Short Course:
Topics on Cyber-Physical Control Systems

Karl H. Johansson
ACCESS Linnaeus Center & School of Electrical Engineering
KTH Royal Institute of Technology, Sweden

Slides and papers available at http://people.kth.se/~kallel

Department of Electronic & Computer Engineering
Hong Kong University of Science and Technology, July-August 2015

Course Outline

Jul 20: What is a cyber-physical system?
Jul 20: Event-based control of networked systems
Jul 22: Cyber-secure networked control systems

Aug 3: Distributed control of multi-agent systems

Aug 7, 11:30am, IAS Lecture Theater: IAS Lecture
“Cyber-physical systems: why connecting the physical world?”
Distributed control of multi-agent systems

Outline

• Introduction
• Distributed control: local model information
• Distributed control: local interactions
• Conclusions
Outline

• Introduction
• Distributed control: local model information
  – Why cannot we assume global model information?
  – How robust can networked controllers be?
• Distributed control: local interactions
  – How much network interaction is needed?
  – How fast convergence is possible?
• Conclusions

Acknowledgements

Presentation based on joint papers with

• Farhad Farokhi (U Melbourne), Cedric Langbort (UIUC)

• Guodong Shi (ANU), Bo Li (CAS), Alexandre Proutiere (KTH), Mikael Johansson (KTH), John Baras (U Maryland)

Funding sources:
Outline

• Introduction

• Distributed control: local model information
  – Why cannot we assume global model information?
  – How robust can networked controllers be?

• Distributed control: local interactions
  – How much network interaction is needed?
  – How fast control is possible?

• Conclusions
Multi-Layer Dynamic Network Models

Cognitive & Human

Optimization & Control

Information & Communication

Collaborating drivers
Operator supervision
Traffic flow optimization
Autonomous vehicle platoons
Wireless communication infrastructure
Vehicle-to-vehicle information flows
Research Challenges

How deal with incomplete global knowledge of plant model?
How robust can networked controllers be to such uncertainties?
How much local interaction is needed to propagate information?
Tradeoff between convergence speed and number of neighbors?

Outline

• Introduction

• Distributed control: local model information
  – Why cannot we assume global model information?
  – How robust can networked controllers be?

• Distributed control: local interactions

• Conclusions
Example

\[ x_1(k + 1) = a_{11}x_1(k) + a_{12}x_2(k) + u_1(k) \]
\[ x_2(k + 1) = a_{21}x_1(k) + a_{22}x_2(k) + u_2(k) \]

\[ J = \sum_{k=1}^{\infty} \|x(k)\|^2 + \|u(k)\|^2 \]

Keep \( J \) small, when
- Controller 1 knows only \( a_{11} \) and \( a_{12} \)
- Controller 2 knows only \( a_{21} \) and \( a_{22} \)

\[ u_1(k) = -a_{11}x_1(k) - a_{12}x_2(k) \]
\[ u_2(k) = -a_{21}x_1(k) - a_{22}x_2(k) \]

achieves \( J \leq 2J^* \)

No limited plant model information strategy can do better.

Why Limited Plant Model Information?

Complexity
Controllers are easier to implement and maintain if they mainly depend on local model information

Availability
The model of other subsystems is not available at the time of design

Privacy
Competitive advantages not to share private model information
Networked Control System

Plant Graph
Networked Control System
Plant Graph

\[ x_i(k+1) = A_{ii}x_i(k) + \sum_{j \neq i} A_{ij}x_j(k) + B_{ii}u_i(k) \]

Plant: \( P = (A,B,x_0) \in \mathcal{A} \times \mathbb{R} \times \mathbb{R}^n \)
\[ x_i \in \mathbb{R}^n_i \text{ and } u_i \in \mathbb{R}^u_i \]

\[ \mathcal{A} = \{ A \in \mathbb{R}^{n_i \times n_i} | A_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i,j \leq q \text{ such that } (s_p)_{ij} = 0 \} \]

\[ s_p = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} A_{11} & A_{12} & 0_{n_1 \times n_3} \\ 0_{n_2 \times n_1} & A_{22} & A_{23} \\ 0_{n_3 \times n_2} & A_{32} & A_{33} \end{bmatrix} \]
Plant Graph

\[ x_i(k+1) = A_i x_i(k) + \sum_{j \neq i} A_{ij} x_j(k) + B_i u_i(k) \]

