Zeno Hybrid Systems*

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Abstract

The interacting continuous and discrete dynamics in hybrid systems may lead to Zeno executions, which are solutions of the system having infinitely many discrete transitions in finite time. Although physical systems do not show Zeno behavior, models of real systems may be Zeno due to modeling abstraction. It is hard to analyze such models with the existing theory. Since abstraction is an important tool in the hierarchical design of hybrid systems, one would like to determine when it may lead to Zeno models. Zeno hybrid systems are studied in detail in the paper. Necessary and sufficient conditions for the existence of Zeno executions are given. The Zeno set is introduced as the ω limit set of a Zeno execution. Properties of the Zeno set are derived for a fairly large class of hybrid systems.

Keywords: Hybrid automata; Zeno execution; Zeno sets

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1 Introduction

Hybrid systems have proved to be an effective tool for the modeling, analysis and design of a large number of evolving technological systems, in which digital devices interact with an analog environment. Systems of this type are common in embedded computation, robotics, mechatronics, avionics, and process control. Thanks to the rapid advances in computer technology, hybrid systems are becoming increasingly relevant and important and consequently have attracted considerable research interest. However, despite recent progress, there are a number of fundamental properties of hybrid systems that have not been investigated to sufficient detail. These include existence and uniqueness of executions, which have only recently been addressed [1, 2, 3, 4]. Another such issue is Zeno executions.

Roughly speaking, an execution of a hybrid system is called *Zeno*, if it takes infinitely many discrete transitions in a finite time interval. Physical systems are, of course, not Zeno, but a hybrid model of a physical system may be Zeno, due to modeling over-abstraction. Since abstraction is an important tool for handling complex systems, understanding when it leads to Zeno hybrid systems is essential, for example for the development of simulation tools for hybrid systems. Zeno hybrid systems, or systems "close" to Zeno, make computer simulations imprecise and time-consuming. Most simulation packages developed for hybrid systems, such as Dymola [5], Omola [6], and SHIFT [7], get stuck when a large number of discrete transitions take place within a short time interval. It is therefore important to understand the Zeno phenomenon in order to develop efficient computational tools for hybrid systems.

It is difficult to draw conclusions about Zeno systems using the available theory. Zeno hybrid automata have been studied to some extent in the theoretical computer science literature [8, 9, 10, 11, 12]. The continuous dynamics in those cases, however, are quite limited. Zeno hybrid automata with more general nonlinear vector fields have only recently been investigated [13, 14, 15, 16]. The lack of theoretical results has often lead researchers to impose non-Zeno assumptions by default. For example, this is the case in recent work on hybrid control design [17, 18, 19, 20]. The work presented in this paper is a first step towards building a suite of results to characterize Zeno hybrid systems. Our results are useful, for instance, when designing hybrid controllers. Since a unified theory for hybrid control design does not yet exist, one has to prove that the closed-loop system is well-posed on a case by case basis; this includes proving that the system is non-Zeno (see for example [21]).

The main contribution of the paper is to present some fundamental properties of Zeno executions and Zeno hybrid automata. We introduce the Zeno set as the ω limit set of a Zeno execution. A complete characterization of the Zeno set is given for a few quite general classes of hybrid systems. The features of the resets turn out to be very important. For example, we show that if the resets are all identity maps or all contracting on the guard, the continuous part of the state converges. We also investigate the conditions under which there are no Zeno execution. In particular, it is proved that for hybrid automata with identity resets on the guard, if the guards and the interior of the domains are disjoint and if the boundaries of the domains for any cycles are also disjoint, then the hybrid automata do not accept Zeno executions.

The outline of the paper is as follows. In Section 2 we introduce the notation and the definitions of hybrid automata and execution (Section 2.1), followed by a number of examples of Zeno hybrid automata from the areas of modeling, simulation, verification, and control (Section 2.2). The examples are used to motivate the analysis in the rest of the paper. Zeno hybrid automata and Zeno sets are introduced in Section 3, and some properties of Zeno sets are discussed in Section 3.2. Section 3.3 presents both necessary and sufficient conditions for the existence of Zeno executions. A summary and conclusions are given in Section 4. To maintain the flow of the paper all technical proofs are given in the appendix.

2 Background and Motivation

2.1 Hybrid Automata, Executions and Underlying Assumptions

For a finite collection V of variables, let $\mathbf{V}$ denote the set of valuations of these variables. We use lower case letters to denote both variables and their valuations. We refer to variables whose set of valuations is finite or countable as *discrete* and to variables whose set of valuations is a subset of a Euclidean space as *continuous*. For a set of continuous variables X with $\mathbf{X} = \mathbb{R}^n$ for $n \geq 0$, we assume that $\mathbf{X}$ is given the Euclidean metric topology, and use $\|\cdot\|$ to denote the Euclidean norm. For a set of discrete variables Q, we assume that $\mathbf{Q}$ is given the discrete topology (every subset is an open set), generated by the metric $d_D(q, q') = 0$ if $q = q'$ and $d_D(q, q') = 1$ if $q \neq q'$. We denote the valuations of the union of Q and X by $\mathbf{Q} \times \mathbf{X}$, which is given the product topology generated by the metric $d((q, x), (q', x')) = d_D(q, q') + \|x - x'\|$. We assume that a subset U of a topological space is given the induced topology, and we use $\overline{U}$ to denote its closure, U^o its interior, $\partial U = \overline{U} \setminus U^o$ its boundary, U^c its complement, $|U|$ its cardinality, and $P(U)$ the set of all subsets of U.

The following definitions are based on [22, 13, 2].

Definition 1 (Hybrid Automaton) *A hybrid automaton H is a collection $H = (Q,\ X,$ $\mathrm{Init},\ f,\ D,\ E,\ G,\ R)$, where*

- *Q is a finite collection of discrete variables;*
- *X is a finite collection of continuous variables with $\mathbf{X} = \mathbb{R}^n$;*
- *$\mathrm{Init} \subseteq \mathbf{Q} \times \mathbf{X}$ is a set of initial states;*
- *$f : \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$ is a vector field;*
- *$D : \mathbf{Q} \to P(\mathbf{X})$ is map assigning to each $q \in \mathbf{Q}$ a subset of $\mathbf{X}$ called the domain*[1] *of q;*
- *$E \subset \mathbf{Q} \times \mathbf{Q}$ is a set of edges;*
- *$G : E \to P(\mathbf{X})$ is a map assigning to each edge $e \in E$ a subset of $\mathbf{X}$ called the guard of e; and*
- *$R : E \times \mathbf{X} \to P(\mathbf{X})$ is a reset map, assigning to each edge $e \in E$ and each $x \in \mathbf{X}$ a subset of $\mathbf{X}$.*

[1] The domain is sometimes called the invariant set in the hybrid system literature in computer science.

