Short Course:
Topics on Cyber-Physical Control Systems

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Course Outline

Jul 20: What is a cyber-physical system?
Jul 20: Event-based control of networked systems
Jul 22: Cyber-secure networked control systems
Aug 3: Distributed control of multi-agent systems

Aug 7, 11:30am, IAS Lecture Theater: IAS Lecture "Cyber-physical systems: why connecting the physical world?"

Distributed control of multi-agent systems

Outline

- Introduction
- Distributed control: local model information
- Distributed control: local interactions
- Conclusions

Outline

- Introduction
- Distributed control: local model information
  - Why cannot we assume global model information?
  - How robust can networked controllers be?
- Distributed control: local interactions
  - How much network interaction is needed?
  - How fast convergence is possible?
- Conclusions

Acknowledgements

Presentation based on joint papers with

- **Farhad Farokhi** (U Melbourne), **Cedric Langbort** (UIUC)
- **Guodong Shi** (ANU), **Bo Li** (CAS), **Alexandre Proutiere** (KTH), **Mikael Johansson** (KTH), **John Baras** (U Maryland)

Funding sources:

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- Distributed control: local model information
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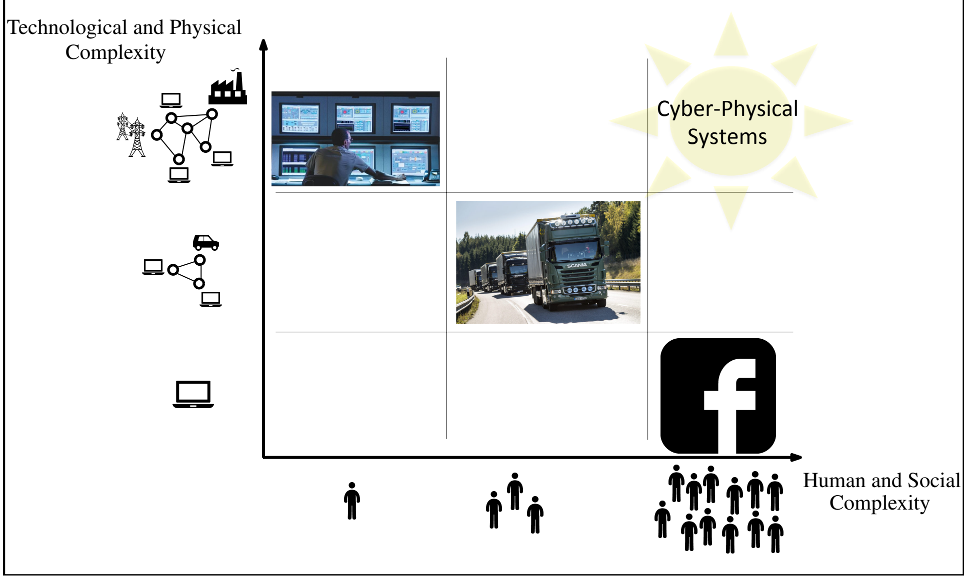
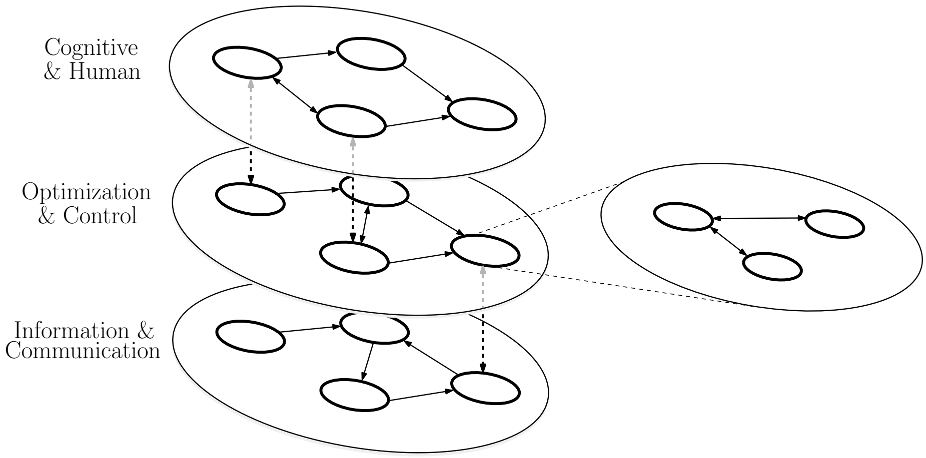
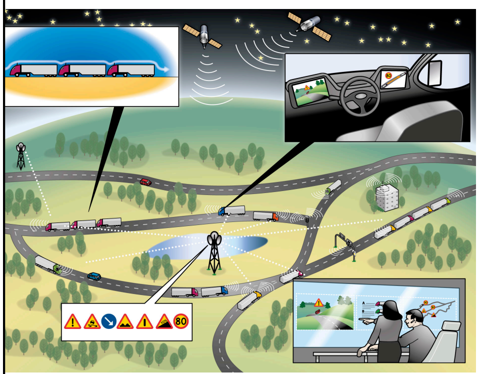
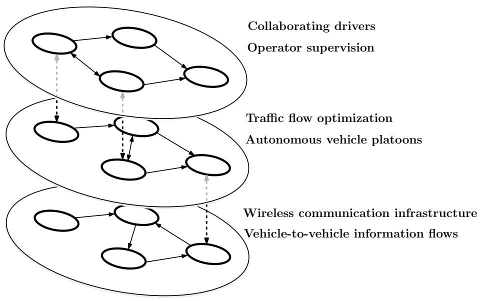


Multi-Layer Dynamic Network Models

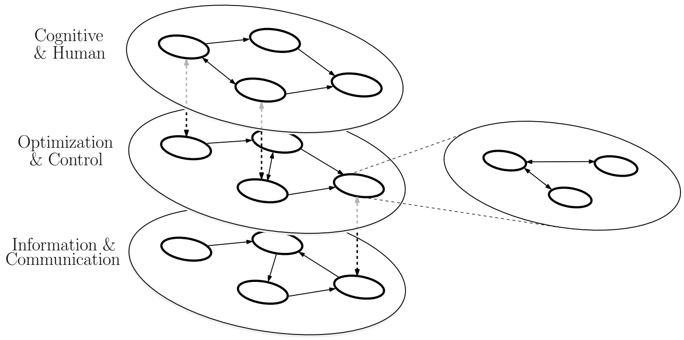






Research Challenges

How deal with incomplete global knowledge of plant model?
How robust can networked controllers be to such uncertainties?
How much local interaction is needed to propagate information?
Tradeoff between convergence speed and number of neighbors?