Plant: \( P = (A, B, x_0) \in \mathcal{A} \times \mathcal{S} \times \mathbb{R}^n \)

\[ x_i \in \mathbb{R}^{n_i} \text{ and } u_i \in \mathbb{R}^q \]

\[ \mathcal{A} = \{ A \in \mathbb{R}^{n \times n} | A_{ij} \geq 0, A_{ij} \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \} \]

\[ S_p = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} A_{11} & A_{12} & 0_{n_1 \times n_3} \\ 0_{n_2 \times n_1} & A_{22} & A_{23} \\ 0_{n_3 \times n_1} & A_{32} & A_{33} \end{bmatrix} \]

\[ \mathcal{S} = \{ B \in \mathbb{R}^{n \times n} | s(B) \geq \epsilon, B_{ij} \geq 0 \text{ for all } 1 \leq i, j \leq q \} \]

\[ B = \begin{bmatrix} B_{11} & 0_{n_1 \times n_2} & 0_{n_1 \times n_3} \\ 0_{n_2 \times n_1} & B_{22} & 0_{n_2 \times n_3} \\ 0_{n_3 \times n_1} & 0_{n_3 \times n_2} & B_{33} \end{bmatrix} \]

Control Graph

\[ u(k) = K x(k) \]

\[ \mathcal{S} = \{ K \in \mathbb{R}^{n \times n} | K_{ij} \geq 0, K_{ij} \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_K)_{ij} = 0 \} \]

\[ S_K = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]

\[ K = \begin{bmatrix} K_{11} & 0_{n_1 \times n_2} & 0_{n_1 \times n_3} \\ 0_{n_2 \times n_1} & K_{22} & 0_{n_2 \times n_3} \\ 0_{n_3 \times n_1} & 0_{n_3 \times n_2} & K_{33} \end{bmatrix} \]
Design Graph

\[ K = \Gamma(P) = \Gamma(A, B) \]

The map \( [\Gamma_{i1} \ \cdots \ \Gamma_{iq}] \) is only a function of \( \{[A_{j1} \ \cdots \ A_{jq}], B_{ij} | (s_{C})_{ij} \neq 0\} \).

\[ S_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

Design Graph

\[ K = \Gamma(P) = \Gamma(A, B) \]

The map \( [\Gamma_{i1} \ \cdots \ \Gamma_{iq}] \) is only a function of \( \{[A_{j1} \ \cdots \ A_{jq}], B_{ij} | (s_{C})_{ij} \neq 0\} \).

\[ S_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

\( [\Gamma_{i1} \ \Gamma_{i2} \ \Gamma_{i3}] \) is a function of \( \{[A_{21} \ A_{22} \ A_{23}], B_{22}, [A_{31} \ A_{32} \ A_{33}], B_{33}\} \)
Climate Control Example

Plant Graph:

Design Graph:

Performance Metric

The competitive ratio of a control design method $\Gamma$ is defined as

$$\rho(\Gamma) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(\mathcal{K}^*(P))}$$
Performance Metric

The **competitive ratio** of a control design method $\Gamma$ is defined as

$$r_p(\Gamma) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A, B))}{J_P(K^*(P))}$$

A control design method $\Gamma'$ is said to **dominate** another control design method $\Gamma$ if

$$J_P(\Gamma'(A, B)) \leq J_P(\Gamma(A, B)),$$

for all $P = (A, B, x_0) \in \mathcal{P}$

with strict inequality holding for at least one plant.

When no such $\Gamma'$ exists, we say that $\Gamma$ is **undominated**.

---

Performance Metric

The **competitive ratio** of a control design method $\Gamma$ is defined as

$$r_p(\Gamma) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A, B))}{J_P(K^*(P))}$$

A control design method $\Gamma'$ is said to **dominate** another control design method $\Gamma$ if

$$J_P(\Gamma'(A, B)) \leq J_P(\Gamma(A, B)),$$

for all $P = (A, B, x_0) \in \mathcal{P}$

with strict inequality holding for at least one plant.

When no such $\Gamma'$ exists, we say that $\Gamma$ is **undominated**.

$$J_P(K) = \sum_{k=1}^{\infty} x(k)^T Q x(k) + \sum_{k=0}^{\infty} u(k)^T R u(k)$$

$Q$ and $R$ are block-diagonal positive definite matrices.
Performance Metric

The **competitive ratio** of a control design method $\Gamma$ is defined as

$$\tau_P(\Gamma) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))}$$

A control design method $\Gamma'$ is said to dominate another control design method $\Gamma$ if

$$J_P(\Gamma(A,B)) \leq J_P(\Gamma'(A,B)), \quad \text{for all } P = (A,B,x_0) \in \mathcal{P}$$

with strict inequality holding for at least one plant.

