Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 51-9 (2018) 124–129

Stochastic Control Formulation of The Car Overtake Problem ⋆

Aneesh Raghavan *, **Jieqiang Wei** **, **John S Baras** *, **Karl Henrik Johansson** **

* *Institute for Systems Research, University of Maryland, College Park, USA(e-mail: raghava, baras@umd.edu)*
** *Department of Automatic Control, KTH, Royal Institute of Technology, Stockholm, Sweden(e-mail: jieqiang, kallej@kth.se)*

Abstract: In this paper, we consider the classic car overtake problem. There are three cars, two moving along the same direction in the same lane while the third car moves in the direction opposite to that of the first two cars in the adjacent lane. The objective of the trailing car is to overtake the car in front of it avoiding collision with the other cars in the scenario. The information available to the trailing car is the relative position, relative velocities with respect to other cars and its position and past actions. The relative position and relative velocity information is corrupted by noise. Given this information, the car needs to make a decision as to whether it wants to overtake or not. We present a control algorithm for the car which minimizes the probability of collision with both the cars. We also present the results obtained by simulating the above scenario with the control algorithm. Through simulations, we study the effect of the variance of the measurement noise and the time at which the decision is made on the probability of collision.

Keywords: Stochastic control, probability of collision, overtake

1. INTRODUCTION

In recent years, there has been significant interest and research activity in the area of cooperative control of multiple autonomous vehicles. The ultimate goal in automating the driving process is to reduce accidents and improve safety. Among all the scenarios, overtaking maneuver is one of the most dangerous ones, especially in two-way roads, due to non-cooperative behaviors from the drivers, lack of distance, poor visibility, etc.

Due to the high demand from the practice, many results about overtaking are available in the literature. Roughly speaking, the overtaking problem can be categorized in the study of the lane-change maneuver, including changing lanes on a highway, leaving the road, or overtaking, which is one of the most thoroughly investigated automatic driving operations for autonomous vehicles after trajectory tracking, see e.g., Böhm et al. (2011), Petrov and Nashashibi (2014) and the references within. In Sezer (2017), the overtaking problem is formulated using the tools from the Mixed Observable Markov Decision Process (MOMDP),which provides optimum strategy considering the uncertainties in the problem. However, the methodology suffers from high complexity. In Vinel et al. (2012), the authors consider the scenario in which the driver is in the loop and the proposed system helps for a safe overtaking by cooperative perception.

We consider the following scenario. There are three cars, Car 1, Car 2 and Car 3 (Figure 1). Car 1 and Car 2 are moving in same lane while Car 3 is moving in the adjacent lane in the direction opposite to the other two cars. There is no physical barrier between the two lanes. Car 1 is trailing Car 2. When Car 1 is close “enough” to Car 2, Car 1 has to decide if it should overtake Car 2 or not while avoiding collision with both the cars. If Car 1 decides to overtake, it needs to determine its trajectory of overtaking as well. At every time instant, Car 1 measures its relative speed and velocity with respect to the other two cars. These measurements are corrupted by noise. Based on noisy measurements of relative distance and velocity with respect to Car 2 and Car 3, Car 1 needs to make the decision.

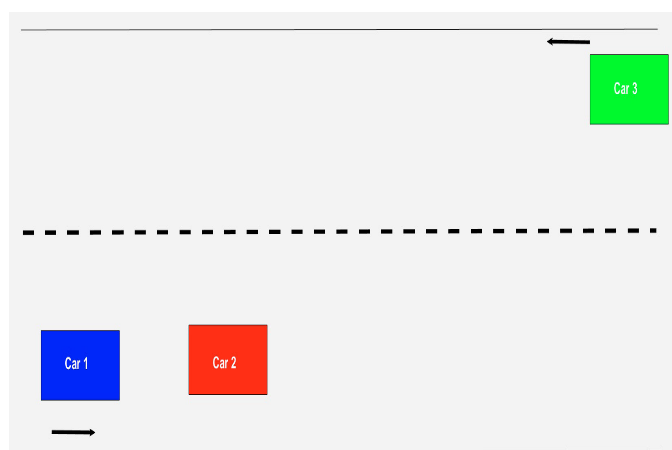

Fig. 1. Schematic

⋆ Research supported by ARO grant W911NF-15-1-0646 and by DARPA through ARO grant W911NF-14-1-0384.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2018.07.021