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- Conclusions

Example

$$x_1(k+1) = a_{11}x_1(k) + a_{12}x_2(k) + u_1(k)$$
$$x_2(k+1) = a_{21}x_1(k) + a_{22}x_2(k) + u_2(k)$$

$$J = \sum_{k=1}^{\infty} \|x(k)\|^2 + \|u(k)\|^2$$

Keep J small, when

Controller 1 knows only a_{11} and a_{12}
Controller 2 knows only a_{21} and a_{22}

$$u_1(k) = -a_{11}x_1(k) - a_{12}x_2(k)$$
$$u_2(k) = -a_{21}x_1(k) - a_{22}x_2(k)$$

achieves $J \leq 2J^*$

No limited plant model information strategy can do better.

Langbort & Delvenne, 2011

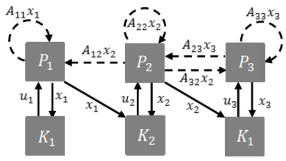
Why Limited Plant Model Information?

Complexity
Controllers are easier to implement and maintain if they mainly depend on local model information

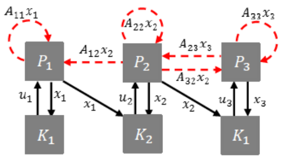
Availability
The model of other subsystems is not available at the time of design

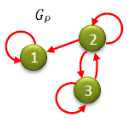
Privacy
Competitive advantages not to share private model information

Networked Control System



Networked Control System





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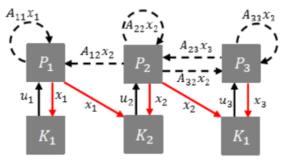
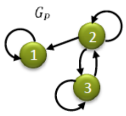
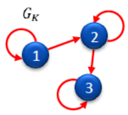


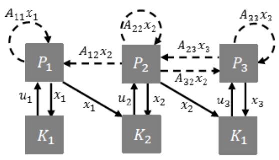


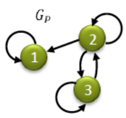
Plant Graph



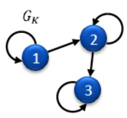
Control Graph

Networked Control System

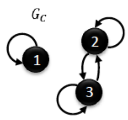




Plant Graph



Control Graph

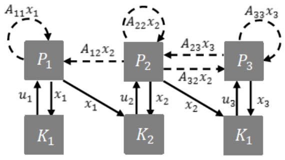


Design Graph

Physical Constraints

Model Information Limitations

Plant Graph

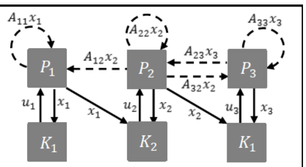


$$x_i(k+1) = A_{ii}x_i(k) + \sum_{j \neq i} A_{ij}x_j(k) + B_{ii}u_i(k)$$

Plant: $P = (A, B, x_0) \in \mathcal{A} \times \mathcal{B} \times \mathbb{R}^n$

$x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{n_i}$

Plant Graph



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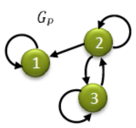
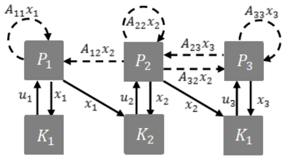
$x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{n_i}$

$$\mathcal{A} = \{ A \in \mathbb{R}^{n \times n} | A_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_P)_{ij} = 0\}$$



$$S_P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} A_{11} & A_{12} & 0_{n_1 \times n_3} \\ 0_{n_2 \times n_1} & A_{22} & A_{23} \\ 0_{n_3 \times n_1} & A_{32} & A_{33} \end{bmatrix}$$

Plant Graph



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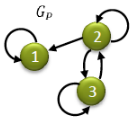

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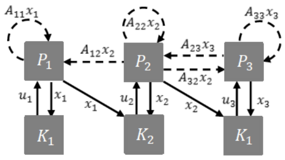
$$\mathcal{B} = \{ B \in \mathbb{R}^{n \times n} \mid \underline{\sigma}(B) \geq \epsilon, B_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i \neq j \leq q \}$$

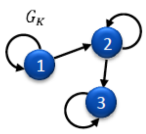
$$B = \begin{bmatrix} B_{11} & 0_{n_1 \times n_2} & 0_{n_1 \times n_3} \\ 0_{n_2 \times n_1} & B_{22} & 0_{n_2 \times n_3} \\ 0_{n_3 \times n_1} & 0_{n_3 \times n_2} & B_{33} \end{bmatrix}$$

Control Graph



$$u(k) = Kx(k)$$

$$\mathcal{K} = \{ K \in \mathbb{R}^{n \times n} | K_{ij} \ = \ 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i,j \leq q \text{ such that } (s_K)_{ij} = 0 \}$$



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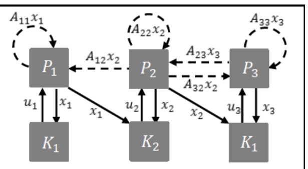
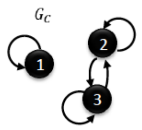
Design Graph



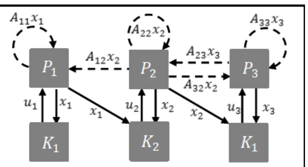
$$K = \Gamma(P) = \Gamma(A, B)$$

The map $[\Gamma_{i1} \quad \cdots \quad \Gamma_{iq}]$ is only a function of $\{[A_{j1} \quad \cdots \quad A_{jq}], B_{jj} \mid (s_C)_{ij} \neq 0\}$.



$$S_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Design Graph



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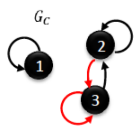
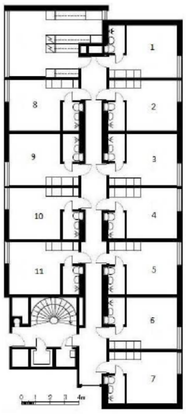


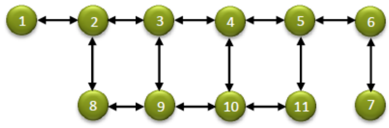
$$S_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$[\Gamma_{31} \quad \Gamma_{32} \quad \Gamma_{33}]$ is a function of $\{[A_{21} \quad A_{22} \quad A_{23}], B_{22}, [A_{31} \quad A_{32} \quad A_{33}], B_{33}\}$

Climate Control Example



Plant Graph:



Design Graph:

Performance Metric

The **competitive ratio** of a control design method Γ is defined as

$$r_{\mathcal{P}}(\Gamma) = \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))}$$

Performance Metric

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A control design method Γ' is said to **dominate** another control design method Γ if

$$J_P\big(\Gamma'(A,B)\big) \leq J_P\big(\Gamma(A,B)\big), \qquad \text{for all } P = (A,B,x_0) \in \mathcal{P}$$

with strict inequality holding for at least one plant.

