proof: First, assume that there is a solution $\hat{\mathbf{x}}$ to System 1.
Then $\mathbf{p}^{\mathsf{T}}\mathbf{y} = (\mathbf{B}\hat{\mathbf{x}})^{\mathsf{T}}\mathbf{y} = \hat{\mathbf{x}}^{\mathsf{T}}(\mathbf{B}^{\mathsf{T}}\mathbf{y})$, which is ≥ 0 if $\mathbf{B}^{\mathsf{T}}\mathbf{y} \geq \mathbf{0}$.

Next, assume that there is no solution to System 1.
Let $\hat{\mathbf{y}} = \mathbf{B}\hat{\mathbf{x}} - \mathbf{p}$, where $\hat{\mathbf{x}}$ is an optimal solution to (3).
Then $\hat{\mathbf{y}} \neq \mathbf{0}$, since there is no solution to System 1.
From the previous slide, $\hat{\mathbf{x}} \geq \mathbf{0}$, $\mathbf{B}^{\mathsf{T}}\hat{\mathbf{y}} \geq \mathbf{0}$ and $\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\hat{\mathbf{y}} = 0$, which implies that $\mathbf{p}^{\mathsf{T}}\hat{\mathbf{y}} = (\mathbf{B}\hat{\mathbf{x}} - \hat{\mathbf{y}})^{\mathsf{T}}\hat{\mathbf{y}} = \hat{\mathbf{x}}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\hat{\mathbf{y}} - \hat{\mathbf{y}}^{\mathsf{T}}\hat{\mathbf{y}} = 0 - \|\,\hat{\mathbf{y}}\,\|^2 < 0$.
Thus, $\hat{\mathbf{y}}$ is a solution to System 2.

QP problem on standard form, revisited

Now return to the original problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{F}\mathbf{x} + \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{4}$$

with $\mathcal{F} = \{\, \mathbf{x} \in \mathsf{Q}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b} \ \text{and} \ \mathbf{x} \geq \mathbf{0} \,\}$ = the feasible region
and $\mathcal{D} = \{\, \mathbf{d} \in \mathsf{Q}^n \mid \mathbf{d} \geq \mathbf{0},\ \mathbf{A}\mathbf{d} = \mathbf{0},\ \mathbf{F}\mathbf{d} = \mathbf{0} \ \text{and} \ \mathbf{c}^{\mathsf{T}}\mathbf{d} < 0 \,\}$.
Let $\mathcal{G} = \{\, (\mathbf{y}, \mathbf{v}) \in \mathsf{Q}^m \times \mathsf{Q}^n \mid \mathbf{A}^{\mathsf{T}}\mathbf{y} - \mathbf{F}\mathbf{v} \leq \mathbf{c} \,\}$.
Then Farkas lemma implies that $\mathcal{D} = \emptyset$ if and only if $\mathcal{G} \neq \emptyset$.

Thus, there is at least one optimal solution to (4) if and only if
$\mathcal{F} \neq \emptyset$ and $\mathcal{G} \neq \emptyset$.

Saddle points

Assume that

X is a given non-empty set in Q^n,

Y is a given non-empty set in Q^m, and

$h : X \times Y \to$ Q is a given function.

The point $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \in X \times Y$ is said to be a saddle point to $h(\mathbf{x}, \mathbf{y})$ (for minimizing w.r.t. $\mathbf{x} \in X$ and maximizing w.r.t. $\mathbf{y} \in Y$) if

$$h(\hat{\mathbf{x}}, \mathbf{y}) \leq h(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \leq h(\mathbf{x}, \hat{\mathbf{y}}) \text{ for all } (\mathbf{x}, \mathbf{y}) \in X \times Y.$$

Primal and dual problems

Let the sets $\mathcal{F} \subseteq X$ and $\mathcal{G} \subseteq Y$ be defined by

$$\mathcal{F} = \{\mathbf{x} \in X \mid \max_{\mathbf{y} \in Y} h(\mathbf{x}, \mathbf{y}) \text{ exists}\},$$
$$\mathcal{G} = \{\mathbf{y} \in Y \mid \min_{\mathbf{x} \in X} h(\mathbf{x}, \mathbf{y}) \text{ exists}\}.$$

Further, let the functions $f : \mathcal{F} \to \mathbb{Q}$ and $g : \mathcal{G} \to \mathbb{Q}$ be defined by

$$f(\mathbf{x}) = \max_{\mathbf{y} \in Y} h(\mathbf{x}, \mathbf{y}),$$
$$g(\mathbf{y}) = \min_{\mathbf{x} \in X} h(\mathbf{x}, \mathbf{y}).$$

Finally, let the primal problem P and the dual problem D be defined by

P : minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in \mathcal{F}$.