When no such $\Gamma'$ exists, we say that $\Gamma$ is **undominated**.

$$J_P(K) = \sum_{k=1}^{\infty} x(k)^T Q x(k) + \sum_{k=0}^{\infty} u(k)^T R u(k)$$

$Q$ and $R$ are block-diagonal positive definite matrices.

**Remark:** When $G_P$ is a complete graph

$$K^*(P) = -(R + B^T X B)^{-1} B^T X A$$

$$A^T X A - A^T X B (R + B^T X B)^{-1} B^T X A - X + Q = 0$$

Assumptions

- All subsystems are fully actuated:

  $$B \in \mathbb{R}^{n \times n} \text{ and } \sigma(B) \geq \epsilon > 0.$$ 

- $G_P$ contains no isolated node.

- $G_C$ contains all self-loops.

- To simplify the presentation, fix $\epsilon = 1$ and $Q = R = I$. 

Problem Formulation

Find the best control design strategy with limited model information:

\[
\min_{\Gamma \in \mathcal{E}} r_P(\Gamma)
\]

Characterize the influence from
- Plant structure \((G_p)\)
- Controller communication \((G_K)\)
- Model limitation \((G_C)\)

Deadbeat Control Design

\[
\Gamma^A(A, B) = -B^{-1}A
\]

Subcontroller \(i\) depends only on subsystem \(i\)'s model:
\[
\begin{bmatrix}
\Gamma_1^A(A, B) & \cdots & \Gamma_n^A(A, B)
\end{bmatrix} = -B_0^{-1}[A_{12} & \cdots & A_{1n}]
\]

\[
x(k + 1) = Ax(k) + Bu(k) \quad x(0) = x_0,
\]
Deadbeat Control Design

Lemma:\quad G_K \supseteq G_P \implies \tau^*(t^*) = 2

Farokhi et al., 2013

* $G_K \supseteq G_P$ means $E_K \supseteq E_P$, so more controller communications than plant interactions
Deadbeat Control Design

Lemma: \( G_K \supseteq G_P \implies \gamma_P(\tau^A) = 2 \)

* \( G_K \supseteq G_P \) means \( E_K \supseteq E_P \)
* \( J_P(\tau^A(A, B)) \leq 2J_P(K^*(P)) \), so deadbeat never worse than twice the optimal controller

---

If enough controller communication, then a simple (deadbeat) controller is quiet good

Farokhi et al., 2013
Proof sketch of Deadbeat Lemma (1/2)

Show that

\[ \frac{J_p(\Gamma^A(A,B))}{J_p(K^*(P))} \leq 2 \]

Note

\[ \frac{J_p(\Gamma^A(A,B))}{J_p(K^*(P))} \leq \frac{J_p(\Gamma^A(A,B))}{J_p(K^*_\text{centralized}(P))} \]

\[ J_p(\Gamma^A(A,B)) = x_0^t A^T B^{-T} B^{-1} A x_0 \]
\[ J_p(K^*_\text{centralized}(P)) = x_0^t (X - I) x_0, \quad X = A^T X A - A^T X B (I + B^T X B)^{-1} B^T X A + I \]
\[ \sigma(B) \geq \epsilon = 1 \implies \frac{J_p(\Gamma^A(A,B))}{J_p(K^*_\text{centralized}(P))} \leq \frac{x_0^t A^T A x_0}{2} \]
\[ \sigma(B) \geq \epsilon = 1 \implies X \geq \frac{1}{2} A^T A + I \implies J_p(K^*_\text{centralized}(P)) \geq \frac{1}{2} x_0^t A^T A x_0 \]

Proof sketch of Deadbeat Lemma (2/2)

Show that upper bound of \( J_p(\Gamma^A(A,B))/J_p(K^*(P)) \) is achieved

No isolated node in \( G_p \) \( \implies \exists i, j : i \neq j \) and \((s_p)_{ij} \neq 0\)

Fix \( i_1 \in I_1 \) and \( j_1 \in I_j \) and consider \( P = (e_{i_1}, e_{j_1}^T, I, e_{j_1}) \)

\[ J_p(K^*_\text{centralized}(P)) \leq J_p(K^*(P)) \]
\[ K^*_\text{centralized}(P) = -\frac{1}{2} e_{i_1} e_{j_1}^T \]
\[ G_K \supseteq G_p \implies K^*_\text{centralized}(P) \in \pi \]
\[ J_p(K^*_\text{centralized}(P)) \geq J_p(K^*(P)) \]
\[ J_p(K^*_\text{centralized}(P)) = J_p(K^*(P)) = \frac{1}{2} \implies \frac{J_p(\Gamma^A(A,B))}{J_p(K^*(P))} = 2 \]
Plant Graphs with no Sinks