When no such Γ' exists, we say that Γ is **undominated**.

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$$J_P(K) = \sum_{k=1}^{\infty} x(k)^T Q x(k) + \sum_{k=0}^{\infty} u(k)^T R u(k)$$

Q and R are block-diagonal positive definite matrices.

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Q and R are block-diagonal positive definite matrices.

Remark: When G_K is a complete graph

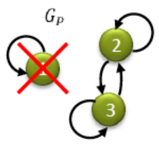
$$K^*(P) = -(R + B^T X B)^{-1} B^T X A$$
$$A^T X A - A^T X B (R + B^T X B)^{-1} B^T X A - X + Q = 0$$

Assumptions

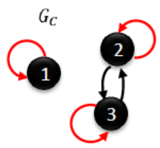
- All subsystems are fully actuated:

  $B \in \mathbb{R}^{n\times n}$ and $\underline{\sigma}(B) \ge \epsilon > 0$.

- G_P contains no isolated node.



- G_C contains all self-loops.

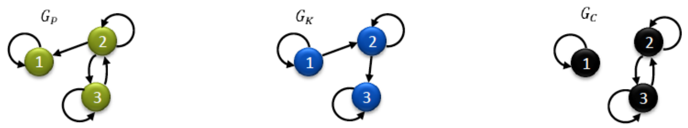


- To simplify the presentation, fix $\epsilon = 1$ and $Q = R = I$.

Problem Formulation

Find the best control design strategy with limited model information:

$$\min_{\Gamma \in \mathcal{C}} r_{\mathcal{P}}(\Gamma)$$



Characterize the influence from

- Plant structure (G_P)
- Controller communication (G_K)
- Model limitation (G_C)

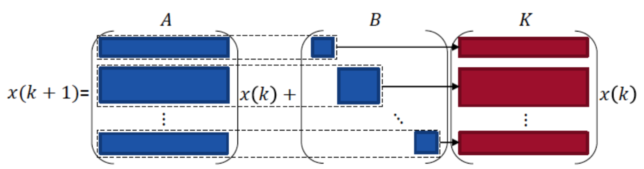
Farokhi et al., 2013

Deadbeat Control Design

$$\Gamma^{\Delta}(A, B) = -B^{-1}A$$

Subcontroller i depends only on subsystem i's model:

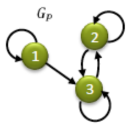
$$[\Gamma_{i1}^{\Delta}(A,B) \quad \cdots \quad \Gamma_{iq}^{\Delta}(A,B)] = -B_{ii}^{-1}[A_{i1} \quad \cdots \quad A_{iq}]$$

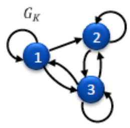


$$x(k+1) = Ax(k) + Bu(k) \; ; \; x(0) = x_0,$$

Deadbeat Control Design

Lemma: $G_K \supseteq G_P \quad \Rightarrow \quad r_{\mathcal{P}}(\Gamma^{\Delta}) = 2$





Farokhi et al., 2013

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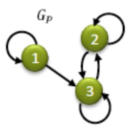
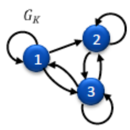




- $G_K \supseteq G_P$ means $E_K \supseteq E_P$, so more controller communications than plant interactions

Farokhi et al., 2013

Deadbeat Control Design

Lemma: $G_K \supseteq G_P \quad \Rightarrow \quad r_{\mathcal{P}}(\Gamma^{\Delta}) = 2$

- $G_K \supseteq G_P$ means $E_K \supseteq E_P$
- $J_P(\Gamma^{\Delta}(A,B)) \leq 2J_P(K^*(P))$, so deadbeat never worse than twice the optimal controller

Farokhi et al., 2013

Deadbeat Control Design

Lemma: $G_K \supseteq G_P \quad \Rightarrow \quad r_{\mathcal{P}}(\Gamma^{\Delta}) = 2$

- $G_K \supseteq G_P$ means $E_K \supseteq E_P$
- $J_P(\Gamma^{\Delta}(A,B)) \leq 2J_P(K^*(P))$

If enough controller communication, then a simple (deadbeat) controller is quiet good

Farokhi et al., 2013

Proof sketch of Deadbeat Lemma (1/2)

Show that

$$\frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*(P))} \leq 2$$

Note

$$\frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*(P))} \leq \frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*_{\text{centralized}}(P))}$$

$$J_P(\Gamma^\Delta(A,B)) = x_0^T A^T B^{-T} B^{-1} A x_0$$

$$J_P(K^*_{\text{centralized}}(P)) = x_0^T (X - I) x_0, \qquad X = A^T X A - A^T X B (I + B^T X B)^{-1} B^T X A + I$$

$$\underline{\sigma}(B) \geq \epsilon = 1 \quad \Longrightarrow \quad J_P(\Gamma^\Delta(A,B)) \leq x_0^T A^T A x_0$$

$$\underline{\sigma}(B) \geq \epsilon = 1 \quad \Longrightarrow \quad X \geq \frac{1}{2} A^T A + I \quad \Longrightarrow \quad J_P(K^*_{\text{centralized}}(P)) \geq \frac{1}{2} x_0^T A^T A x_0$$

Proof sketch of Deadbeat Lemma (2/2)

Show that upper bound of $J_P(\Gamma^\Delta(A,B))/J_P(K^*(P))$ is achieved

No isolated node in $G_P \implies \exists i,j : i \neq j \text{ and } (s_P)_{ij} \neq 0$

Fix $i_1 \in I_i$ and $j_1 \in I_j$ and consider $P = (e_{i_1} e_{j_1}^T, I, e_{j_1})$

$$J_P(K^*_{\text{centralized}}(P)) \leq J_P(K^*(P))$$

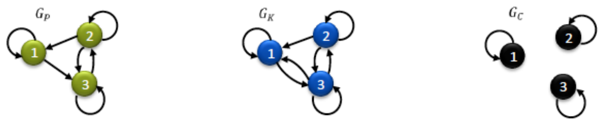
$$K^*_{\text{centralized}}(P) = -\tfrac{1}{2} e_{i_1} e_{j_1}^T$$

$$G_K \supseteq G_P \implies K^*_{\text{centralized}}(P) \in \mathcal{K}$$

$$J_P(K^*_{\text{centralized}}(P)) \geq J_P(K^*(P))$$

$$J_P(K^*_{\text{centralized}}(P)) = J_P(K^*(P)) = \tfrac{1}{2} \implies \frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*(P))} = 2$$