The major contribution of this paper is the stochastic control formulation of this problem and the control algorithm based on probability of collision calculation. The probability space constructed by the trailing car is based on the joint distribution of the noise in its measurements and its initial state. The actions of the other cars are considered as exogenous random variables. Based on the statistics of the measurement noise, a numerical method to compute probability of Car 1 colliding with Car 2 and probability of Car 1 colliding with Car 3 while overtaking is discussed. A control algorithm, where the probability of collision is minimized is presented. Through simulations the effect of measurement noise and time at which the decision is made on the probability of collision is studied.

Notation: Let $\mathcal{V}$ denote the set of admissible speeds for the cars and it is assumed to be a finite set. Let $\mathcal{W}$ denote the set of admissible angular speeds for Car 1 and it is also assumed to be a finite set. Let $\mathcal{M}$ denote the set of possible times that Car 1 could spend in the alternate lane while overtaking. d denotes the breadth of the lanes while L denotes minimum safety distance between cars. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space for Car 1.

2. PROBLEM FORMULATION

2.1 System Model

The dynamics of Car 1 is modeled by the unicycle (*Dubin's*) model:

$$\begin{aligned} x_1(k+1) &= x_1(k) + v_1(k)\cos(\theta_1(k)),\\ y_1(k+1) &= y_1(k) + v_1(k)\sin(\theta_1(k)),\\ \theta_1(k+1) &= \theta_1(k) + \omega(k). \end{aligned}$$

$x_1(k)$, $y_1(k)$ denote the longitudinal and lateral coordinates of Car 1 at time k. $\theta_1(k)$ denotes the orientation of Car 1 at time k. The longitudinal coordinate at time zero is, $x_1(0) \backsim \mathcal{N}(0, \Sigma_1)$, while the lateral coordinate and orientation are $y_1(0) = \frac{d}{2}, \theta_1(0) = 0$. The actions taken by Car 1 at time k are, $v_1(k)$ and $\omega(k)$. For Car 2 and Car 3, their y coordinate is fixed. They traverse along the x direction with speed that could be time varying. The dynamics of Car 2 is described as:

$$\begin{aligned} x_2(k+1) &= x_2(k) + v_2(k),\\ y_2(k) &= \frac{d}{2},\ \theta_2(k) = 0. \end{aligned}$$

The initial longitudinal coordinate of Car 2 is random, $x_2(0) \backsim \mathcal{N}(\tilde{x}_2, \Sigma_2)$. $\tilde{x}_2$ is larger than 0 and Σ_2 is chosen small enough, so that $x_2(0) > x_1(0)$. The dynamics of Car 3 is described as:

$$\begin{aligned} x_3(k+1) &= x_3(k) + v_3(k),\\ y_3(k) &= \frac{3d}{2},\ \theta_3(k) = 0. \end{aligned}$$

The initial longitudinal coordinate of Car 3 is random, $x_3(0) \backsim \mathcal{N}(\tilde{x}_3, \Sigma_3)$, $\tilde{x}_3$ is much greater than $\tilde{x}_2$. Σ_3 is chosen small enough, so that $x_3(0) > x_2(0)$. The actions taken by Car 2 and Car 3 are $v_2(k)$ and $v_3(k)$ respectively. The first set of observations collected by Car 1 are its relative positions with respect to Car 2 and Car 3:

$$\begin{aligned} z_1(k) &= x_2(k) - x_1(k) + W_1(k),\ k \geq 0,\\ z_2(k) &= x_3(k) - x_1(k) + W_2(k),\ k \geq 0. \end{aligned}$$