We refer to $(q, x) \in \mathbf{Q} \times \mathbf{X}$ as the *state* of H. Throughout the paper, it is assumed that $|\mathbf{Q}| < \infty$ and that f is Lipschitz continuous in its second argument. Further, we assume that for all $e \in E$, $G(e) \neq \emptyset$ and for all $x \in G(e)$, $R(e, x) \neq \emptyset$.[2] A hybrid automaton can be represented by a directed graph $(\mathbf{Q}, E)$, with vertices $\mathbf{Q}$ and edges E. For an example, see Figure 1. For each vertex, $q \in \mathbf{Q}$, we specify a vector field, $f(q, \cdot)$ and a domain, $D(q)$. For each edge we specify a guard, $G(e)$, and a reset map, $R(e, \cdot)$ (which is suppressed if $R(e, x) = \{x\}$). The discrete part of the initial state is indicated by a double circle and the continuous part by an arrow. Since there is a unique graphical representation for each hybrid automaton, we will use the corresponding graph as a formal definition in the examples.

Definition 2 (Hybrid Time Trajectory) *A hybrid time trajectory is a finite or infinite sequence of intervals* $\tau = \{I_i\}_{i=0}^N$, *such that*

- $I_i = [\tau_i, \tau_i']$ *for all* $0 \leq i < N$,
- *if* $N < \infty$ *then either* $I_N = [\tau_N, \tau_N']$ *or* $I_N = [\tau_N, \tau_N')$,
- $\tau_i \leq \tau_i'$ *for all* i *and* $\tau_i' = \tau_{i+1}$ *for all* $0 \leq i < N$.

A hybrid time trajectory is a sequence of intervals of the real line, whose end points overlap. The interpretation is that the end points of the intervals are the times at which discrete transitions take place. Note that $\tau_i = \tau_i'$ is allowed, therefore multiple discrete transitions may take place at the same time. Since all hybrid automata that will be discussed are time invariant, we assume, without loss of generality, that $\tau_0 = 0$. Hybrid time trajectories can extend to infinity if τ is an infinite sequence or if it is a finite sequence ending with an interval of the form $[\tau_N, \infty)$.

For a hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$, let $\langle\tau\rangle$ denote the set $\{0, 1, \ldots, N\}$ if τ is finite and $\{0, 1, \ldots\}$ if τ is infinite. We use q and x to denote the time evolution of the discrete and continuous state, respectively (with a slight abuse of notation). Here q is a map from $\langle\tau\rangle$ to

[2] This can be done without loss of generality, since if a hybrid automaton violates these conditions, one can construct another hybrid automaton that accepts exactly the same set of executions and satisfies the conditions [2].

$\mathbf{Q}$ and $x = \{x^i : i \in \langle\tau\rangle\}$ is a collection of C^1 maps. An execution is now defined as a triple $\chi = (\tau, q, x)$ in the following way.

Definition 3 (Execution) *An execution of a hybrid automaton H is a collection $\chi = (\tau, q, x)$, where τ is a hybrid time trajectory, $q : \langle\tau\rangle \to \mathbf{Q}$ is a map, and $x = \{x^i : i \in \langle\tau\rangle\}$ is a collection of C^1 maps $x^i : I_i \to \mathbf{X}$, such that*

- $(q(0), x^0(0)) \in \mathrm{Init}$,
- *for all $i \in \langle\tau\rangle$ and for all $t \in I_i$, $\dot{x}^i(t) = f\big(q(i), x^i(t)\big)$ and for all $t \in [\tau_i, \tau_i')$, $x^i(t) \in D(q(i))$,*
- *for all $i \in \langle\tau\rangle$, $e = (q(i), q(i+1)) \in E$, $x^i(\tau_i') \in G(e)$, and $x^{i+1}(\tau_{i+1}) \in R(e, x^i(\tau_i'))$.*

We say a hybrid automaton *accepts* an execution χ. For an execution $\chi = (\tau, q, x)$, we use $(q_0, x_0) = \big(q(0), x^0(0)\big)$ to denote the initial state of χ. The *execution time* $\mathcal{T}(\chi)$ is defined as $\mathcal{T}(\chi) = \sum_{i=0}^{N}(\tau_i' - \tau_i) = \lim_{i\to N} \tau_i'$, where $N+1$ is the number of intervals in the hybrid time trajectory. An execution is called *finite* if τ is a finite sequence ending with a compact interval, it is called *infinite* if τ is either an infinite sequence or if $\mathcal{T}(\chi) = \infty$, and it is called *Zeno* if it is infinite but $\mathcal{T}(\chi) < \infty$. For a Zeno execution χ, we call $\tau_\infty = \mathcal{T}(\chi)$ the *Zeno time*. We use $\mathcal{E}_H^\infty(q_0, x_0)$ to denote the set of all infinite executions of H with initial condition $(q_0, x_0) \in \mathrm{Init}$. All the hybrid automata considered in this paper are assumed to be *non-blocking*, in the sense that $\mathcal{E}_H^\infty(q_0, x_0) \neq \emptyset$ for all $(q_0, x_0) \in \mathrm{Init}$. Conditions for determining when this is the case are given in [2].

A state $(q, x) \in \mathbf{Q} \times \mathbf{X}$ is called *reachable* by H, if there exists a finite execution $\chi = (\tau, q, x)$ with $\tau = \{I_i\}_{i=0}^N$ and $(q(N), x^N(\tau_N')) = (q, x)$. We use Reach_H to denote the set of states reachable by a hybrid automaton H. Throughout this paper, we assume that $\mathrm{Reach}_H \subseteq \bigcup_{q\in\mathbf{Q}} \{q\} \times \overline{D(q)}$. Conditions under which this is the case can be established using invariant assertions, proved by induction arguments over the length of the executions [2].

2.2 Motivating Examples

It should be noted that the Zeno phenomenon is a strictly hybrid phenomenon in the sense that both continuous dynamics and discrete dynamics are needed to generate Zeno executions. In this section, we illustrate how Zeno hybrid automata appear in different areas of hybrid systems. In particular, we give examples of Zeno hybrid automata in modeling, simulation, verification, and control.

Modeling and Simulation

Hybrid automata provide a natural framework for developing models with abstracted dynamics. This is a useful approach when analyzing complex systems, and for control design it often leads to an appealing hierarchical system structure. However, if the abstraction is not done carefully, erroneous conclusions may be derived from the model, as illustrated by the following example.

Example 1 (Water Tank System)

Consider the water tank system of Alur and Henzinger [9], shown in Figure 1. Here x_i denotes the volume of water in Tank i, and $v_i > 0$ denotes the constant flow of water out of Tank i. Let w denote the constant flow of water into the system, directed exclusively to either Tank 1 or Tank 2 at each point in time. The objective is to keep the water volumes above r_1 and r_2, respectively (assuming that $x_1(0) > r_1$ and $x_2(0) > r_2$). This is to be achieved by a switched control strategy that switches the inflow to Tank 1 whenever $x_1 \leq r_1$ and to Tank 2 whenever $x_2 \leq r_2$. It is straightforward to show that the unique infinite execution that the hybrid automaton accepts for each initial state is Zeno, if $\max\{v_1, v_2\} < w < v_1 + v_2$. The Zeno time is $\tau_\infty = \big(x_1(0) + x_2(0) - r_1 - r_2\big)/(v_1 + v_2 - w)$. Of course, a real implementation of the water tank system cannot be Zeno, but instead one or both of the tanks will drain. The Zeno model does not capture this. The actual scenario depends on the dynamics of the switch, which in the model was assumed to be instantaneous. For further discussions on this example, see [13].

