# Distributed Event-Based Control

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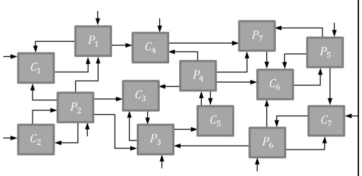
# Motivation

Networked control systems have time-varying communications influencing their global performance

Network topology depends on
- Internal events: states, controls
- External events: disturbances, outages



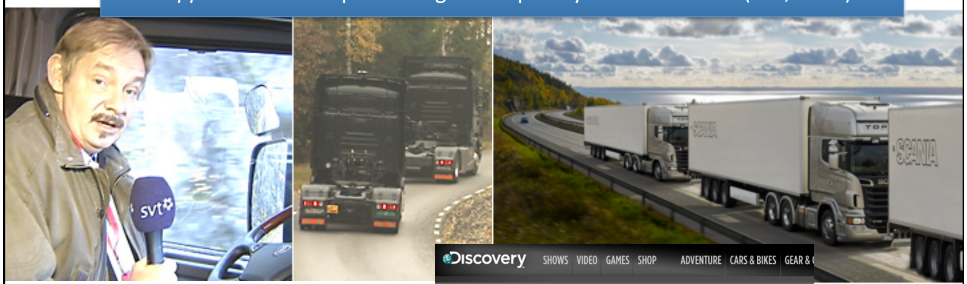
Need for integrated design of control and communication



Process industry

# Truck Platooning for Fuel Reduction

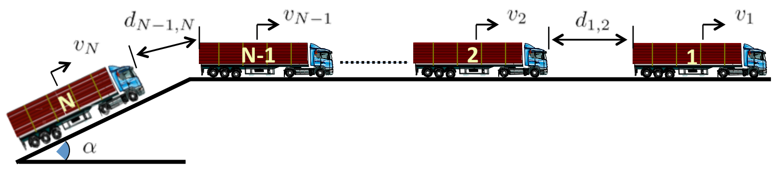
*Rapport* on vehicle platooning developed by KTH and Scania (Oct, 2011)





PhD student Assad Alam on *Discovery Channel* (Jan, 2012)

# Networked Control in Platooning




- Platooning control applications require **collaborative actions**
  - Fuel-efficient adaptive cruise controllers
  - Collaborative route planning
  - Autonomous safety maneuvers
- Vehicles need **accurate estimates** of neighboring vehicles' states and actions
- Control performance is tightly coupled to how well data (position, velocity, breaking estimates) are communicated across the platoon

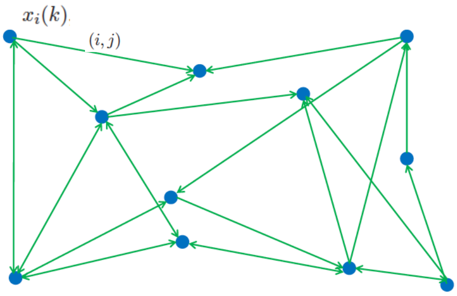
- How does the communication influence the system performance?
- What is an efficient communication strategy for specific control tasks?

# Mathematical Model

Directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Node set $\mathcal{V} = \{1, 2, \ldots, n\}$

Arc $e = (i, j) \in \mathcal{E}$




Time-varying graph process

$$\mathcal{G}_k(\omega) = (\mathcal{V}, \mathcal{E}_k(\omega)), k = 0, 1, \ldots$$

To each node $i \in \mathcal{V}$, associate a scalar state $x_i(k)$

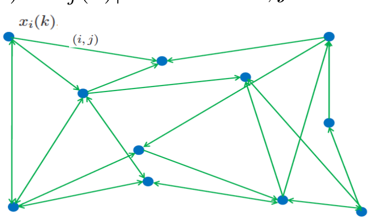
$x_i$ updates based on own computation and neighbor information

$$\mathcal{N}_i(k) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}_k\} \cup \{i\}$$


# Objective

Control the states to agreement: $\lim_{k \to \infty} |x_i(k) - x_j(k)| = 0$ for all $i, j \in \mathcal{V}$