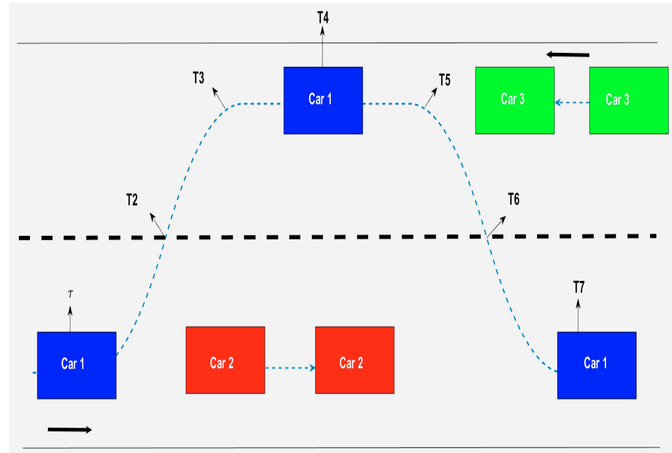

Fig. 2. Sample trajectory during overtake

Once the cars take their respective actions, the second set of observations collected by Car 1 are the relative velocities with respect to Car 2 and Car 3.

$$\begin{aligned} z_3(k) &= v_2(k) - v_1(k) + W_3(k),\ k \geq 0,\\ z_4(k) &= v_3(k) - v_1(k) + W_4(k),\ k \geq 0. \end{aligned}$$

$W_i(k),\ i = 1,2,3,4$ are white Gaussian processes with zero mean and variances $\sigma_i^2(k)$. $\{W_i\}_{k\geq 1}$, $x_1(0)$, $x_2(0)$ and $x_3(0)$ are assumed to be independent. Let $\mathcal{I}_1(k)$ denote the information available to Car 1 before taking its action. Then,

$$\begin{aligned} \mathcal{I}_1(k) =& \{x_1(n), v_1(n), y_1(n), \theta_1(n), z_1(n), z_2(n)\}_{n=0}^{n=k}\\ &\cup \{z_3(n), z_4(n)\}_{n=0}^{n=k-1}. \end{aligned}$$

After the three cars take their respective actions, $z_3(k)$ and $z_4(k)$ also become available. Hence the new information set available to Car 1 is

$$\begin{aligned} \mathcal{I}_2(k) =& \{x_1(n), v_1(n), y_1(n), \theta_1(n), z_1(n), z_2(n)\}_{n=0}^{n=k}\\ &\cup \{z_3(n), z_4(n)\}_{n=0}^{n=k}. \end{aligned}$$

2.2 Control Problem

In this section we formulate the control problem for Car 1. As a solution to the overtake problem, Car 1 could wait for Car 3 to pass by and then it would have to take into account only collision with Car 2 while making the decision to overtake. We do not consider such a solution as a feasible solution. Apart from choosing v_1 and ω_1 at every time instant, Car 1 needs to take other decisions as well. The first objective of Car 1 is to find the *decision time*, i.e., the time at which it should decide to overtake or not. We denote the decision time by τ. At τ, Car 1 should make the decision to overtake or not. We note the decision by $\mathcal{D}$. After the decision is taken, Car 1 needs to decide its trajectory for overtaking. In our study we restrict ourselves to trajectories generated as follows: the trajectories are characterized by v_1, ω_1 and t_1. v_1 denotes the constant speed of Car 1 through out the overtake. ω_1, the magnitude of the angular velocity of Car 1 during the overtake. t_1 denotes the time spent by Car 1 in the alternate lane. Let $\Delta = \frac{1}{\omega_1}\arccos(1 - \frac{d\omega_1}{2v_1})$. Δ is approximately the time taken by Car 1 to go from $y = \frac{d}{2}$ to $y = d$. Then,

$$\omega(k) = \omega_1,\ \tau \le k < \tau + \Delta\ [T2],$$
$$\omega(k) = -\omega_1,\ \tau + \Delta\ [T2] \le k < \tau + 2\Delta\ [T3],$$
$$\omega(k) = 0,\ \tau + 2\Delta\ [T3] \le k < \tau + 2\Delta + t_1\ [T5],$$
$$\omega(k) = -\omega_1,\ \tau + 2\Delta + t_1\ [T5] \le k < \tau + 3\Delta + t_1\ [T6],$$
$$\omega(k) = \omega_1,\ \tau + 3\Delta + t_1\ [T6] \le k < \tau + 4\Delta + t_1\ [T7].$$

A sample trajectory generated using such a law is shown in Figure 2. Note that, in our formulation we do not give Car 1 the freedom to return to the original lane during the process of overtaking.