It is difficult to run efficient computer simulations for systems that show a large number of discrete transitions during a short time interval. Often, either the numerical error or the simulation time (or both) will be unsatisfactory [13, 14]. One class of systems where this problem arises is mechanical systems with friction. If the friction is modeled as Coulomb friction, then the frictional force F_f is given by $F_f = -K \operatorname{sgn} v$, where v is the relative velocity of the contact surfaces and $K > 0$ is a constant. We may easily model a system with Coulomb friction as a hybrid automaton, where the domains and the guards depend on the sign of the velocity. Frictional systems sometimes have so called stick–slip motion, which means that the motions is divided into two phases both of non-zero duration: one when $v \neq 0$ and one when $v = 0$ (for a simple example see [23]). For the hybrid automaton, the latter corresponds to a Zeno behavior, because it implies that the velocity switches infinitely fast between a positive and a negative value. Resolving Zenoness by introducing a new discrete state, which has a vector field given by the continuous dynamics corresponding to the sticking motion, has been proposed for simple examples [24, 25], but so far no rigorous method seems to have been developed. In many cases, such a method would speed up the simulation considerably. Other simulation methods to avoid Zeno include time-stepping methods [26]. It should be pointed out that modeling of friction and impacts for rigid bodies is, of course, by itself a very active field, where advanced mathematical tools are used to handle contacts in a consistent way [27]. The example here is meant as an illustration of the multi-domain modeling approach taken in hybrid systems.

Analysis and Verification

Most of the verification methods proposed for hybrid systems seek to determine whether the set of states reachable by a hybrid automaton satisfies certain properties. For example, *model checking* techniques involve computer algorithms that "explore" the set of reachable states automatically. This approach, developed in theoretical computer science for purely discrete systems, has been extended to timed automata [28], multi-rate automata [29], hybrid automata with constant differential inclusions [30] and, most recently, classes of hybrid automata with linear vector fields [31]. *Deductive* techniques, on the other hand, seek to directly establish properties of the executions of the hybrid automaton, by proving, for example, invariant assertions [32]. Though the analysis is not completely automated in this case, the proofs may be assisted by theorem provers [33, 34].

These verification techniques may lead to misleading claims when applied to Zeno hybrid automata, since the set of states reachable by a Zeno model may not reflect the states reachable by the underlying physical system. For example, for the water tank hybrid automaton (which is in fact a multi-rate automaton) one can show that for all the reachable states the water in the tanks will be above the desired low water marks. Clearly this cannot be the case for the physical system, when the rate at which water is added to the tanks, w, is less than the total rate at which water is removed, $v_1 + v_2$, i.e., when the hybrid automaton model is Zeno.

Similar problems are encountered when one tries to extend Lyapunov type analysis techniques to hybrid systems. This has led researchers in this area to explicitly add assumptions requiring the system to be non-Zeno [18, 17].

Safe and Optimal Control

Methods for designing controllers that ensure that a hybrid system is safe, in the sense that it does not reach an undesirable configuration, have also been developed. These methods, motivated by earlier work on purely discrete and purely continuous systems, have been extended to timed automata [35], hybrid automata with constant differential inclusions [36], and hybrid automata with nonlinear vector fields [22]. All the proposed approaches suffer from the drawback that they allow the controller to "cheat" by forcing the system to be Zeno, and thereby hiding the fact that unsafe states can be reached. This has again forced researchers to a-priori introduce non-Zeno assumptions. As an example consider the following problem from free flight air traffic management [37].

Example 2 (Aircraft Conflict Resolution)

Consider two aircraft moving at the same constant altitude along straight line trajectories. Introduce a set of coordinates that centers one aircraft at the origin and let (x_1, x_2) denote the relative coordinates of the other aircraft. The dynamics of this system is given by $\dot{x}_1 = -v_e + v_p \cos\theta$, $\dot{x}_2 = v_p \sin\theta$, where $v_e(t) \in [\underline{v}_e, \overline{v}_e] \subset (0,\infty)$ is the velocity of the first aircraft, $v_p(t) \in [\underline{v}_p, \overline{v}_p] \subset (0,\infty)$ the velocity of the second, and $\theta \in (-\pi,\pi)$ the constant relative orientation of the aircrafts. If v_e is treated as a control signal and v_p as a disturbance signal, the model describes a pursuit-evasion game with the first aircraft being the evader and the second the pursuer. The evader would like to prevent the pursuer from getting closer than a certain distance, which would define an unsafe flight configuration. Solving the game [37] leads to a saddle solution described by the hybrid automaton in Figure 2, where $\psi(x) = \operatorname{sgn} x_1 \sin\theta + \operatorname{sgn} x_2 \cos\theta$. It is easy to verify that all executions accepted by this hybrid automaton avoid the unsafe set. The configuration may, however, still be unsafe in practice. The reason is that the hybrid automaton accepts Zeno executions, for example, governed by the initial states depicted in the figure. They correspond to a situation where the evader constantly switches its velocity between $v_e = \underline{v}_e$ and $v_e = \overline{v}_e$. This is, of course, not realistic, because there are some dynamics involved in the switching. If this controller was implemented in practice, the system would most likely reach the unsafe set through a chattering trajectory.

Zeno type behavior may also arise for certain classes of optimal control problems.

Example 3 (Fuller's Phenomenon)

The hybrid automaton in Figure 3 generates the optimal controls for the problem of minimizing the performance index $\int_0^\infty x_1^p(t)\,dt$, $p > 1$, with respect to the dynamics $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, $x(0) = x_0 \neq (0,0)$, and the control constraint $|u(t)| \leq 1$. The domains and the guards involve the constant $C = C(p) \in (0, 1/2)$. It is possible to show that this hybrid automaton is Zeno [38]. In optimal control, this is referred to as Fuller's phenomenon [39].

3 Zeno Hybrid Automata and Zeno Executions

Zeno hybrid automata are defined in this section. The notion of Zeno set is introduced as the ω limit set for infinite executions with finite execution time. A number of examples are presented to illustrate the characteristics of the Zeno set. For a related discussion see [13, 14, 15, 16].

Definition 4 (Zeno Hybrid Automaton) *A hybrid automaton H is Zeno if there exists an initial state $(q_0, x_0) \in \mathrm{Init}$, such that all executions in $\mathcal{E}_H^\infty(q_0, x_0)$ are Zeno.*

Many models of real systems are Zeno, for example, the hybrid systems discussed in Section 2.2. An example of a Zeno execution is given next.

Example 4 (Multi-Level Bouncing Ball)

The hybrid automaton in Figure 4 (which is a variant of the bouncing ball automaton of [13]) accepts the execution illustrated by the simulation to the right. The continuous part of the execution is shown for $x_0 = (2, 0)$. It is easily checked that the hybrid automaton is Zeno by explicitly deriving the sequence of time intervals $\{\tau_i' - \tau_i\}_{i=1}^\infty$, which is a converging geometric series.

It is clear that Zenoness is due to the interplay between the vector field, the resets, and the guards. For example, if in Example 4 the resets of x_2 is replaced by $x_2 := x_2/(dx_2 - 1)$, where $d = 1/\sqrt{20x_1(0)}$, then it is easy to verify that $\{\tau_i' - \tau_i\}_{i=1}^\infty$ diverges as $\sum_{i=1}^\infty \{1/i\}$. Hence, the hybrid automaton will not accept any Zeno executions in this case.