D : maximize $g(\mathbf{y})$ subject to $\mathbf{y} \in \mathcal{G}$.

Connection between P, D and saddle points

Some easily proven facts:

- If $\mathbf{x} \in \mathcal{F}$ and $\mathbf{y} \in \mathcal{G}$ then $g(\mathbf{y}) \leq h(\mathbf{x}, \mathbf{y}) \leq f(\mathbf{x})$.

- If $\hat{\mathbf{x}} \in \mathcal{F}$, $\hat{\mathbf{y}} \in \mathcal{G}$ and $f(\hat{\mathbf{x}}) = g(\hat{\mathbf{y}})$ then
  $\hat{\mathbf{x}}$ is an optimal solution to P, and
  $\hat{\mathbf{y}}$ is an optimal solution to D.

- $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a saddle point to $h(\mathbf{x}, \mathbf{y})$ if and only if
  $\hat{\mathbf{x}} \in \mathcal{F}$, $\hat{\mathbf{y}} \in \mathcal{G}$ and $f(\hat{\mathbf{x}}) = g(\hat{\mathbf{y}})$.

- $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a saddle point to $h(\mathbf{x}, \mathbf{y})$ if and only if
  $\hat{\mathbf{x}}$ is an optimal solution to P,
  $\hat{\mathbf{y}}$ is an optimal solution to D,
  and the optimal values to P and D are equal.

A quadratic convex-concave saddle funktion $h(\mathbf{x},\mathbf{y})$

Assume that $X = \{\mathbf{x} \in \mathsf{Q}^n \mid \mathbf{x} \geq \mathbf{0}\}$ and $Y = \{\mathbf{y} \in \mathsf{Q}^m \mid \mathbf{y} \geq \mathbf{0}\}$, while $h(\mathbf{x},\mathbf{y})$ is a general quadratic function which is convex in $\mathbf{x}$ (for each fixt $\mathbf{y} \in Y$) and concave in $\mathbf{y}$ (for each fixt $\mathbf{x} \in X$):

$$h(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

where $\mathbf{h}_1 \in \mathsf{Q}^n$ and $\mathbf{h}_2 \in \mathsf{Q}^m$ are given vectors, while $\mathbf{H}_{11} \in \mathsf{Q}^{n\times n}$, $\mathbf{H}_{12} \in \mathsf{Q}^{n\times m}$, $\mathbf{H}_{21} \in \mathsf{Q}^{m\times n}$ and $\mathbf{H}_{22} \in \mathsf{Q}^{m\times m}$ are given matrices such that $\mathbf{H}_{11}$ and $\mathbf{H}_{22}$ are symmetric and $\mathbf{H}_{21} = \mathbf{H}_{12}^{\mathsf{T}}$.
The requirements that $h(\mathbf{x},\mathbf{y})$ should be convex in $\mathbf{x}$ and concave in $\mathbf{y}$ are fulfilled if and only if $\mathbf{H}_{11}$ is positive semidefinite and $\mathbf{H}_{22}$ is negative semidefinite.

A quadratic convex-concave saddle funktion $h(\mathbf{x}, \mathbf{y})$

The notations are now changed as follows:

$$\begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{b} \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{F} & -\mathbf{A}^\mathsf{T} \\ -\mathbf{A} & -\mathbf{G} \end{bmatrix}$$

where both $\mathbf{F}$ and $\mathbf{G}$ are symmetric and positive semidefinite. Then

$$h(\mathbf{x}, \mathbf{y}) = \mathbf{c}^\mathsf{T}\mathbf{x} + \mathbf{b}^\mathsf{T}\mathbf{y} - \mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} + \tfrac{1}{2}\,\mathbf{x}^\mathsf{T}\mathbf{F}\mathbf{x} - \tfrac{1}{2}\,\mathbf{y}^\mathsf{T}\mathbf{G}\mathbf{y},$$

while, as already mentioned above,

$$X = \{\mathbf{x} \in \mathsf{Q}^n \mid \mathbf{x} \geq \mathbf{0}\} \;\; \text{and} \;\; Y = \{\mathbf{y} \in \mathsf{Q}^m \mid \mathbf{y} \geq \mathbf{0}\}.$$

Saddle point conditions

Assume that X, Y and $h(\mathbf{x},\mathbf{y})$ are defined as above.