Hence the control problem for Car 1 is to find Decision time (τ), the overtake decision ($\mathcal{D}$), and a triple (v_1, ω_1, t_1) which has the least probability of collision.

3. SOLUTION

To find its control policy, Car 1 needs to estimate the position and velocity of Car 2 and Car 3 from the information available to it.

3.1 Estimation

Car 1 estimates the position and velocity of Car 2 and Car 3 using the available information. The position of Car 2 can be estimated as follows : Let $\tilde{z}_1(k) = z_1(k) + x_1(k)$ and $\tilde{z}_3(k) = z_3(k) + v_1(k)$. Then,

$$x_2(k+1) = x_2(k) + \tilde{z}_3(k) - W_3(k),$$
$$\tilde{z}_1(k) = x_2(k) + W_1(k).$$

Let,

$$\mathbb{E}_{\mathbb{P}}[x_2(k+1)|\mathcal{I}_2(k)] = \bar{x}_2(k+1),\ \mathbb{E}_{\mathbb{P}}[x_2(k)|\mathcal{I}_1(k)] = \hat{x}_2(k).$$

Using $\mathcal{I}_1(k)$, and Kalman filtering,

$$\hat{x}_2(k) = \bar{x}_2(k) + G(k)(\tilde{z}_1(k) - \bar{x}_2(k)),$$
$$G(k) = \frac{P(k)}{\sigma_1^2(k)},\ Q(k+1) = P(k) + \sigma_3^2(k),$$
$$P(k) = \frac{Q(k)\sigma_1^2(k)}{\sigma_1^2(k) + Q(k)},\ P(0) = \Sigma_2.$$

Once the actions are taken by the cars, $\mathcal{I}_2(k)$ is available. Since $v_2(k)$ is treated as an exogenous random variable, $\mathbb{E}[W_3(k)|\mathcal{I}_2(k)]$ is set to zero (i.e., $\mathbb{E}[W_3(k)]$).

$$\bar{x}_2(k+1) = \hat{x}_2(k) + \tilde{z}_3(k).$$

The best estimate of velocity of Car 2 is given by :

$$\hat{v}_2(k) = \mathbb{E}_{\mathbb{P}}[x_2(k+1) - x_2(k)|\mathcal{I}_2(k)] =$$
$$\bar{x}_2(k+1) - \hat{x}_2(k) = \tilde{z}_3(k).$$

Let $e_2(k) = x_2(k) - \hat{x}_2(k)$. $e_2(k) \backsim \mathcal{N}(0, P(k))$. Similarly the position and velocity of Car 3 can be estimated by Car 1: Let $\tilde{z}_2(k) = z_2(k) + x_1(k)$ and $\tilde{z}_4(k) = z_4(k) + v_1(k)$. Then,

$$\bar{x}_3(k+1) = \hat{x}_3(k) + \tilde{z}_4(k),$$
$$\hat{x}_3(k) = \bar{x}_3(k) + H(k)(\tilde{z}_2(k) - \bar{x}_3(k)),$$
$$H(k) = \frac{R(k)}{\sigma_2^2(k)},\ S(k+1) = R(k) + \sigma_4^2(k),$$
$$R(k) = \frac{S(k)\sigma_2^2(k)}{\sigma_2^2(k) + S(k)},\ R_0 = \Sigma_3.$$
$$\hat{v}_3(k) = \mathbb{E}_{\mathbb{P}}[x_3(k+1) - x_3(k)|\mathcal{I}_2(k)]$$
$$= \bar{x}_3(k+1) - \hat{x}_3(k) = \tilde{z}_4(k).$$

Let $e_3(k) = x_3(k) - \hat{x}_3(k)$. $e_3(k) \backsim \mathcal{N}(0, R(k))$.