Plant Graphs with no Sinks

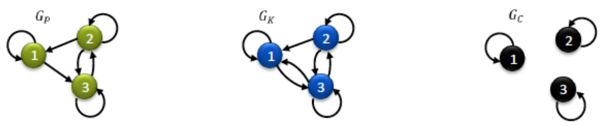


Theorem: $\left.\begin{array}{c} G_P \text{ has no sink} \\ G_K \supseteq G_P \\ G_C \text{ is fully disconnected} \end{array}\right\} \Rightarrow \; r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Delta}) \quad \forall\, \Gamma \in \mathcal{C}$

When G_P has no sink, there is no control design strategy Γ with a better competitive ratio $r_P(\Gamma) = \sup_{P \in \mathcal{P}} J_P(\Gamma(A,B))/J_P(K^*(P))$ than deadbeat Γ^{Δ}

Farokhi et al., 2013

Plant Graphs with no Sinks



Theorem: $\left.\begin{array}{c} G_P \text{ has no sink} \\ G_K \supseteq G_P \\ G_C \text{ is fully disconnected} \end{array}\right\} \Rightarrow \; r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Delta}) \quad \forall\, \Gamma \in \mathcal{C}$

$\left.\begin{array}{c} G_K \supseteq G_P \\ G_C \text{ is fully disconnected} \end{array}\right\} \Rightarrow \quad G_P \text{ has no sink} \;\Leftrightarrow\; \Gamma^{\Delta} \text{ is undominated}$

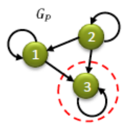
When G_P has no sink, there is no control design strategy Γ that is always better than deadbeat Γ^{Δ} for all P.

Farokhi et al., 2013

Modified Deadbeat Control Design

When G_P has $c \geq 1$ sinks, let its adjacency matrix be

$$S_P = \begin{bmatrix} (S_P)_{11} & 0_{(q-c)\times q} \\ (S_P)_{21} & (S_P)_{22} \end{bmatrix}$$



Introduce the modified deadbeat control design strategy:

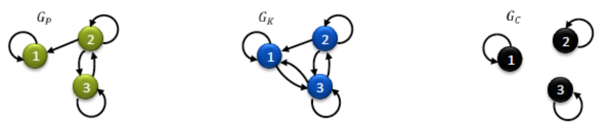
$$[\Gamma^{\Theta}_{i1}(A,B) \quad \cdots \quad \Gamma^{\Theta}_{iq}(A,B)] = \begin{cases} -B_{ii}^{-1}[A_{i1} \quad \cdots \quad A_{iq}] & i \text{ is not a sink} \\ -(I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} [A_{i1} \quad \cdots \quad A_{iq}] & i \text{ is a sink} \end{cases}$$

$$A_{ii}^T X_{ii} A_{ii} - A_{ii}^T X_{ii} B_{ii} (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} A_{ii} - X_{ii} + I = 0$$

Lemma: $G_K \supseteq G_P \Rightarrow r_{\mathcal{P}}(\Gamma^{\Theta}) = \begin{cases} 2 & (S_P)_{11} \neq 0 \\ 1 & (S_P)_{11} = 0 \text{ and } (S_P)_{22} = 0 \end{cases}$

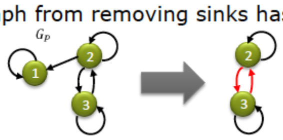
Farokhi et al., 2013

Plant Graphs with Sinks



Theorem: $\left.\begin{array}{r} (S_P)_{11} \text{ is nondiagonal} \\ G_K \supseteq G_P \\ G_C \text{ is fully disconnected} \end{array}\right\} \Rightarrow r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta}) \quad \forall\, \Gamma \in \mathcal{C}$

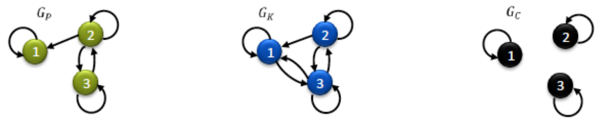
- $(S_P)_{11}$ nondiagonal means that the subgraph from removing sinks has at least one edge between two nodes



When G_P has at least one sink, there is no control design strategy Γ with a better competitive ratio than modified deadbeat Γ^{Θ}

Farokhi et al., 2013

Plant Graphs with Sinks



Theorem:

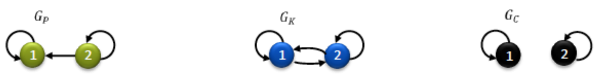
$$\left.\begin{array}{r}(S_P)_{11} \text{ is nondiagonal} \\ G_K \supseteq G_P \\ G_C \text{ is fully disconnected}\end{array}\right\} \Rightarrow \quad r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta}) \quad \forall\, \Gamma \in \mathcal{C}$$

$$\left.\begin{array}{r}G_K \supseteq G_P \\ G_C \text{ is fully disconnected}\end{array}\right\} \Rightarrow \quad \Gamma^{\Theta} \text{ is undominated}$$

When G_P has at least one sink, there is no control design strategy Γ that is always better than modified deadbeat Γ^{Θ} for all P.

Farokhi et al., 2013

Example



$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \qquad \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

- $K^*(P) = -(I + X)^{-1} X A$ — $A^T X A - A^T X (I + X)^{-1} X A + I = X$
- $\Gamma^{\Delta}(A, B) = -\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$ — $J_P(\Gamma^{\Delta}(A, B)) \leq 2 J_P(K^*(P))$
- $\Gamma^{\Theta}(A, B) = -\begin{bmatrix} w a_{11} & w a_{12} \\ 0 & a_{22} \end{bmatrix}$ — $J_P(\Gamma^{\Theta}(A, B)) \leq J_P(\Gamma^{\Delta}(A, B)) \leq 2 J_P(K^*(P))$ and undominated

$$w = \frac{a_{11}^2 - 2 + \sqrt{a_{11}^4 + 4}}{2 a_{11}^2}$$

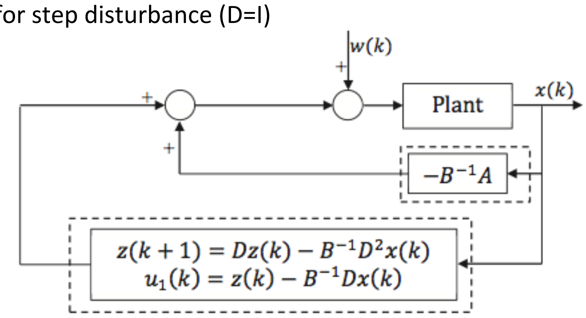
Disturbance Accommodation

$$x(k+1) = Ax(k) + B(u(k) + w(k))\ ;\ x(0) = x_0,$$
$$w(k+1) = Dw(k)\ ;\ w(0) = w_0$$