Theorem:

\[ G_p \text{ has no sink} \quad G_K \supseteq G_p \quad G_C \text{ is fully disconnected} \implies \tau_P(\Gamma) \geq \tau_P(\Gamma^A) \quad \forall \Gamma \in \mathcal{G} \]

When \( G_p \) has no sink, there is no control design strategy \( \Gamma \) with a better competitive ratio \( \tau_P(\Gamma) = \sup_{P \in \mathcal{P}} \frac{f_P(\Gamma(A,B))}{f_P(K^*(P))} \) than deadbeat \( \Gamma^A \)

Farokhi et al., 2013

Plant Graphs with no Sinks

Theorem:

\[ G_p \text{ has no sink} \quad G_K \supseteq G_p \quad G_C \text{ is fully disconnected} \implies \tau_P(\Gamma) \geq \tau_P(\Gamma^A) \quad \forall \Gamma \in \mathcal{G} \]

\[ \quad \quad \quad \quad \quad \quad G_K \supseteq G_p \quad G_C \text{ is fully disconnected} \implies G_p \text{ has no sink} \; \Gamma^A \text{ is undominated} \]

When \( G_p \) has no sink, there is no control design strategy \( \Gamma \) that is always better than deadbeat \( \Gamma^A \) for all \( P \).

Farokhi et al., 2013
Modified Deadbeat Control Design

When $G_p$ has $c \geq 1$ sinks, let its adjacency matrix be

$$S_p = \begin{pmatrix} (S_p)_{11} & (S_p)_{12} \\ (S_p)_{21} & (S_p)_{22} \end{pmatrix}$$

Introduce the modified deadbeat control design strategy:

$$\begin{pmatrix} r_1^{\Theta}(A, B) \\ r_2^{\Theta}(A, B) \end{pmatrix} = \begin{cases} -B_{11}^{-1}A_{11} \cdots A_{c1} & i \text{ is not a sink} \\ \left( I + B_{11}^{T}X_nB_{11} \right)^{-1}B_{11}^{T}X_nA_{11} \cdots A_{c1} & i \text{ is a sink} \end{cases}$$

$$A_{11}^{T}X_nA_{11} - A_{11}^{T}X_nB_{11} \left( I + B_{11}^{T}X_nB_{11} \right)^{-1}B_{11}^{T}X_nA_{11} - X_n + I = 0$$

Lemma: $G_K \supseteq G_p \implies r_p(\Gamma^{\Theta}) = \begin{cases} 2 & (S_p)_{11} \neq 0 \\ 1 & (S_p)_{11} = 0 \text{ and } (S_p)_{22} = 0 \end{cases}$

Farokhi et al., 2013

Plant Graphs with Sinks

Theorem: $(S_p)_{11}$ is nondiagonal $\implies r_p(\Gamma^{\Theta}) \geq r_p(\Gamma^{\Theta})$ $\forall \Gamma \in \mathcal{E}$

$(S_p)_{11}$ nondiagonal means that the subgraph from removing sinks has at least one edge between two nodes

Farokhi et al., 2013

When $G_p$ has at least one sink, there is no control design strategy $\Gamma^{\Theta}$ with a better competitive ratio than modified deadbeat $\Gamma^{\Theta}$
Plant Graphs with Sinks

Theorem: \( (G_p)_1 \) is nondiagonal
\( G_K \supseteq G_p \)
\( G_C \) is fully disconnected
\[ \Rightarrow r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^0) \] \( \forall \Gamma \in \mathcal{E} \)
\( G_K \supseteq G_p \)
\( G_C \) is fully disconnected
\[ \Rightarrow \Gamma^0 \text{ is undominated} \]

When \( G_p \) has at least one sink, there is no control design strategy \( \Gamma \) that is always better than modified deadbeat \( \Gamma^0 \) for all \( P \).