Then $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ is a saddle point, i.e. satisfies the inequalities
$h(\hat{\mathbf{x}},\mathbf{y}) \leq h(\hat{\mathbf{x}},\hat{\mathbf{y}}) \leq h(\mathbf{x},\hat{\mathbf{y}})$ for all $(\mathbf{x},\mathbf{y}) \in X\times Y$,
if and only if the following conditions hold:

$$
\begin{aligned}
&\mathbf{A}\hat{\mathbf{x}} + \mathbf{G}\hat{\mathbf{y}} \geq \mathbf{b},\\
&\mathbf{A}^{\mathsf{T}}\hat{\mathbf{y}} - \mathbf{F}\hat{\mathbf{x}} \leq \mathbf{c},\\
&\hat{\mathbf{x}} \geq \mathbf{0},\\
&\hat{\mathbf{y}} \geq \mathbf{0},\\
&\hat{\mathbf{y}}^{\mathsf{T}}(\mathbf{A}\hat{\mathbf{x}} + \mathbf{G}\hat{\mathbf{y}} - \mathbf{b}) = 0,\\
&\hat{\mathbf{x}}^{\mathsf{T}}(\mathbf{c} - \mathbf{A}^{\mathsf{T}}\hat{\mathbf{y}} + \mathbf{F}\hat{\mathbf{x}}) = 0.
\end{aligned}
$$

The natural pair of primal and dual QP problems

With X, Y and $h(\mathbf{x},\mathbf{y})$ as above, the primal problem is equivalent to the following QP problem in the variables $\mathbf{x} \in \mathsf{Q}^n$ and $\mathbf{u} \in \mathsf{Q}^m$:

$$\begin{aligned} \mathsf{P}: \quad & \text{minimize} && \mathbf{c}^\mathsf{T}\mathbf{x} + \tfrac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{F}\mathbf{x} + \tfrac{1}{2}\mathbf{u}^\mathsf{T}\mathbf{G}\mathbf{u} \\ & \text{subject to} && \mathbf{A}\mathbf{x} + \mathbf{G}\mathbf{u} \geq \mathbf{b}, \\ &&& \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

while the dual problem is equivalent to the following QP problem in the variables $\mathbf{y} \in \mathsf{Q}^m$ and $\mathbf{v} \in \mathsf{Q}^n$:

$$\begin{aligned} \mathsf{D}: \quad & \text{maximize} && \mathbf{b}^\mathsf{T}\mathbf{y} - \tfrac{1}{2}\mathbf{y}^\mathsf{T}\mathbf{G}\mathbf{y} - \tfrac{1}{2}\mathbf{v}^\mathsf{T}\mathbf{F}\mathbf{v} \\ & \text{subject to} && \mathbf{A}^\mathsf{T}\mathbf{y} - \mathbf{F}\mathbf{v} \leq \mathbf{c}, \\ &&& \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Properties of the natural pair of primal and dual QP problems

Let $\mathcal{F}$ and $\mathcal{G}$ denote the feasible regions for P and D, respectively, i.e.

$\mathcal{F} = \{ (\mathbf{x}, \mathbf{u}) \in \mathsf{Q}^n \times \mathsf{Q}^m \mid \mathbf{Ax} + \mathbf{Gu} \geq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \}$ and

$\mathcal{G} = \{ (\mathbf{y}, \mathbf{v}) \in \mathsf{Q}^m \times \mathsf{Q}^n \mid \mathbf{A}^\mathsf{T}\mathbf{y} - \mathbf{Fv} \leq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \}$.

Then the following equivalences hold:

$\mathcal{F} \neq \emptyset$ and $\mathcal{G} \neq \emptyset \quad \Leftrightarrow$

There is at least one optimal solution to P $\quad \Leftrightarrow$

There is at least one optimal solution to D $\quad \Leftrightarrow$

There is at least one saddle point.

Moreover:

If $\mathcal{F} \neq \emptyset$ and $\mathcal{G} \neq \emptyset$ then the optimal values of P and D are equal.

More properties of the natural pair of QP problems

Some additional (easily proven) results:

- Assume that $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is optimal to P and $(\hat{\mathbf{y}}, \hat{\mathbf{v}})$ is optimal to D.
  Then $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a saddle point.

- Assume that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a saddle point.
  Then $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is optimal to P if and only if $\mathbf{G}\hat{\mathbf{u}} = \mathbf{G}\hat{\mathbf{y}}$,
  and $(\hat{\mathbf{y}}, \hat{\mathbf{v}})$ is optimal to D if and only if $\mathbf{F}\hat{\mathbf{v}} = \mathbf{F}\hat{\mathbf{x}}$.
  In particular, $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is optimal to P and $(\hat{\mathbf{y}}, \hat{\mathbf{x}})$ is optimal to D.

- $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ is optimal to P if and only if there is vector $\hat{\mathbf{y}}$
  such that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a saddle point and $\mathbf{G}\hat{\mathbf{u}} = \mathbf{G}\hat{\mathbf{y}}$.
  $(\hat{\mathbf{y}}, \hat{\mathbf{v}})$ is optimal to D if and only if there is vector $\hat{\mathbf{x}}$
  such that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a saddle point and $\mathbf{F}\hat{\mathbf{v}} = \mathbf{F}\hat{\mathbf{x}}$.