Also called *consensus*, *rendezvous*, *formation*, etc




Local update law

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) x_j(k)$$

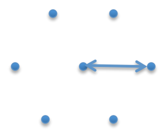
Prototype model for a collaborative control problem with coupled network and node dynamics

**Related work on Markov chains, belief evolution, consensus algorithms, distributed control etc:**
Hajnal (1958), Wolfowitz (1963), DeGroot (1974), Tsitsiklis, Bertsekas & Athans (1986), Jadbabaie, Lin & Morse (2003), Moreau (2005), Ren & Beard (2005), Golub & Jackson (2007), Cao, Anderson & Morse (2008), Acemoglu, Ozdaglar & ParandehGheib (2010), etc

# A Gossip Algorithm

At each $k$, select a pair of nodes that "gossip":

$$x_i(k+1) = \begin{cases} [x_i(k) + x_j(k)]/2 & \text{if } (i,j) \text{ or } (j,i) \text{ is selected} \\ x_i(k) & \text{otherwise} \end{cases}$$



Equivalently

$$x(k+1) = A_k x(k)$$

where $A_k \in \mathcal{A}$ with

$$\mathcal{A} = \{I - (e_i - e_j)(e_i - e_j)^T/2 : i, j \in \mathcal{V}\}$$

and $e_m$ is the unit vector

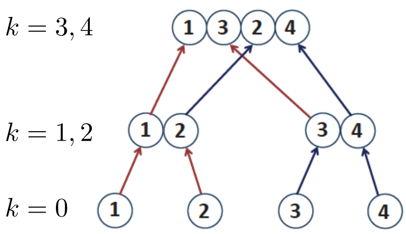
Can the gossiping pairs be selected to achieve finite-time convergence ?

Various bounds on the convergence time to asymptotic consensus, e.g., Karp et al. (2000), Kempe et al. (2003), Boyd et al., (2006), Shah (2008), Liu et al. (2011)

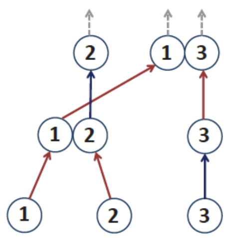
# Gossiping Convergence: Examples

$$x_i(k+1) = \begin{cases} [x_i(k) + x_j(k)]/2 & \text{if } (i,j) \text{ or } (j,i) \text{ is selected} \\ x_i(k) & \text{otherwise} \end{cases}$$

Convergence in 4 steps for n=4 nodes



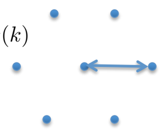

No finite-time convergence for n=3 nodes



# Finite-Time Convergence of Gossiping

**Theorem:** There exists a deterministic gossip algorithm $\{A_k\}_{k=0}^{\infty}$ that ensures global finite-time convergence if and only if there exists an integer $m \geq 0$ such that the number of nodes $n = 2^m$.

$$x(k+1) = A_k x(k)$$



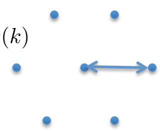
The proof is constructive and provides a gossip algorithm reaching (fastest?) global convergence in $(n \log_2 n)/2$ steps

Shi et al. (2012)

# Impossibility of Finite-Time Convergence

**Theorem:** Suppose there exists no integer $m \geq 0$ such that $n = 2^m$. Then, for almost all initial values, it is impossible to find a gossip algorithm $\{A_k\}_{k=0}^{\infty}$ that reaches finite-time convergence.

$$x(k+1) = A_k x(k)$$

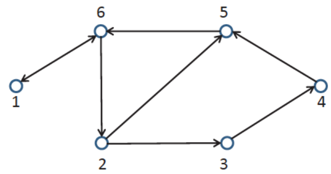


Shi et al. (2012)

## Distributed Averaging and Maximizing

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

$\eta_k \in [0,1]$ and $\alpha_k \in [0, 1-\eta_k]$




$\eta_k \equiv 0,\ \alpha_k \equiv 0$: distributed maximizing
$\eta_k \equiv 0,\ \alpha_k \equiv 1$: distributed minimizing
$\eta_k \in (0,1],\ \alpha_k \in [0, 1-\eta_k]$: distributed weighted averaging

## Impossibilities of Convergence

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

Averaging algorithms: $\eta_k \in (0,1],\ \alpha_k \in [0, 1-\eta_k]$

**Theorem:** For every averaging algorithm, **finite-time** convergence fails for all initial conditions except for the consensus manifold.