Farokhi et al., 2013

Example

\[
\begin{align*}
G_p & : \begin{bmatrix} x_1(k + 1) \\ x_2(k + 1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \\
G_C & : \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},
\end{align*}
\]

- \( K^*(P) = -(I + X)^{-1}XA \)
- \( A^TXA - A^TX(I + X)^{-1}XA + I = X \)
- \( \Gamma^0(A, B) = -\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \)
- \( J_P(\Gamma^0(A, B)) \leq 2J_P(K^*(P)) \)
- \( \Gamma(A, B) = -\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \)
- \( J_P(\Gamma(A, B)) \leq J_P(K^*(A, B)) \leq 2J_P(K^*(P)) \)

and undominated
Disturbance Accommodation

\[ x(k+1) = Ax(k) + B(u(k) + w(k)) \; ; \; x(0) = x_0, \]
\[ w(k+1) = Dw(k) \; ; \; w(0) = w_0 \]

Deadbeat controller with deadbeat observer is undominated

\[ \Gamma^D (A, B, D) = \begin{bmatrix} D & -B^{-1}D^2 \\ I & -B^{-1}(A + D) \end{bmatrix} \]

Corresponds to PI control for step disturbance (D=1)

Statistical model information

Designs with full model information (FMI), limited (exact) model information (LMI), statistical model information (SMI)

Example

\[ \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \]

\[ E\{a_{11}\} = 2.0 \; \text{and} \; E\{(a_{11} - E\{a_{11}\})^2\} = 0.4, \]
\[ E\{a_{12}\} = 1.0 \; \text{and} \; E\{(a_{12} - E\{a_{12}\})^2\} = 0.1, \]

Etc.

\[ \sup_{x_0 \in \mathbb{R}^n} \frac{x_0^T P_{\text{LMI}} x_0}{x_0^T P_{\text{FMI}} x_0} = 1.0088 \leq 1 + 1/\epsilon^2 = 2, \]

\[ \sup_{x_0 \in \mathbb{R}^n} \frac{x_0^T P_{\text{SMI}} x_0}{x_0^T P_{\text{LMI}} x_0} = 2.3607. \]
Adaptive Controllers

Consider a general (nonlinear) adaptive controller with limited model information

\[ u(k) = K(F_k) \]
\[ F_k = \sigma(\{x(t)\}_{t=0}^{k} \cup \{u(t)\}_{t=0}^{k-1}) \]

Then, there exists a control design method \( K = \Gamma^*(P) \) such that

\[ J_P(\Gamma^*(P)) \leq J_P(K^*(P)) \]

where \( K^*(P) \) is the optimal controller with full model information

- It is possible to achieve a competitive ratio equal to one for an adaptive controller with limited plant model information
- Proof is constructive, uses adaptation algorithm of [Campi & Kumar, 1998]

Outline

- Introduction
- Distributed control: local model information
  - Distributed control: local interactions
    - How much network interaction is needed?
    - How fast convergence is possible?
- Conclusions
Mathematical Model

Directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \)
- Node set \( \mathcal{V} = \{1, 2, \ldots, n\} \)
- Arc \( e = (i, j) \in \mathcal{E} \)

Time-varying graph process \( \mathcal{G}_k(\omega) = (\mathcal{V}, \mathcal{E}_k(\omega)), k = 0, 1, \ldots \)

To each node \( i \in \mathcal{V} \), associate a scalar state \( x_i(k) \)

\( x_i \) updates based on own computation and neighbor information

\[ \mathcal{N}_i(k) = \{ j \in \mathcal{V} : (j, i) \in \mathcal{E}_k \} \cup \{ i \} \]

Prototype model for a collaborative control problem with coupled network and node dynamics

Objective

Control the states to agreement: \( \lim_{k \to \infty} |x_i(k) - x_j(k)| = 0 \) for all \( i, j \in \mathcal{V} \)

Also called consensus, rendezvous, formation, etc

Local update law

\[ x_i(k + 1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k)x_j(k) \]

Related work on Markov chains, belief evolution, consensus algorithms, distributed control etc:
Symmetric Gossip Algorithm

At each $k$, select a pair of nodes that “gossip”:

$$x_i(k+1) = \begin{cases} \frac{[x_i(k) + x_j(k)]}{2} & \text{if } (i, j) \text{ or } (j, i) \text{ is selected} \\ x_i(k) & \text{otherwise} \end{cases}$$

Equivalently:

$$x(k+1) = P_k x(k),$$

where

$$P_k \in \left\{ I - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : i, j \in V \right\}$$

with $e_m$ being the $n$-dimensional unit vector whose $m$’th component is 1.