3.1 ω limit sets and Zeno sets

To investigate the limiting properties of infinite executions, we generalized the concept of ω limit set [40] to hybrid automata.

Definition 5 (ω Limit Set) *A point $(\hat{q}, \hat{x}) \in \mathbf{Q} \times \mathbf{X}$ is an ω limit point of an infinite execution $\chi = (\tau, q, x)$, if there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in I_{i_n}$ for some $i_n \in \langle\tau\rangle$, such that as $n \to \infty$, $\theta_n \to \mathcal{T}(\chi)$ and $(q(i_n), x^{i_n}(\theta_n)) \to (\hat{q}, \hat{x})$. The ω limit set $S_\chi \subset \mathbf{Q} \times \mathbf{X}$ of an execution χ is the set of all ω limit points of χ.*

The Zeno set is introduced as the ω limit set of an infinite execution that has finite execution time.

Definition 6 (Zeno Set) *An ω limit point of a Zeno execution is called a Zeno point. The Zeno set of a Zeno execution is the set of all Zeno points.*

We use $Z_\infty \subset \mathbf{Q} \times \mathbf{X}$ to denote the Zeno set. In other words, Z_∞ consists of all cluster points of sequences $\{(q(i_n), x^{i_n}(\theta_n))\}_{n=0}^{\infty}$ with $\theta_n \in I_{i_n}$ and $i_n \in \langle \tau \rangle$ such that $\theta_n \to \tau_\infty$ as $n \to \infty$. We write $\mathbf{Q}_\infty = \{q \in \mathbf{Q} : \exists\, x \in \mathbf{X}, (q,x) \in Z_\infty\}$ for the discrete part of Z_∞ and $\mathbf{X}_\infty = \{x \in \mathbf{X} : \exists\, q \in \mathbf{Q}, (q,x) \in Z_\infty\}$ for the continuous part (notice that, in general, $Z_\infty \neq \mathbf{Q}_\infty \times \mathbf{X}_\infty$). The Zeno set can be empty, finite, countable, or even uncountable. For all the examples in the Section 2.2 the continuous state converged to a unique value, i.e. $Z_\infty \subset \mathbf{Q} \times \{\hat{x}\}$ for some $\hat{x} \in \mathbf{X}$. In Example 4, we have $Z_\infty = \{(q_1,(0,0)), (q_2,(3,0)), (q_2,(5,0))\}$, so that $\mathbf{Q}_\infty = \mathbf{Q}$ and $\mathbf{X}_\infty = \{(0,0),(3,0),(5,0)\}$. We present a few more examples to illustrate other possibilities.

Example 5 (Uncountable Zeno Set)

Consider a hybrid automaton H accepting a Zeno execution with $Z_\infty = \{(\hat{q}, \hat{x})\}$. Modify this hybrid automaton into a new hybrid automaton H' by adding two components (x_e, x_f) to the continuous state of H. For x_e and x_f, let the continuous dynamics in all discrete states be $\dot{x}_e = \dot{x}_f = 0$, the resets for all edges be

$$\begin{pmatrix} x_e \\ x_f \end{pmatrix} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_e \\ x_f \end{pmatrix}$$

where θ is a rational constant and the initial condition be $\{(1,0)\}$. The guards and the domains of H' are the obvious extensions of those of H. Then, the Zeno set of H' is $Z'_\infty = \{\hat{q}\} \times \{(\hat{x}, x_e, x_f) \in \mathbf{X} \times \mathbb{R}^2 : x_e^2 + x_f^2 = 1\}$. Hence, the Zeno set is an uncountable set.

Example 6 (Empty Zeno Set)

Consider a hybrid automaton that accepts a Zeno execution with $Z_\infty = \{(\hat{q}, \hat{x})\}$. Append a component x_e to the continuous state with trivial dynamics $\dot{x}_e = 0$, reset $x_e := 2x_e$ and initial condition $x_e(0) = 1$. Then, for all $\{\theta_n\}_{n=0}^{\infty}$ with $\theta_n \in I_{i_n}$ and $i_n \in \langle \tau \rangle$ such that as $n \to \infty$, $\theta_n \to \tau_\infty$, the sequence $\{x_e^{i_n}(\theta_n)\}_{n=0}^{\infty}$ has no cluster point. The modified hybrid automaton has no Zeno point (its Zeno set is empty), since the augmented continuous state blows up.

The discrete part of the Zeno set, $\mathbf{Q}_\infty$, will be visited infinitely often by a Zeno execution. A discrete state being visited infinitely often is, however, not necessarily in $\mathbf{Q}_\infty$, as shown by the following example.

Example 7 (Discrete States Visited Infinitely Often)

Consider the hybrid automaton in Figure 5. It is easy to see that it accepts a Zeno execution with Zeno time $\tau_\infty = 1$. The Zeno set is $Z_\infty = \{(q_1, (0, 1))\}$. The discrete state q_2 is visited infinitely often by the Zeno execution, but still $q_2 \notin \mathbf{Q}_\infty$. The reason for this is that x_2 blows up in q_2.

Lemma 2 in Section 3.3 gives conditions under which a discrete state that is visited infinitely often belongs to $\mathbf{Q}_\infty$.

For most hybrid automata, the discrete evolution of Zeno executions becomes periodic as the Zeno time is approached. However, in [41] hybrid automata that do not exhibit this periodic behavior are presented.

3.2 Properties of Zeno Sets

Determining the structure of the Zeno set can be very important in some cases. For example, if it turns out that the continuous state converges ($|\mathbf{X}_\infty| = 1$), one may hope to define extensions of the Zeno execution beyond τ_∞ using regularization techniques [13]. To study such properties of the Zeno set we introduce the following definitions. The reset R is called an *identity on* G if for all $(q, q') \in E$ and for all $x \in G(q, q')$, $R(q, q', x) = \{x\}$. R is called *non-expanding on* G if there exists $\delta \in [0, 1]$ such that for all $(q, q') \in E$, for all $x \in G(q, q')$ and for all $x' \in R(q, q', x)$

$$\|x'\| \leq \delta \|x\|$$

R is called *contracting on* G if the same is true for some $\delta \in [0, 1)$. Finally, R is called *non-contracting on* G if for all $(q, q') \in E$, for all $x \in G(q, q')$ and for all $x' \in R(q, q', x)$, $\|x'\| \geq \|x\|$.

In the proofs, the following lemma is used, which is an immediate generalization of the corresponding result for continuous-time systems [40, Prop. 5.3].

Lemma 1 *Consider a hybrid automaton with R non-expanding on G. Then, there exists $c > 0$ such that for all executions $\chi = (\tau, q, x)$, all $n \in \langle \tau \rangle$ and all $t \in I_n$, $\|x^n(t)\| \leq \big(\|x_0\| + 1\big)e^{ct} - 1$. If instead R is non-contracting on G, then $\|x^n(t)\| \geq \big(\|x_0\| + 1\big)e^{-ct} - 1$.*

For continuous-time systems, Lipschitz continuity of the vector field excludes the possibility for finite escape time. Lemma 1 allows us to draw a similar conclusion for hybrid systems whose reset is non-expanding on the guards. This, in particular, implies that all Zeno executions of a hybrid system remain bounded. Therefore, since the vector fields are assumed to be Lipschitz continuous, there exists some $K > 0$ such that for all $n \in \langle \tau \rangle$ and $t \in I_n$, $\|f(q(n), x^n(t))\| \leq K$. Moreover, for sequences $\{\theta_n\}_{n=0}^{\infty}$ with $\theta_n \in I_{i_n}$ and $i_n \in \langle \tau \rangle$ such that $\theta_n \to \tau_\infty$ as $n \to \infty$, the Weierstrass theorem implies that $\{q(i_n), x^{i_n}(\theta_n)\}_{n=0}^{\infty}$ has at least one cluster point. Therefore, Zeno executions of hybrid automata whose resets are non-expanding on the guards, have at least one Zeno point ($Z_\infty \neq \emptyset$).