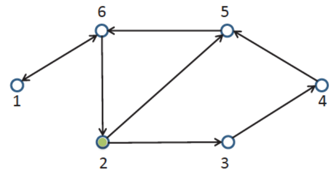
**Theorem:** For every averaging algorithm, **asymptotic** convergence fails for all initial conditions except for the consensus manifold if $\sum_{k=0}^{\infty}(1-\eta_k) < \infty$.

These results are independent of the network topology

# Convergence of Maximizing Algorithms

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

Maximizing algorithms: $\nu_k \equiv \alpha_k \equiv 0$



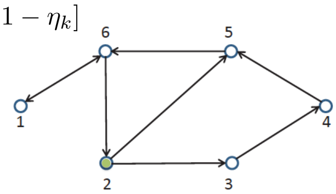

**Theorem:** Suppose $\mathcal{G}_k \equiv \mathcal{G}_*$ is a fixed graph. Global finite-time convergence is achieved if and only if $\mathcal{G}_*$ is strongly connected.

Shi & J (2012)

# Convergence of Averaging Algorithms

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

Averaging algorithms: $\eta_k \in (0, 1]$, $\alpha_k \in [0, 1 - \eta_k]$



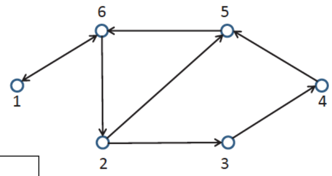

**Theorem:** Suppose $\mathcal{G}_k \equiv \mathcal{G}_*$ is a fixed graph and $\alpha_k \equiv \alpha > 0$. Global asymptotic convergence is achieved if and only if $\mathcal{G}_*$ has a root.

Shi & J (2012)

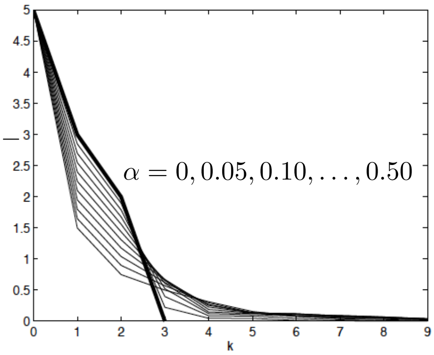
# Example

$$x_i(k+1) = \alpha \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1-\alpha) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

- $0 < \alpha < 1$: global **asymptotic** consensus
- $\alpha = 0$ or $\alpha = 1$: global **finite-time** consensus




$\max_{i \in \mathcal{V}} x_i(k) - \min_{i \in \mathcal{V}} x_i(k)$




Shi & J (2012)

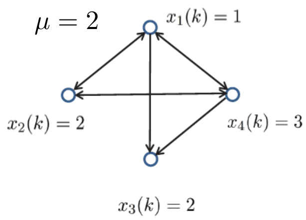
# State-Dependent Nearest-Value Graphs

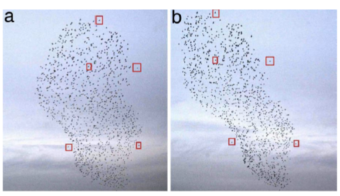
Fix positive integer $\mu$

Neighbors of node $i \in \mathcal{V}$ are nodes in the union of

$$\mathcal{N}_i^-(k) = \{\text{nearest } \mu \text{ neighbors } j \in \mathcal{V} \text{ with } x_j(k) < x_i(k) \text{ and distinct values}\}$$

$$\mathcal{N}_i^+(k) = \{\text{nearest } \mu \text{ neighbors } j \in \mathcal{V} \text{ with } x_j(k) > x_i(k) \text{ and distinct values}\}$$




Motivated from recent studies of starlings collective behavior [Ballerini et al., PNAS, 2008]

# Finite-time Convergence

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

**Theorem:** Consider a nearest-value graph and an averaging algorithm with $\eta_k \equiv 0$ and $\alpha_k \in (0,1)$.

(i) If $n \leq 2\mu$, then global finite-time consensus is achieved.