Various bounds on the convergence time to asymptotic consensus, e.g., Karp et al. (2000), Kempe et al. (2003), Boyd et al., (2006), Shah (2008)

---

Gossiping Convergence: Examples

$$x_i(k+1) = \begin{cases} \frac{[x_i(k) + x_j(k)]}{2} & \text{if } (i, j) \text{ or } (j, i) \text{ is selected} \\ x_i(k) & \text{otherwise} \end{cases}$$

Convergence in 4 steps for $n=4$ nodes for all initial values

No finite-time convergence for $n=3$ nodes for almost all initial values
Definition of Finite-time Convergence

\[ x(k + 1) = P_k x(k) \]

A symmetric gossip algorithm \( \{P_k\}_0^\infty \) converges globally in finite time if there exists an integer \( T \geq 0 \) such that

\[ \text{rank}(P_T \ldots P_0) = 1. \]

Finite-Time Convergence of Symmetric Gossiping

**Theorem**

There exists a symmetric gossip algorithm that converges globally in finite time if and only if \( n = 2^m \) for some integer \( m \geq 0 \).
Finite-Time Convergence of Symmetric Gossiping

**Theorem**
There exists a symmetric gossip algorithm that converges globally in finite time if and only if \( n = 2^m \) for some integer \( m \geq 0 \).

**“Proof”**
- Sufficiency: Induction over \( n \)
- Necessity: Contradiction using a particular initial value

Proof is constructive: for \( n = 2^m \) it provides a **fastest algorithm** converging in \( (n \log_2 n)/2 \) steps

Shi et al., 2015

---

Impossibility of Finite-Time Convergence of Symmetric Gossiping

**Theorem**
Suppose there exists no integer \( m \geq 0 \) such that \( n = 2^m \). Then for almost all initial values (under standard Lebesgue measure), it is impossible to find a symmetric gossip algorithm to reach finite-time convergence under the given initial value.

Initial value
\[
x_1(0) = 1 \quad x_2(0) = 3 \quad x_3(0) = 2
\]
yields finite-time convergence, but is an exception.

Shi et al., 2015
Asymmetric Gossip Algorithm

\[ x_i(k + 1) = x_i(k) \]
\[ x_j(k + 1) = \frac{1}{2} r_i(k) + \frac{1}{2} x_j(k) \]

Equivalently

\[ x(k + 1) = P_k x(k), \]

where

\[ P_k \in \left\{ I - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : i, j \in V \right\} \cup \left\{ I - \frac{e_i(e_i - e_j)^T}{2} : i, j \in V \right\} \]

with \( e_m \) being the \( n \)-dimensional unit vector whose \( m \)’th component is 1.

---

Finite-Time Convergence of Asymmetric Gossiping

**Theorem**

For any network with \( n \) nodes, there always exists a gossip algorithm with asymmetric updates that converges globally in finite time.

Consider a network with \( n = 2^m + r \) nodes for \( 0 < r < 2^m \). A fastest gossip algorithm allowing asymmetric updates reaches convergence using \( mn + 2r \) node updates.

Shi et al., 2015
## Other Distributed Averaging Algorithms

\[
x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in N_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in N_i(k)} x_j(k)
\]

\[
\eta_k \in [0, 1] \text{ and } \alpha_k \in [0, 1 - \eta_k]
\]

- \(\eta_k \equiv 0, \alpha_k \equiv 0\): distributed maximizing
- \(\eta_k \equiv 0, \alpha_k \equiv 1\): distributed minimizing
- \(\eta_k \in (0, 1], \alpha_k \in [0, 1 - \eta_k]\): distributed weighted averaging

---

## Impossibilities of Convergence

\[
x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in N_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in N_i(k)} x_j(k)
\]

Averaging algorithms: \(\eta_k \in (0, 1], \alpha_k \in [0, 1 - \eta_k]\)

**Theorem:** For every averaging algorithm, **finite-time** convergence fails for all initial conditions except for the consensus manifold.

**Theorem:** For every averaging algorithm, **asymptotic** convergence fails for all initial conditions except for the consensus manifold if \(\sum_{k=0}^{\infty} (1 - \eta_k) < \infty\).
Convergence of Maximizing Algorithms

\[
x_i(k + 1) = \eta_k x_i(k) + \alpha_k \min_{j \in N_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in N_i(k)} x_j(k)
\]

Maximizing algorithms: \( \eta_k \equiv \alpha_k \equiv 0 \)

**Theorem:** Suppose \( G_k \equiv G \) is a fixed graph. Global finite-time convergence is achieved if and only if \( G \) is strongly connected.