For identity resets, the continuous part of the Zeno set is a single point, as is shown next.

Theorem 1 *For all Zeno executions of a hybrid automaton with R identity on G,* $|\mathbf{X}_\infty| = 1$.

A similar result holds if R is contracting on G.

Theorem 2 *For all Zeno executions of a hybrid automaton with R contracting on G,* $\mathbf{X}_\infty = \{0\}$.

Notice that the definition of contracting given above implicitly requires that $R(q, q', 0) = \{0\}$ for all $(q, q') \in E$ such that $0 \in G(q, q')$. The result can be extended to cases where the resets share any common fixed point, x^*, and are contracting after a change of coordinates taking x to $x - x^*$. This would, for example, allow us to extend Theorem 2 to cover appropriate classes of affine functions.

3.3 Existence of Zeno Executions

Since Zeno executions do not reflect the true behavior of a physical system, it is important to study under what conditions hybrid automata accept only Zeno executions. Sufficient and necessary conditions for this are presented in this section.

First, recall that every hybrid automaton can be associated with a directed graph $(\mathbf{Q}, E)$. It is obvious that a hybrid automaton is Zeno only if that graph has a cycle. The following observation is also straightforward.

Proposition 1 *If there exists a finite collection of states* $\{(q_i, x_i)\}_{i=1}^K$ *such that*

- $(q_1, x_1) = (q_K, x_K)$*;*
- $x_{i+1} = R(q_i, q_{i+1}, x_i)$ *for all* $i = 1, \ldots, K-1$*; and*
- $(q_i, x_i) \in \text{Reach}_H$ *for some* $i = 1, \ldots, K$*;*

then the hybrid automaton accepts a Zeno execution.

Example 7 shows that it is possible for a discrete state to be visited infinitely often by a Zeno execution, but still not appear in $\mathbf{Q}_\infty$. However, if the reset is non-expanding on the guards, this is not the case.

Lemma 2 *For all Zeno executions* $\chi = (\tau, q, x)$ *of a hybrid automaton with* R *non-expanding on* G*, there exists some* M *such that for all* $i \geq M$*,* $q(i) \in \mathbf{Q}_\infty$*.*

Lemma 2 can be used to establish the location of Zeno points with respect to the domains.

Theorem 3 *Consider a hybrid automaton with* R *non-expanding on* G *and assume it accepts a Zeno execution with Zeno set* $Z_\infty = \{(q_i, x_i)\}_{i=1}^m$ *for some* $m > 0$*. If* $G(q, q') \cap D(q)^o = \emptyset$ *for all* $(q, q') \in E$ *with* $q, q' \in \mathbf{Q}_\infty$*, then* $x_i \in \partial D(q_i)$ *for all* $i = 1, \ldots, m$*.*

A consequence of Theorem 3 is that $\mathbf{X}_\infty \subseteq \bigcap_{i=1}^{|\mathbf{Q}_\infty|} \partial D(q_i)$. The following non-Zeno condition follows directly from the theorem.

Corollary 1 *A hybrid automaton with* R *identity on* G *accepts no Zeno executions if*

- $G(q, q') \cap D(q)^o = \emptyset$ *for all* $(q, q') \in E$*,*
- *for all cycles* $\{q_i\}_{i=1}^K$ *with* $q_K = q_1$ *and* $(q_i, q_{i+1}) \in E$*,* $1 \leq i \leq K-1$*,* $\bigcap_{i=1}^{K-1} \partial D(q_i) = \emptyset$*.*

It is interesting to notice that the standing assumption by Tavernini in [42] is implied by the two conditions in Corollary 1. Under this assumption, it is proved that each solution has finitely many switching points in finite time, i.e., the system is non-Zeno.

The second condition in Corollary 1, on disjoint boundaries of the domains, can be replaced by the two assumptions that the boundaries only meet at a single point which is an equilibrium point for the vector field in each discrete state. Without loss of generality this point can be assumed to be the origin.

Theorem 4 *For a hybrid automaton with R identity on G, an execution $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$ with $x_0 \neq 0$ is not Zeno if*

- $G(q,q') \cap D(q)^o = \emptyset$ *for all* $(q,q') \in E$,
- *for all cycles* $\{q_i\}_{i=1}^K$ *with* $q_K = q_1$ *and* $(q_i, q_{i+1}) \in E$, $1 \leq i \leq K-1$, $\bigcap_{i=1}^{K-1} \partial D(q_i) = \{0\}$ *and* $f(q_i, 0) = 0$.

Note that the first assumption of Theorems 3 and 4 are fulfilled for systems under logic-based switching [43], piecewise linear systems [20], complementarity systems [44] and mixed logical dynamical systems [45]. Hence, these two results provide assumptions to guarantee non-Zenoness for these large classes of hybrid systems. Similar result to Corollary 1 and Theorem 4 hold also for hybrid automata with contracting resets having the origin as a fixed point.

4 Conclusions

Motivated by a number of examples appearing in different applications of hybrid systems, we have studied some properties of Zeno executions. We saw that it is important to understand Zeno in order to develop efficient tools for modeling, verification, simulation, and design of hybrid systems. The Zeno set was introduced to capture the limiting behavior of Zeno executions. In general, the Zeno set can have a complex structure. It was proved, however, that for hybrid automata having either only identity resets or only resets contracting on the guard, the continuous part of the Zeno set is a singleton. These hybrid systems include, for example, feedback control systems with logic-based switching and complementarity systems. For such systems, our results guarantee that every Zeno execution converges to a single point in the continuous state space. It may then be possible to extend the execution beyond the Zeno time in a consistent way [13, 3]. Both necessary and sufficient conditions for a hybrid automaton to accept Zeno executions were presented. In particular, it was proved that for hybrid automata with identity resets on the guard, if the guards and the interior of the domains are disjoint and if the boundaries of the domains are also disjoint, then the hybrid automaton does not accept Zeno executions. Moreover, it was shown that the last condition can be replaced by that the boundaries of the domains have the single intersection point being the origin, which should also be an equilibrium point for the vector field in all discrete states.