Deadbeat controller with deadbeat observer is undominated

$$\Gamma^{\Delta}(A, B, D) = \left[\begin{array}{c|c} D & -B^{-1}D^2 \\ \hline I & -B^{-1}(A+D) \end{array}\right]$$

Corresponds to PI control for step disturbance (D=I)



Statistical model information

Designs with full model information (FMI), limited (exact) model information (LMI), statistical model information (SMI)

Example

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\mathbb{E}\{a_{11}\} = 2.0 \text{ and } \mathbb{E}\{(a_{11} - \mathbb{E}\{a_{11}\})^2\} = 0.4,$$
$$\mathbb{E}\{a_{12}\} = 1.0 \text{ and } \mathbb{E}\{(a_{12} - \mathbb{E}\{a_{12}\})^2\} = 0.1,$$

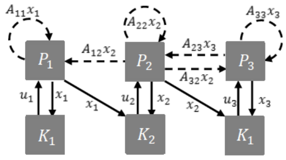
Etc.

$$\sup_{x_0 \in \mathbb{R}^n} \frac{x_0^\top P^{\mathrm{LMI}} x_0}{x_0^\top P^{\mathrm{FMI}} x_0} = 1.0088 \leq 1 + 1/\epsilon^2 = 2,$$

$$\sup_{x_0 \in \mathbb{R}^n} \frac{x_0^\top P^{\mathrm{SMI}} x_0}{x_0^\top P^{\mathrm{LMI}} x_0} = 2.3607.$$

Farokhi & J, TAC, 2015

Adaptive Controllers



Consider a general (nonlinear) adaptive controller with limited model information

$$u(k) = \mathbf{K}(\mathcal{F}_k)$$

$$\mathcal{F}_k = \sigma(\{x(t)\}_{t=0}^{k} \cup \{u(t)\}_{t=0}^{k-1})$$

Then, there exists a control design method $\mathbf{K} = \Gamma^*(P)$ such that

$$J_P(\Gamma^*(P)) \overset{as}{=} J_P(\mathbf{K}^*(P))$$

where $\mathbf{K}^*(P)$ is the optimal controller with full model information

- It is possible to achieve a competitive ratio equal to one for an **adaptive** controller with limited plant model information
- Proof is constructive, uses adaptation algorithm of [Campi & Kumar, 1998]

Farokhi & J, SCL, 2015

Outline

- Introduction
- Distributed control: local model information
- Distributed control: local interactions
  - How much network interaction is needed?
  - How fast convergence is possible?
- Conclusions

Mathematical Model

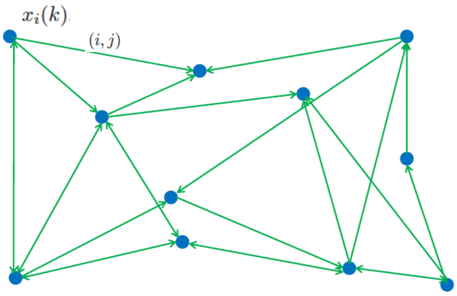
Directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Node set $\mathcal{V} = \{1, 2, \ldots, n\}$

Arc $e = (i, j) \in \mathcal{E}$

Time-varying graph process

$$\mathcal{G}_k(\omega) = (\mathcal{V}, \mathcal{E}_k(\omega)), k = 0, 1, \ldots$$



To each node $i \in \mathcal{V}$, associate a scalar state $x_i(k)$

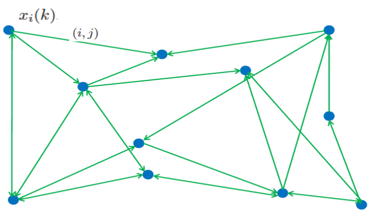
x_i updates based on own computation and neighbor information

$$\mathcal{N}_i(k) = \{j \in \mathcal{V} : \ (j, i) \in \mathcal{E}_k\} \cup \{i\}$$

Objective

Control the states to agreement: $\lim_{k \to \infty} |x_i(k) - x_j(k)| = 0$ for all $i, j \in \mathcal{V}$

Also called *consensus*, *rendezvous*, *formation*, etc



Local update law

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) x_j(k)$$

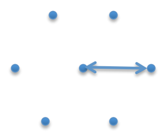
Prototype model for a collaborative control problem with coupled network and node dynamics

Related work on Markov chains, belief evolution, consensus algorithms, distributed control etc:
Hajnal (1958), Wolfowitz (1963), DeGroot (1974), Tsitsiklis, Bertsekas & Athans (1986), Jadbabaie, Lin & Morse (2003), Moreau (2005), Ren & Beard (2005), Golub & Jackson (2007), Cao, Anderson & Morse (2008), Acemoglu, Ozdaglar & ParandehGheib (2010), etc

Symmetric Gossip Algorithm

At each k, select a pair of nodes that "gossip":

$$x_i(k+1) = \begin{cases} [x_i(k) + x_j(k)]/2 & \text{if } (i,j) \text{ or } (j,i) \text{ is selected} \\ x_i(k) & \text{otherwise} \end{cases}$$



Equivalently

$$x(k+1) = P_k x(k),$$

where

$$P_k \in \left\{ I - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : i, j \in \mathsf{V} \right\}$$

with e_m being the n-dimensional unit vector whose m'th component is 1.

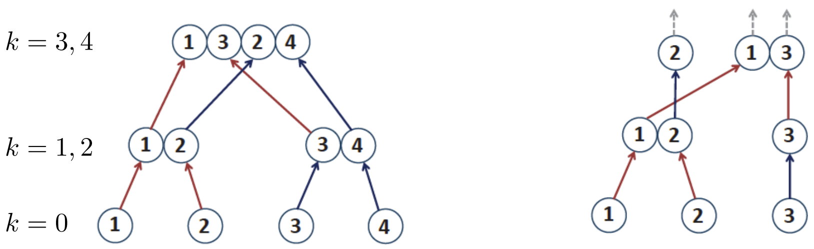
Various bounds on the convergence time to asymptotic consensus, e.g., Karp et al. (2000), Kempe et al. (2003), Boyd et al., (2006), Shah (2008)

Gossiping Convergence: Examples

$$x_i(k+1) = \begin{cases} [x_i(k) + x_j(k)]/2 & \text{if } (i,j) \text{ or } (j,i) \text{ is selected} \\ x_i(k) & \text{otherwise} \end{cases}$$