(ii) If $n > 2\mu$, then no finite-time consensus is achieved for almost all initial conditions.

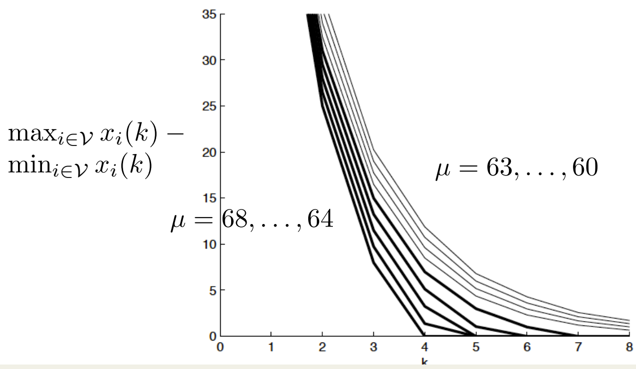
Finite-time convergence only with sufficiently many neighbors

Shi & J (2012)

# Example

$$x_i(k+1) = \alpha \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \alpha) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

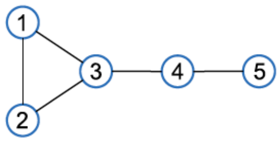
$n = 128$ nodes and $\alpha = 1/2$

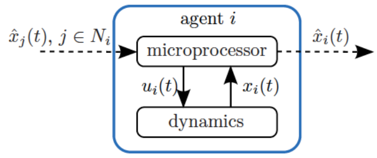



Finite-time convergence for only for many neighbors: μ≥64

Shi & J (2012)

# State-dependent Event-based Scheduling of Measurement Communications




$$\dot{x}_i(t) = u_i(t)$$

$$u_i(t) = -\sum_{j \in N_i} (\hat{x}_i(t) - \hat{x}_j(t))$$

**Event-based broadcasting**

$$\hat{x}_i(t) = x_i(t_k^i),\ t \in [t_k^i, t_{k+1}^i[$$

$$0 \le t_0^i \le t_1^i \le t_2^i \le \cdots$$

$$t_{k+1}^i = \inf\{t :\ t > t_k^i, f_i(t) > 0\}$$

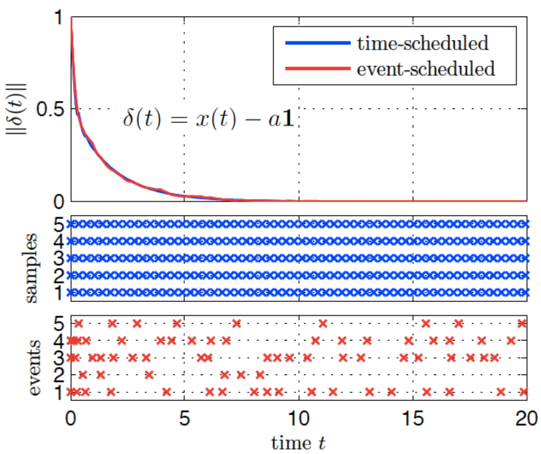
$$f_i(t, e_i(t)) = |e_i(t)| - (c_0 + c_1 e^{-\alpha t})$$

$$e_i(t) = \hat{x}_i(t) - x_i(t)$$

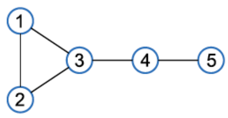
Practical consensus is achieved if $0<\alpha<\lambda_2(L)$

Seyboth et al. (2011)

# Event-based vs Periodic Communication




Graph:



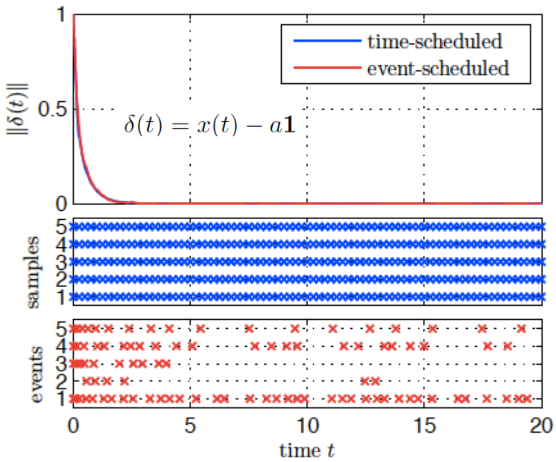

Sampling periods:

- Time-scheduling:
  $\tau_s = 0.350$
  $\tau_{max} = 0.480$
- Event-scheduling:
  $\tau_{mean} = 1.389$

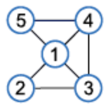
$\tau_{max}$ : largest stabilizing sampling period, see *G. Xie et al., ACC2009*

Seyboth et al. (2011)

# Event-based vs Periodic Communication




Graph:



Sampling periods:

- Time-scheduling:
  $\tau_s = 0.250$
  $\tau_{max} = 0.400$
- Event-scheduling:
  $\tau_{mean} = 1.053$

$\tau_{max}$ : largest stabilizing sampling period, see *G. Xie et al., ACC2009*

Seyboth et al. (2011)

# Conclusions

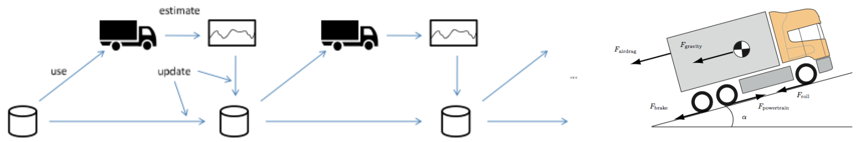
- Controls and communications are coupled in many applications
- Fundamental limits for some control objectives and network protocols
  - Finite-time and asymptotic convergence, gossiping and averaging
- Tradeoffs between communication capacity and control computations

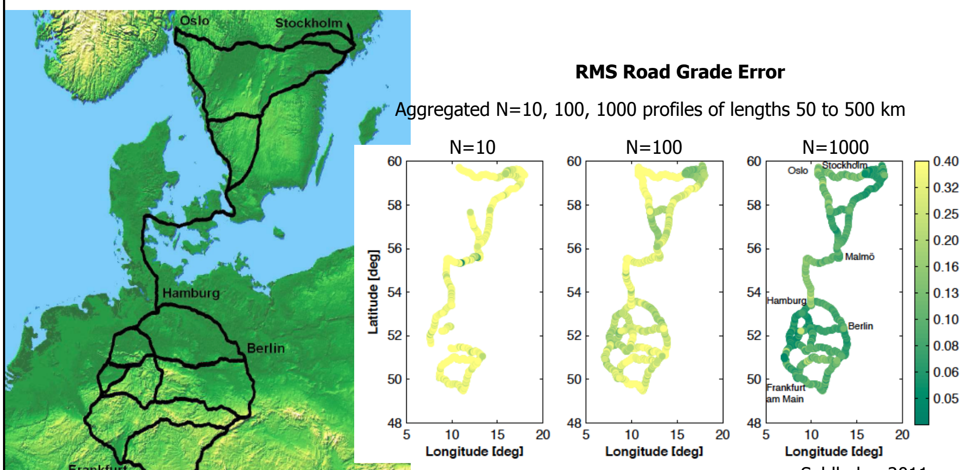
**Extensions**

- Stochastic dynamics and networks
- PHY, MAC, NET models


http://www.ee.kth.se/~kallej

# Collaborative Road Grade Estimation




Sahlholm, 2011