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Appendix

Proof of Lemma 1

By Lipschitz continuity, there exists $c > 0$ such that for all $n \in \langle \tau \rangle$ and $t \in I_n$, $\|f(q(n), x^n(t))\| \leq c\,(\|x^n(t)\| + 1)$. Since $\|x\|^2 = x^T x$ it follows that

$$\left|\frac{d\|x\|^2}{dt}\right| = 2\|x\| \left|\frac{d\|x\|}{dt}\right| = 2|x^T \dot{x}| \leq 2\|x\|\|\dot{x}\|,$$

so that

$$\left|\frac{d\|x\|}{dt}\right| \le \|\dot{x}\| = \|f(q,x)\| \le c(\|x\|+1)$$

$$-c\left(\|x\|+1\right) \le \frac{d\|x\|}{dt} \le c\left(\|x\|+1\right).$$

Applying the Bellman–Gronwall Lemma [40] twice, we have

$$\left(\|x^n(\tau_n)\|+1\right)e^{-c(t-\tau_n)} \le \|x^n(t)\|+1 \le \left(\|x^n(\tau_n)\|+1\right)e^{c(t-\tau_n)}, \quad t \in I_n \text{ and } n \in \langle\tau\rangle.$$

By the assumption on non-expanding resets, we have $\|x^n(\tau_n)\| \le \|x^{n-1}(\tau'_{n-1})\|$, which yields

$$\begin{aligned}\|x^n(t)\|+1 &\le \left(\|x^{n-1}(\tau'_{n-1})\|+1\right)e^{c(t-\tau_n)} \\ &\le \left(\|x^{n-1}(\tau_{n-1})\|+1\right)e^{c(\tau'_{n-1}-\tau_{n-1})}e^{c(t-\tau_n)}.\end{aligned}$$

By induction,

$$\|x^n(t)\|+1 \le \left(\|x^0(0)\|+1\right)e^{ct},$$

therefore,

$$\|x^n(t)\| \le (\|x_0\|+1)e^{ct}-1.$$

The proof for non-contracting resets is similar.

Proof of Theorem 1

Consider a Zeno execution $\chi = (\tau, q, x)$. For all $\{\theta_n\}_{n=0}^{\infty}$ with $\theta_n \in I_{i_n}$ and $i_n \in \langle\tau\rangle$ such that $\theta_n \to \tau_\infty$ as $n \to \infty$, we have

$$\begin{aligned}x^{i_n}(\theta_n) &= x^{i_n}(\tau_{i_n}) + \int_{\tau_{i_n}}^{\theta_n} f\left(q(i_n), x^{i_n}(s)\right) ds \\ &= x^{i_n}(\tau_{i_n}) + (\theta_n - \tau_{i_n})\left(f_1(q(i_n), x^{i_n}(\xi^1_{i_n})), \dots, f_n(q(i_n), x^{i_n}(\xi^n_{i_n}))\right),\end{aligned}$$

for some $\xi^1_{i_n}, \dots, \xi^n_{i_n} \in I_{i_n}$. Hence for all $k > l \ge 0$,

$$\begin{aligned}x^{i_k}(\theta_k) &= x^{i_l}(\theta_l) + (\tau'_{i_l} - \theta_l)\left(f_1(q(i_l), x^{i_l}(\xi^1_{i_l})), \dots, f_n(q(i_l), x^{i_l}(\xi^n_{i_l}))\right) \\ &\quad + \sum_{i=i_l+1}^{i_k-1} (\tau'_i - \tau_i)\left(f_1(q(i), x^i(\xi^1_i)), \dots, f_n(q(i), x^i(\xi^n_i))\right) \\ &\quad + (\theta_k - \tau_{i_k})\left(f_1(q(i_k), x^{i_k}(\xi^1_{i_k})), \dots, f_n(q(i_k), x^{i_k}(\xi^n_{i_k}))\right),\end{aligned}$$

which gives that

$$\|x^{i_k}(\theta_k) - x^{i_l}(\theta_l)\| \le K \sum_{i=i_l}^{i_k} (\tau_i' - \tau_i),$$

where $K > 0$ is such that $\| \left(f_1(q(n), x^n(\xi_n^1)), \dots, f_n(q(n), x^n(\xi_n^n))\right) \| \le K$ for all $n \in \langle\tau\rangle$ and ξ_n^1, …, $\xi_n^n \in I_n$ (recall that such a K exists by continuity of f and the fact that $\|x\|$ remains bounded). Since $\sum_{n=0}^{\infty}(\tau_n' - \tau_n) < \infty$, $\{x^{i_n}(\theta_n)\}_{n=0}^{\infty}$ is a Cauchy sequence. The space $\mathbf{X} = \mathbb{R}^n$ is complete, so the sequence has a limit $\hat{x} = \lim_{n\to\infty} x^{i_n}(\theta_n)$. Moreover, the following argument shows that this limit is independent of the choice of sequence $\{\theta_n\}_{n=0}^{\infty}$. Consider two sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, $\alpha_n \in I_{i_n}$ and $\beta_n \in I_{j_n}$, i_n, $j_n \in \langle\tau\rangle$ such that $\alpha_n \to \tau_\infty$ and $\beta_n \to \tau_\infty$ as $n \to \infty$. Without loss of generality, suppose that $i_n \ge j_n$,

$$\begin{aligned} x^{i_n}(\alpha_n) = x^{j_n}(\beta_n) &+ (\tau_{j_n}' - \beta_n)\left(f_1(q(j_n), x^{j_n}(\xi_{j_n}^1)), \dots, f_n(q(j_n), x^{j_n}(\xi_{j_n}^n))\right) \\ &+ \sum_{j=j_n+1}^{i_n-1} (\tau_j' - \tau_j)\left(f_1(q(j), x^j(\xi_j^1)), \dots, f_n(q(j), x^j(\xi_j^n))\right) \\ &+ (\alpha_n - \tau_{i_n})\left(f_1(q(i_n), x^{i_n}(\xi_{i_n}^1)), \dots, f_n(q(i_n), x^{i_n}(\xi_{i_n}^n))\right). \end{aligned}$$

This gives that $\|x^{i_n}(\alpha_n) - x^{j_n}(\beta_n)\| \le K\sum_{j=j_n}^{i_n}(\tau_j' - \tau_j)$. Hence, $\|x^{i_n}(\alpha_n) - x^{j_n}(\beta_n)\| \to 0$ as $n \to \infty$, which shows that both sequences have the same limit. This completes the proof.

Proof of Theorem 2

As in the proof of Theorem 1, we get that for a Zeno execution $\chi = (\tau, q, x)$ it holds that

$$\begin{aligned} \|x^{i_n}(\theta_n)\| &\le \|x^{i_n}(\tau_{i_n})\| + \left\| \int_{\tau_{i_n}}^{\theta_n} f\left(q(i_n), x^{i_n}(s)\right) ds \right\| \\ &\le \|x^{i_n}(\tau_{i_n})\| + K(\tau_{i_n}' - \tau_{i_n}). \end{aligned}$$

Using the fact that $\|x^{i_n}(\tau_{i_n})\| \le \delta\|x^{i_n-1}(\tau_{i_n-1}')\|$ for some $\delta \in [0, 1)$, it follows that

$$\begin{aligned} \|x^{i_n}(\theta_n)\| &\le \delta\|x^{i_n-1}(\tau_{i_n-1}')\| + K(\tau_{i_n}' - \tau_{i_n}) \\ &= \delta\left\| x^{i_n-1}(\tau_{i_n-1}) + \int_{\tau_{i_n-1}}^{\tau_{i_n-1}'} f\left(q(i_n-1), x^{i_n-1}(s)\right) ds \right\| + K(\tau_{i_n}' - \tau_{i_n}) \\ &\le \delta\|x^{i_n-1}(\tau_{i_n-1})\| + K\delta(\tau_{i_n-1}' - \tau_{i_n-1}) + K(\tau_{i_n}' - \tau_{i_n}). \end{aligned}$$