Convergence in 4 steps for n=4 nodes for all initial values

No finite-time convergence for n=3 nodes for almost all initial values

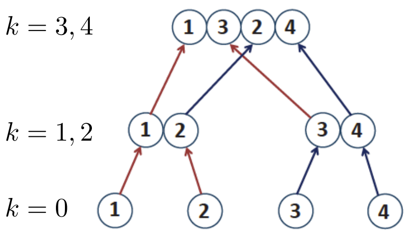


Definition of Finite-time Convergence

$$x(k+1) = P_k x(k)$$

A symmetric gossip algorithm $\{P_k\}_0^\infty$ converges globally in finite time if there exists an integer $T \geq 0$ such that

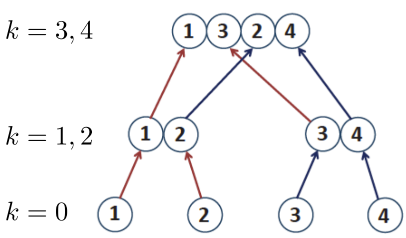
$$\text{rank}\Big(P_T \dots P_0\Big) = 1.$$

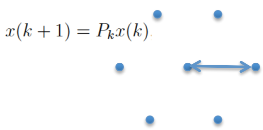


Finite-Time Convergence of Symmetric Gossiping

Theorem

There exists a symmetric gossip algorithm that converges globally in finite time if and only if $n = 2^m$ for some integer $m \geq 0$.





Shi et al., 2015

Finite-Time Convergence of Symmetric Gossiping

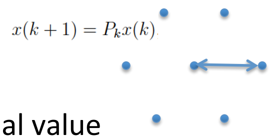
Theorem

There exists a symmetric gossip algorithm that converges globally in finite time if and only if $n = 2^m$ for some integer $m \geq 0$.

"Proof"

Sufficiency: Induction over n

Necessity: Contradiction using a particular initial value



Proof is constructive: for $n = 2^m$ it provides a **fastest algorithm** converging in $(n \log_2 n)/2$ steps

Shi et al., 2015

Impossibility of Finite-Time Convergence of Symmetric Gossiping

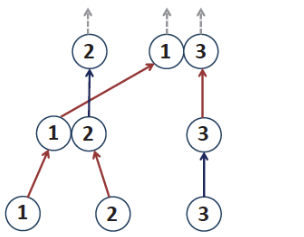
Theorem

Suppose there exists no integer $m \geq 0$ such that $n = 2^m$. Then for almost all initial values (under standard Lebesgue measure), it is impossible to find a symmetric gossip algorithm to reach finite-time convergence under the given initial value.

Initial value

$x_1(0) = 1 \quad x_2(0) = 3 \quad x_3(0) = 2$

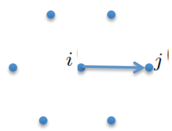
yields finite-time convergence, but is an exception.



Shi et al., 2015

Asymmetric Gossip Algorithm

$$x_i(k+1) = x_i(k)$$
$$x_j(k+1) = \frac{1}{2}x_i(k) + \frac{1}{2}x_j(k)$$



Equivalently

$$x(k+1) = P_k x(k),$$

where

$$P_k \in \left\{I - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : i, j \in \mathsf{V}\right\} \bigcup \left\{I - \frac{e_i(e_i - e_j)^T}{2} : i, j \in \mathsf{V}\right\}$$

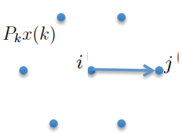
with e_m being the n-dimensional unit vector whose m'th component is 1.

Finite-Time Convergence of Asymmetric Gossiping

Theorem

For any network with n nodes, there always exists a gossip algorithm with asymmetric updates that converges globally in finite time.

$x(k+1) = P_k x(k)$



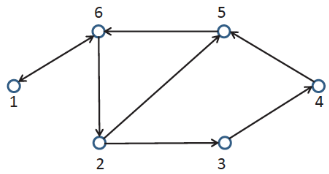
Consider a network with $n = 2^m + r$ nodes for $0 < r < 2^m$. A fastest gossip algorithm allowing asymmetric updates reaches convergence using $mn + 2r$ node updates.

Shi et al., 2015

Other Distributed Averaging Algorithms

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

$\eta_k \in [0,1]$ and $\alpha_k \in [0, 1-\eta_k]$



$\eta_k \equiv 0$, $\alpha_k \equiv 0$: distributed maximizing
$\eta_k \equiv 0$, $\alpha_k \equiv 1$: distributed minimizing
$\eta_k \in (0,1]$, $\alpha_k \in [0, 1-\eta_k]$: distributed weighted averaging

Impossibilities of Convergence

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

Averaging algorithms: $\eta_k \in (0,1]$, $\alpha_k \in [0, 1-\eta_k]$

Theorem: For every averaging algorithm, **finite-time** convergence fails for all initial conditions except for the consensus manifold.

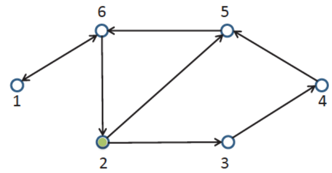
Theorem: For every averaging algorithm, **asymptotic** convergence fails for all initial conditions except for the consensus manifold if $\sum_{k=0}^{\infty}(1-\eta_k) < \infty$.

Shi & J, ACC, 2013

Convergence of Maximizing Algorithms

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

Maximizing algorithms: $\eta_k \equiv \alpha_k \equiv 0$



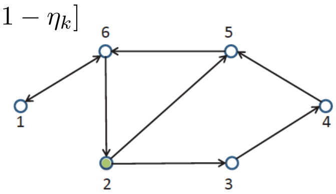
Theorem: Suppose $\mathcal{G}_k \equiv \mathcal{G}_*$ is a fixed graph. Global finite-time convergence is achieved if and only if $\mathcal{G}_*$ is strongly connected.

Shi & J, ACC, 2013

Convergence of Averaging Algorithms

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

Averaging algorithms: $\eta_k \in (0, 1]$, $\alpha_k \in [0, 1 - \eta_k]$



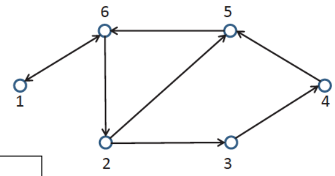
Theorem: Suppose $\mathcal{G}_k \equiv \mathcal{G}_*$ is a fixed graph and $\alpha_k \equiv \alpha > 0$. Global asymptotic convergence is achieved if and only if $\mathcal{G}_*$ has a root.