Convergence of Averaging Algorithms

\[
x_i(k + 1) = \eta_k x_i(k) + \alpha_k \min_{j \in N_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in N_i(k)} x_j(k)
\]

Averaging algorithms: \( \eta_k \in (0, 1], \alpha_k \in [0, 1 - \eta_k] \)

**Theorem:** Suppose \( G_k \equiv G \) is a fixed graph and \( \alpha_k \equiv \alpha > 0 \). Global asymptotic convergence is achieved if and only if \( G \) has a root.
Example

\[ x_i(k+1) = \alpha \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \alpha) \max_{j \in \mathcal{N}_i(k)} x_j(k) \]

- \(0 < \alpha < 1\): global asymptotic consensus
- \(\alpha = 0\) or \(\alpha = 1\): global finite-time consensus

State-Dependent Nearest-Value Graphs

Fix positive integer \(\mu\)

Neighbors of node \(i \in \mathcal{V}\) are nodes in the union of

\(\mathcal{N}_i^-(k) = \{\text{nearest } \mu \text{ neighbors } j \in \mathcal{V} \text{ with } x_j(k) < x_i(k) \text{ and distinct values}\}\)

\(\mathcal{N}_i^+(k) = \{\text{nearest } \mu \text{ neighbors } j \in \mathcal{V} \text{ with } x_j(k) > x_i(k) \text{ and distinct values}\}\)

Motivated from recent studies of bird collective behavior [Ballerini et al, PNAS, 2008]:

In fact, we discover that each bird interacts on average with a fixed number of neighbours (six-seven), rather than with all neighbours within a fixed metric distance.
Finite-time Convergence

\[ x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in N_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in N_i(k)} x_j(k) \]

**Theorem:** Consider a nearest-value graph and an averaging algorithm with \( \eta_k \equiv 0 \) and \( \alpha_k \in (0,1) \).

(i) If \( n \leq 2\mu \), then global finite-time consensus is achieved.

(ii) If \( n > 2\mu \), then no finite-time consensus is achieved for almost all initial conditions.

---

Finite-time convergence only with sufficiently many neighbors

---

**Example**

\[ x_i(k+1) = \alpha \min_{j \in N_i(k)} x_j(k) + (1 - \alpha) \max_{j \in N_i(k)} x_j(k) \]

\( n = 128 \) nodes and \( \alpha = 1/2 \)

---

\( \max_{i \in V} x_i(k) - \min_{i \in V} x_i(k) \)

\( \mu = 63, \ldots, 64 \)

\( \mu = 68, \ldots, 64 \)

---

Shi & J, ACC, 2013
Outline

• Introduction
• Distributed control: local model information
• Distributed control: local interactions
  • Conclusions

Conclusions

• Global plant model information is seldom available in cyber-physical control systems
• A framework to study the effect of (very) limited exchange of plant model information on the performance
• Simpler control strategies vs more communication:

• Finite-time convergence of some low-order protocols

http://people.kth.se/~kallej
What about *dynamic* controllers?
What about *under-actuated* subsystems?
Networked Control System

Plant Graph:
\[ x_i(k + 1) = A_{ii} x_i(k) + \sum_{j \neq i} A_{ij} x_j(k) + B_{ii} u_i(k) + H_{ii} w_i(k) \]
Plant: \( P = (A, B, H) \in \mathcal{P} \times \mathcal{B} \times \mathcal{H} \)
\( x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{n_u}, \) and \( w_i \in \mathbb{R}^{n_w} \)

Control Graph:
\[ K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = C_K (zI - A_K)^{-1} B_K + D_K \]
\( \mathcal{K} = \{ K \in \mathbb{R}^{n_K \times n_K} : K_{ij} = 0 \in (\mathbb{R}^{n_K \times n_K}) \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_K)_{ij} = 0 \} \)

Design Graph:
\[ K = \Gamma(P) = \Gamma(A, B, H) \]
The map \( \{ \tau_1 \cdots \tau_q \} \) is only a function of \( \{ A_{11} \cdots A_{qq}, B_{jj}, D_{jj} \} \) \( (s_K)_{ij} \neq 0 \).

Performance Metric

The competitive ratio of a control design method \( \Gamma \) is defined as
\[ r_p(\Gamma) = \sup_{P \in \mathcal{P}} \frac{J_p(\Gamma(A, B), P)}{J_p(K^*(P), P)} \]
\[ J_p(K) = \left\| \tau_{wy}(z) \right\|_2^2 \]
\( \tau_{wy}(z) \) is the closed-loop transfer function from exogenous input \( w(k) \) to output
\[ y(k) = \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \]
\( \mathcal{C} \) and \( \mathcal{D} \) are full-rank block-diagonal square matrices.
Assumptions

- All subsystems are fully actuated:
  \[ B \in \mathbb{R}^{n \times n} \text{ and } \sigma(B) \geq \epsilon > 0. \]

- \( G_p \) contains no isolated node.