By induction,

$$\|x^{i_n}(\theta_n)\| \le \delta^{i_n}\|x_0\| + K \sum_{m=0}^{i_n} \delta^{i_n-m}(\tau_m' - \tau_m),$$

and

$$\sum_{i_n=0}^{\infty} K \sum_{m=0}^{i_n} \delta^{i_n-m}(\tau'_m - \tau_m) = K \sum_{m=0}^{\infty}(\tau'_m - \tau_m) \sum_{i_n=0}^{\infty} \delta^{i_n} = \frac{K\tau_\infty}{1-\delta} < \infty.$$

Therefore, $K\sum_{m=0}^{i_n} \delta^{i_n-m}(\tau'_m - \tau_m) \to 0$ as $i_n \to \infty$, which yields that $\|x^{i_n}(\theta_n)\| \to 0$ as $n \to \infty$, hence $\mathbf{X}_\infty = \{0\}$.

Proof of Lemma 2

Suppose that for all $M \geq 0$ there exist some $i_n \geq M$, such that $q(\tau_{i_n}) \notin \mathbf{Q}_\infty$. By the assumption that $\mathbf{Q}$ is finite, the sequence $\{q(\tau_{i_n})\}_{n=0}^{\infty}$ has a subsequence $\{q(\tau_{i_{n_m}})\}_{m=0}^{\infty}$ with $q(\tau_{i_{n_m}}) = \hat{q} \notin \mathbf{Q}_\infty$. By Lemma 1, the sequence $\{x^{i_{n_m}}(\tau_{i_{n_m}})\}_{m=0}^{\infty}$ is bounded. Then, there exists some $\hat{x} \in \mathbf{X}$ such that $\lim_{m\to\infty} x(\tau_{i_{n_m}}) = \hat{x}$ (by possibly passing to a subsequence). By the definition of the Zeno set, $(\hat{q}, \hat{x}) \in Z_\infty$, which gives a contradiction.

Proof of Theorem 3

For every $(\hat{q}, \hat{x}) \in Z_\infty$, there exists a sequence $\{\theta_n\}_{n=0}^{\infty}$ with $\theta_n \in I_{i_n}$ and $i_n \in \langle\tau\rangle$ such that as $n \to \infty$, $q(i_n) \to \hat{q}$ and $x^{i_n}(\theta_n) \to \hat{x}$. Notice that, since $\mathbf{Q}$ is given the discrete topology, $q(i_n) \to \hat{q}$ implies that $q(i_n) = \hat{q}$ for n sufficiently large. Moreover, by an argument similar to the one in the proof of Theorem 1, there exists $K > 0$ such that $\|x^{i_n}(\tau'_{i_n}) - x^{i_n}(\theta_n)\| \leq K\|\tau'_{i_n} - \theta_n\|$. Therefore, $\|x^{i_n}(\tau'_{i_n}) - \hat{x}\| \to 0$ as $n \to \infty$. By the standing assumptions, we have that $x^{i_n}(\tau'_{i_n}) \in \overline{D(\hat{q})}$. Moreover, from Lemma 2, $q(i_n) \in \mathbf{Q}_\infty$ if n is sufficiently large. Hence, for n sufficiently large, there exists some $\hat{q}' \in \mathbf{Q}_\infty$ such that $x^{i_n}(\tau'_{i_n}) \in G(\hat{q}, \hat{q}')$, which gives that $x^{i_n}(\tau'_{i_n}) \in G(\hat{q}, \hat{q}') \cap \overline{D(\hat{q})}$. Since $G(\hat{q}, \hat{q}') \cap D(\hat{q})^o = \emptyset$, it follows that $x^{i_n}(\tau'_{i_n}) \in \partial D(\hat{q})$. Moreover, since $\partial D(\hat{q})$ is a closed set, the limit $\hat{x} = \lim_{n\to\infty} x^{i_n}(\tau'_{i_n})$ belongs to $\partial D(\hat{q})$.

Proof of Theorem 4

Assume that $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$ with $x_0 \neq 0$ is a Zeno execution. By Theorem 1, it holds that $Z_\infty = \mathbf{Q}_\infty \times \{\hat{x}\}$ for some $\hat{x} \in \mathbf{X}$. Then by Theorem 3 and the second assumption, it follows that $\hat{x} = 0$. Let L be the largest Lipschitz constant of $f(q, \cdot)$ for all $q \in \mathbf{Q}$. By Lemma 2, there exists some M such that for all $n \geq M$, $q(n) \in \mathbf{Q}_\infty$. Then, from [40, Prop. 5.3], we have that for all $n \in \langle \tau \rangle$, $n \geq M$ and $t \in I_n$,

$$\|x^n(t)\| \geq \|x^n(\tau_n)\| \, e^{-L(t-\tau_n)}.$$

Since $x^n(\tau_n) = x^{n-1}(\tau'_{n-1})$,

$$\|x^n(t)\| \geq \|x^{n-1}(\tau'_{n-1})\| \, e^{-L(t-\tau_n)} \geq \|x^{n-1}(\tau_{n-1})\| \, e^{-L(t-\tau_{n-1})}.$$

Proceeding further,

$$\|x^n(t)\| \geq \|x_0\| \, e^{-Lt}.$$

Since χ is a Zeno execution, $\lim_{n\to\infty} \|x^n(\tau_n)\| \geq \|x_0\| \, e^{-L\tau_\infty} > 0$. This contradicts, however, the fact that $\lim_{n\to\infty} x^n(\tau_n) = \hat{x} = 0$. Hence, the hybrid automaton accepts no Zeno executions.

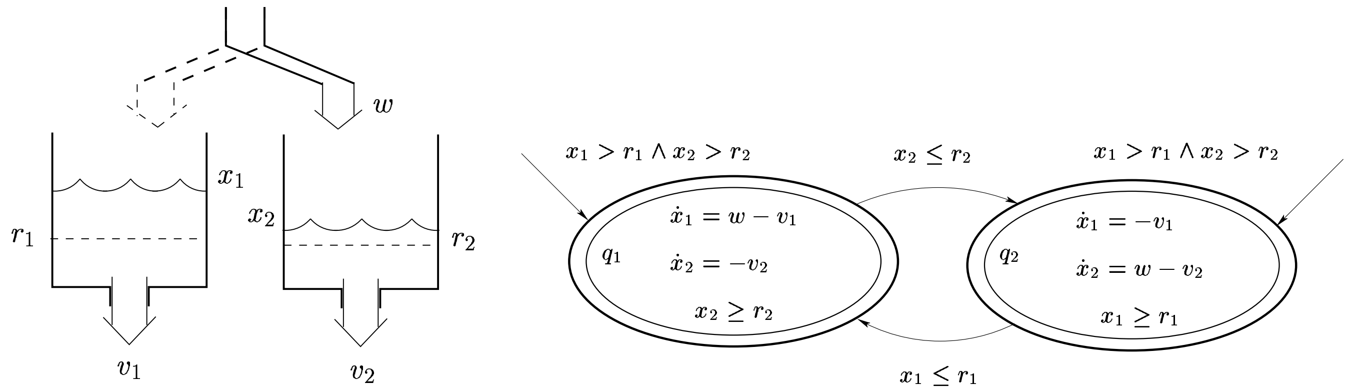

Figure 1: Water tank system and corresponding Zeno hybrid automaton. (JUN ZHANG)