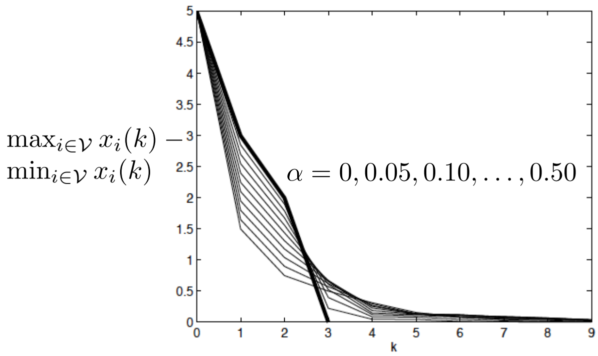
Shi & J, ACC, 2013

Example

$$x_i(k+1) = \alpha \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1-\alpha) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

- $0 < \alpha < 1$: global **asymptotic** consensus
- $\alpha = 0$ or $\alpha = 1$: global **finite-time** consensus





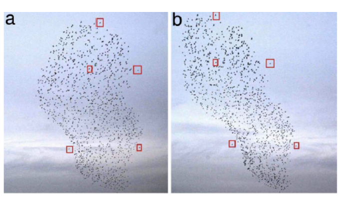
State-Dependent Nearest-Value Graphs

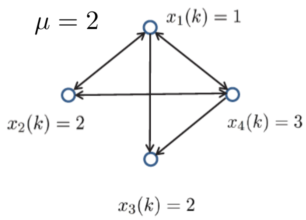
Fix positive integer μ

Neighbors of node $i \in \mathcal{V}$ are nodes in the union of

$$\mathcal{N}_i^-(k) = \{\text{nearest } \mu \text{ neighbors } j \in \mathcal{V} \text{ with } x_j(k) < x_i(k) \text{ and distinct values}\}$$

$$\mathcal{N}_i^+(k) = \{\text{nearest } \mu \text{ neighbors } j \in \mathcal{V} \text{ with } x_j(k) > x_i(k) \text{ and distinct values}\}$$





Motivated from recent studies of bird collective behavior [Ballerini et al, PNAS, 2008]:

In fact, we discover that each bird interacts on average with a fixed number of neighbours (six-seven), rather than with all neighbours within a fixed metric distance.

Finite-time Convergence

$$x_i(k+1) = \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \eta_k - \alpha_k) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

Theorem: Consider a nearest-value graph and an averaging algorithm with $\eta_k \equiv 0$ and $\alpha_k \in (0,1)$.

(i) If $n \leq 2\mu$, then global finite-time consensus is achieved.

(ii) If $n > 2\mu$, then no finite-time consensus is achieved for almost all initial conditions.

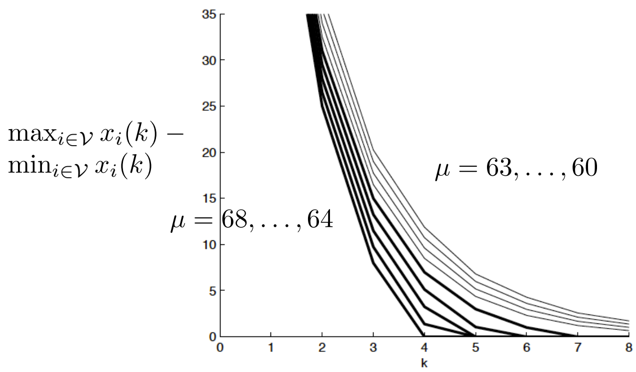
Finite-time convergence only with sufficiently many neighbors

Shi & J, ACC, 2013

Example

$$x_i(k+1) = \alpha \min_{j \in \mathcal{N}_i(k)} x_j(k) + (1 - \alpha) \max_{j \in \mathcal{N}_i(k)} x_j(k)$$

$n = 128$ nodes and $\alpha = 1/2$

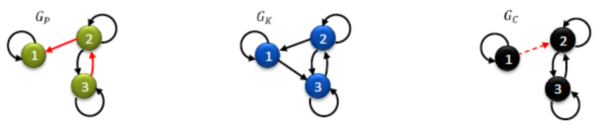


Outline

- Introduction
- Distributed control: local model information
- Distributed control: local interactions
- Conclusions

Conclusions

- **Global plant model information** is seldom available in cyber-physical control systems
- A framework to study the effect of (very) limited exchange of plant model information on the performance
- Simpler **control** strategies **vs** more **communication**:



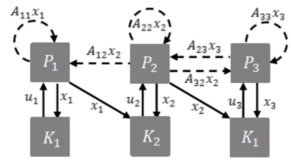
- **Finite-time convergence** of some low-order protocols

http://people.kth.se/~kallej

What about **dynamic** controllers?

What about **under-actutated** subsystems?

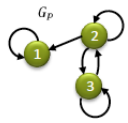
Networked Control System



Plant Graph:

$$x_i(k+1) = A_{ii}x_i(k) + \sum_{j \neq i} A_{ij}x_j(k) + B_{ii}u_i(k) + H_{ii}w_i(k)$$

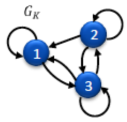
Plant: $P = (A, B, H) \in \mathcal{A} \times \mathcal{B} \times \mathcal{H}$ $\quad x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{n_i}$, and $w_i \in \mathbb{R}^{n_i}$



Control Graph:

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = C_K(zI - A_K)^{-1}B_K + D_K$$

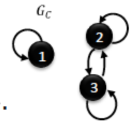
$\mathcal{K} = \{K \in (\mathcal{RL}_\infty)^{n \times n} | K_{ij} = 0 \in (\mathcal{RL}_\infty)^{n_i \times n_j}$ for all $1 \leq i, j \leq q$ such that $(s_K)_{ij} = 0\}$



Design Graph:

$$K = \Gamma(P) = \Gamma(A, B, H)$$

The map $[\Gamma_{i1} \ \cdots \ \Gamma_{iq}]$ is only a function of $\{[A_{j1} \ \cdots \ A_{jq}], B_{jj}, H_{jj} | (s_C)_{ij} \neq 0\}$.



Performance Metric

The **competitive ratio** of a control design method Γ is defined as

$$r_P(\Gamma) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))}$$

$$J_P(K) = \|T_{wy}(z)\|_2^2$$

$T_{wy}(z)$ is the closed-loop transfer function from exogenous input $w(k)$ to output

$$y(k) = \begin{bmatrix} C \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ D \end{bmatrix} u(k)$$

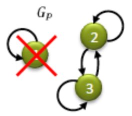
C and D are full-rank block-diagonal square matrices.