- \( G_p, G_K, G_C \) contain all self-loops.

- To simplify the presentation, fix \( \epsilon = 1 \) and \( C = D = I \).

Modified Deadbeat Undominated by Dynamic Controllers

Theorem: \( G_K \) is a complete graph, \( G_C \) is fully disconnected

\[ r_p(I) \geq r_p(I^o) \quad \forall I \in \mathcal{E} \quad \& \quad I^o \text{ is undominated} \]

- \( G_p \) is acyclic
- \( G_x = G_T \)
- \( G_C \) is fully disconnected

\[ r_p(I) \geq r_p(I^o) \quad \forall I \in \mathcal{E} \quad \& \quad I^o \text{ is undominated} \]

If enough controller communication, static controller \( I^o(A, B) \) suffices to outperform more complex controller

This is true even though \( K^*(P) \) is dynamic.
Extension to Under-actuated Systems

\( \mathcal{G}' = \{ B \in \mathbb{R}^{m \times m} \mid c(B) \geq e, B_{ij} = 0 \in \mathbb{R}^{n \times m_j} \text{ for all } 1 \leq i \neq j \leq q \} \)

- If node \( i \) is a sink, assume:
  - \( \text{rank}(B_{ii}) = m_i \leq n_i \)
  - \((A_{ii}, B_{ii})\) is controllable
  - \( \text{span}(A_{ij}) \subseteq \text{span}(B_{ii}) \) for all \( j \neq i \)

- If node \( i \) is not a sink, assume:
  - \( m_i = n_i \)

Example: Vehicle Platooning

\[
\begin{bmatrix}
\dot{v}_1(t) \\
d_{12}(t) \\
\dot{v}_2(t) \\
d_{23}(t) \\
\dot{v}_3(t)
\end{bmatrix} =
\begin{bmatrix}
-g_1/m_1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 0 & g_2/m_2 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & g_3/m_3
\end{bmatrix}
\begin{bmatrix}
v_1(t) \\
d_{12}(t) \\
v_2(t) \\
d_{23}(t) \\
v_3(t)
\end{bmatrix} +
\begin{bmatrix}
b_1/m_1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} u_1(t) +
\begin{bmatrix}
0 & b_2/m_2 \\
0 & 0 \\
0 & 0 \\
0 & b_3/m_3
\end{bmatrix} u_2(t) +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} u_3(t) +
\begin{bmatrix}
w_1(t) \\
w_2(t) \\
w_3(t) \\
w_4(t) \\
w_5(t)
\end{bmatrix}
\]

\( g_1 = g_2 = g_3 = 0.1 \quad \text{and} \quad b_1 = b_2 = b_3 = 1.0 \)
Example: Vehicle Platooning

Regulating inter-vehicle distances $d_{12}$ and $d_{23}$

\[
\begin{align*}
\dot{v}_1(t) &= -g_1 m_1 + 0 0 0 0 0 \\
\dot{d}_{13}(t) &= 1 0 0 0 0 \\
\dot{v}_2(t) &= 0 0 0 0 0 \\
\dot{d}_{23}(t) &= 0 0 0 0 0 \\
\dot{v}_3(t) &= 0 0 0 0 0 \\
\end{align*}
\]

where $m = [m_1, m_2, m_3]^T \in [0.5, 1.0]^3$.

Find control design strategy $\Gamma$ that

\[
\min_{\Gamma} \max_{\alpha} \|T_{zw}(s; \Gamma, m)\|_{\infty}
\]

where $m = [m_1, m_2, m_3]^T \in [0.5, 1.0]^3$ and $\Gamma$ belongs to the set of polynomials of $m_i$, $i = 1, 2, 3$, up to the second order.
Example: Vehicle Platooning

Control Design with Local Model Information
\[ \max_{\alpha \in \mathcal{A}} \left\| T_{zw} (s; \Gamma_{\text{local}}, \alpha) \right\|_{\infty} = 4.7905 \]

Control Design with Limited Model Information
\[ \max_{\alpha \in \mathcal{A}} \left\| T_{zw} (s; \Gamma_{\text{limited}}, \alpha) \right\|_{\infty} = 3.5533 \] \[ 25.8\% \]

Control Design with Full Model Information
\[ \max_{\alpha \in \mathcal{A}} \left\| T_{zw} (s; \Gamma_{\text{full}}, \alpha) \right\|_{\infty} = 3.3596 \] \[ 5.4\% \]