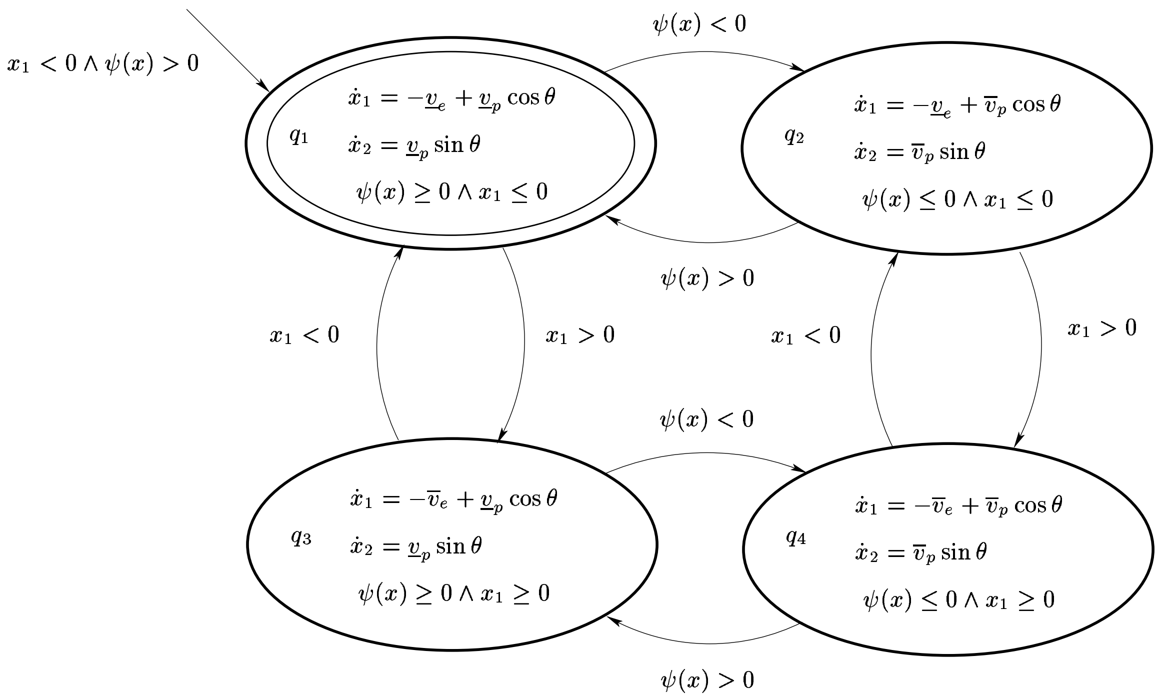

Figure 2: Zeno hybrid automaton describing a conflict between two aircrafts. (JUN ZHANG)

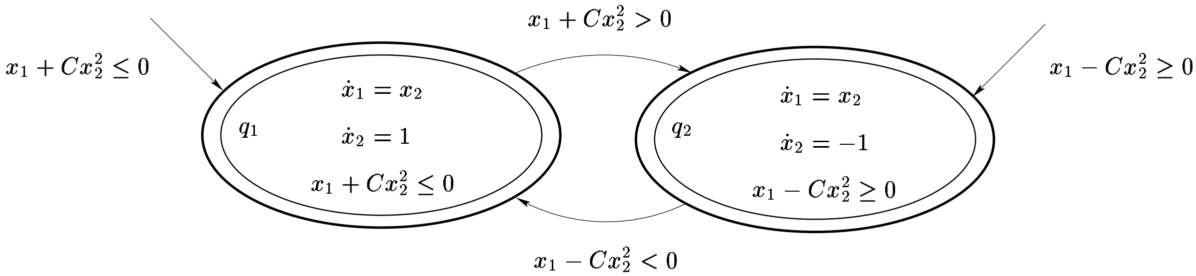

Figure 3: Zeno hybrid automaton describing Fuller's phenomenon. (JUN ZHANG)

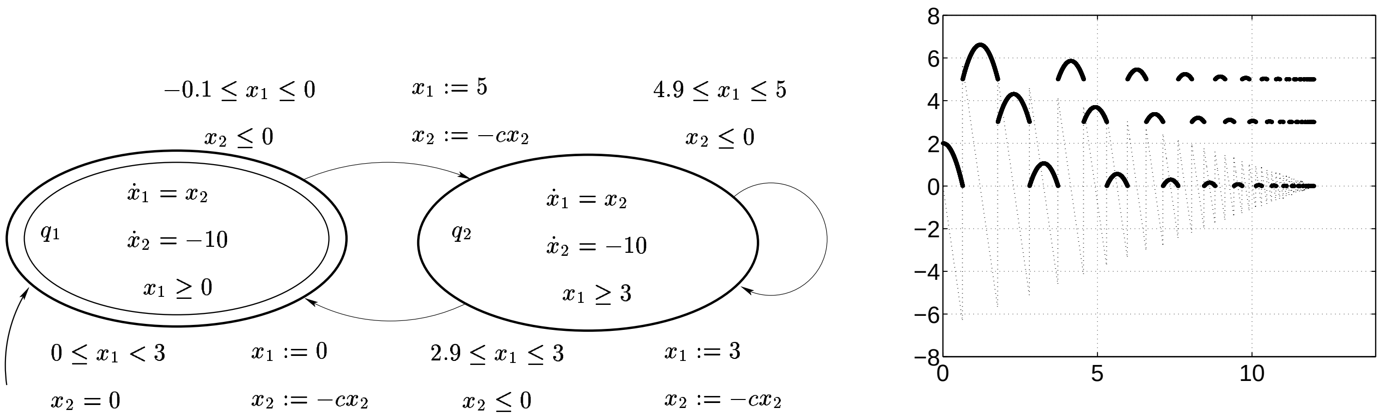

Figure 4: Zeno hybrid automaton together with the continuous part of an execution (x_1 solid and x_2 dotted). (JUN ZHANG)

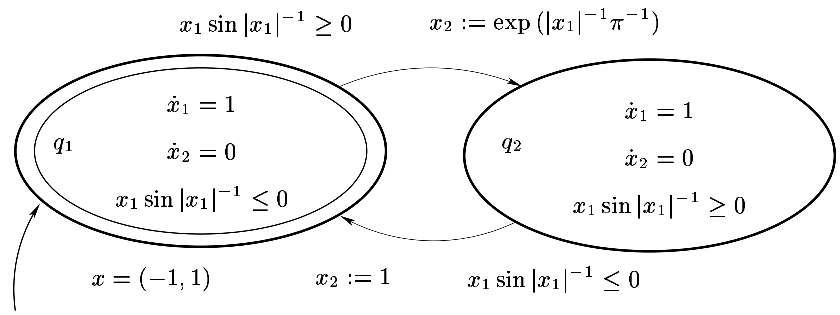

Figure 5: Zeno hybrid automaton in Example 7. (JUN ZHANG)