Assumptions

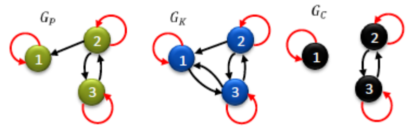
- All subsystems are fully actuated:

  $B \in \mathbb{R}^{n\times n}$ and $\underline{\sigma}(B) \geq \epsilon > 0.$

- G_P contains no isolated node.

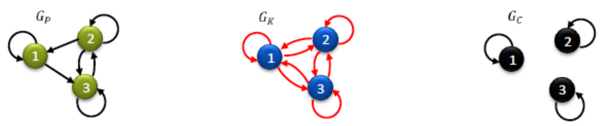


- G_P, G_K, G_C contain all self-loops.



- To simplify the presentation, fix $\epsilon = 1$ and $C = D = I$.

Modified Deadbeat Undominated by Dynamic Controllers



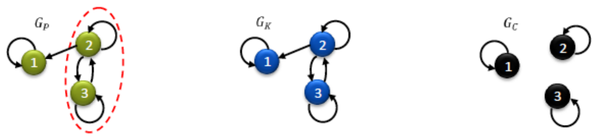
Theorem: $\left.\begin{array}{r} G_K \text{ is a complete graph} \\ G_C \text{ is fully disconnected} \end{array}\right\} \Rightarrow r_P(\Gamma) \geq r_P(\Gamma^{\ominus}) \quad \forall \Gamma \in \mathscr{C} \quad \& \quad \Gamma^{\ominus}$ is undominated

$\left.\begin{array}{r} G_P \text{ is acyclic} \\ G_K \supseteq G_P \\ G_C \text{ is fully disconnected} \end{array}\right\} \Rightarrow r_P(\Gamma) \geq r_P(\Gamma^{\ominus}) \quad \forall \Gamma \in \mathscr{C} \quad \& \quad \Gamma^{\ominus}$ is undominated

If enough controller communication, static controller $\Gamma^{\ominus}(A, B)$ suffices to outperform more complex controller

This is true even though $K^*(P)$ is dynamic

Extension to Under-actuated Systems



$$\mathcal{B}' = \{B \in \mathbb{R}^{n \times m} \mid \underline{\sigma}(B) \geq \epsilon, B_{ij} = 0 \in \mathbb{R}^{n_i \times m_j} \text{ for all } 1 \leq i \neq j \leq q\}$$

- If node i is a sink, assume:
  - $\mathrm{rank}(B_{ii}) = m_i \leq n_i$
  - (A_{ii}, B_{ii}) is controllable
  - $\mathrm{span}(A_{ij}) \subseteq \mathrm{span}(B_{ii})$ for all $j \neq i$
- If node i is not a sink, assume:
  - $m_i = n_i$

Example: Vehicle Platooning

Regulating inter-vehicle distances d_{12} and d_{23}

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{d}_{12}(t) \\ \dot{v}_2(t) \\ \dot{d}_{23}(t) \\ \dot{v}_3(t) \end{bmatrix} = \begin{bmatrix} -\varrho_1/m_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\varrho_2/m_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\varrho_3/m_3 \end{bmatrix} \begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} b_1/m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_2/m_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_3/m_3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \\ w_5(t) \end{bmatrix}$$

$\varrho_1 = \varrho_2 = \varrho_3 = 0.1$ and $b_1 = b_2 = b_3 = 1.0$

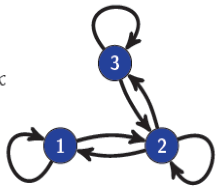
Example: Vehicle Platooning

Regulating inter-vehicle distances d_{12} and d_{23}

$$\begin{bmatrix} \dot v_1(t) \\ \dot d_{12}(t) \\ \dot v_2(t) \\ \dot d_{23}(t) \\ \dot v_3(t) \end{bmatrix} = \begin{bmatrix} -\varrho_1/m_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\varrho_2/m_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\varrho_3/m_3 \end{bmatrix} \begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} b_1/m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_2/m_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_3/m_3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \\ w_5(t) \end{bmatrix}$$

$\varrho_1 = \varrho_2 = \varrho_3 = 0.1$ and $b_1 = b_2 = b_3 = 1.0$

$$u_1(k) = \Gamma_1(m) \begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \end{bmatrix} \quad u_2(k) = \Gamma_2(m) \begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix} \quad u_3(k) = \Gamma_3(m) \begin{bmatrix} v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix} \quad \mathcal{G}_{\mathcal{K}}$$



where $m = [m_1\, m_2\, m_3]^\top \in [0.5, 1.0]^3$.

Example: Vehicle Platooning

Regulating inter-vehicle distances d_{12} and d_{23}

$$\begin{bmatrix} \dot v_1(t) \\ \dot d_{12}(t) \\ \dot v_2(t) \\ \dot d_{23}(t) \\ \dot v_3(t) \end{bmatrix} = \begin{bmatrix} -\varrho_1/m_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\varrho_2/m_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\varrho_3/m_3 \end{bmatrix} \begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} b_1/m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_2/m_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_3/m_3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \\ w_5(t) \end{bmatrix}$$

$\varrho_1 = \varrho_2 = \varrho_3 = 0.1$ and $b_1 = b_2 = b_3 = 1.0$

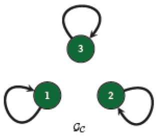
$$z(t) = \begin{bmatrix} d_{12}(t) & d_{23}(t) & u_1(t) & u_2(t) & u_3(t) \end{bmatrix}^\top$$

Find control design strategy Γ that

$$\min_{\Gamma} \max_{\alpha} \|T_{zw}(s; \Gamma, m)\|_\infty$$

where $m = [m_1\, m_2\, m_3]^\top \in [0.5, 1.0]^3$ and Γ belongs to the set of polynomials of m_i, $i = 1, 2, 3$, up to the second order.

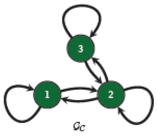
Example: Vehicle Platooning



Control Design with Local Model Information

$$\max_{\alpha \in \mathcal{A}} \; \left\| T_{zw} \left(s; \Gamma^{\text{local}}, \alpha \right) \right\|_{\infty} = 4.7905$$

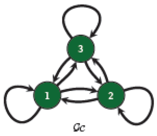
25.8%



Control Design with Limited Model Information

$$\max_{\alpha \in \mathcal{A}} \; \left\| T_{zw} \left(s; \Gamma^{\text{limited}}, \alpha \right) \right\|_{\infty} = 3.5533$$

5.4%



Control Design with Full Model Information

$$\max_{\alpha \in \mathcal{A}} \; \left\| T_{zw} \left(s; \Gamma^{\text{full}}, \alpha \right) \right\|_{\infty} = 3.3596$$