3.2 Decision Time

Using the estimated position, Car 1 considers a distance based definition for τ. τ is defined as :

$$\tau = \min_{k\ge 1}\{k : N \times L \le \hat{x}_2(k) - x_1(k) \le (N+1) \times L\},$$

where N is a natural number. N could be used to characterize the behavior of the driver in Car 1. For a passive driver N could 5 or 8, while for a aggressive driver it could be 1 or 2. Hence the estimated safety distance between Car 1 and Car 2 at τ is greater than or equal to $N \times L$. To guarantee existence of τ, the simulation time step has to be chosen to be less than $\frac{L}{2\times \max \mathcal{V}}$.

3.3 Feasibility

At a given time instant k, when the above definition is satisfied and τ is found, the next goal for Car 1 is to make the decision of overtaking. Note that since the cars have not taken their actions yet, velocity estimates are not available. At τ, given $x_1(\tau), \hat{x}_2(\tau), \hat{v}_2(\tau-1)$, every triple $(v_1, \omega_1, t_1) \in \mathcal{V}\times\mathcal{W}\times\mathcal{M}$ does not guarantee that Car 1 will overtake Car 2. Hence using the information available τ, Car 1 predicts where Car 2 would be at the time it finishes overtaking and the feasible triples are defined accordingly. Let $M \times L$ be the desired safety distance between Car 1 and Car 2 after Car 1 overtakes Car 2.

Definition 3.1. Given, $X_2 = [x_1(\tau); \hat{x}_2(\tau); \hat{v}_2(\tau-1)]$, $(v_1, \omega_1, t_1) \in \mathcal{V} \times \mathcal{W} \times \mathcal{M}$ is said to be feasible for Car 1 with respect to Car 2 if :

$$x_1(\tau) + \frac{2v_1}{\omega_1}\sin(2\omega_1\Delta) + v_1 t_1$$
$$\ge [\hat{x}_2(\tau)] + (\hat{v}_2(\tau-1))(4\Delta + t_1) + M \times L.$$

The L.H.S of the above inequality is the position of Car 1 after it overtakes Car 2 along the trajectory generated by v_1, ω_1 and t_1. The R.H.S of the inequality is the estimated position of Car 2 at the time Car 1 finishes overtaking incremented by the safety distance. For given X_2 , let S_{X_2} denote the set of all feasible triples. At τ, given $x_1(\tau), \hat{x}_3(\tau), \hat{v}_3(\tau-1)$, some triples $(v_1, \omega_1, t_1) \in \mathcal{V} \times \mathcal{W} \times \mathcal{M}$ might lead to collision with Car 3 during the process of overtaking. Car 1 considers a triple (v_1, ω_1, t_1), to be feasible if at $T3$, Car 3 has already passed it or at $T5$, Car 3 is at safe distance from it. Since the exact position and velocity of Car 3 is not known, following definition is also based on the estimate. Let $\bar{M} \times L$ be the desired safety distance between Car 1 and Car 3 when Car 1 overtakes Car 2.

Definition 3.2. Given $X_3 = [x_1(\tau)); \hat{x}_3(\tau)); \hat{v}_3(\tau-1)]$, $(v_1, \omega_1, t_1) \in \mathcal{V} \times \mathcal{W} \times \mathcal{M}$ is said to be feasible for Car 1 with respect to Car 3 if :

$$x_1(\tau) + \frac{v_1}{\omega_1}\sin(2\omega_1\Delta) \ge \hat{x}_3(\tau) + (\hat{v}_3(\tau-1))(2\Delta) + \bar{M} \times L,$$

or $x_1(\tau) + \frac{v_1}{\omega_1}\sin(2\omega_1\Delta) + v_1 t_1 + \bar{M} \times L$

$$\le \hat{x}_3(\tau) + (\hat{v}_3(\tau-1))(2\Delta + t_1).$$

For given X_3 , let S_{X_3} denote the set of all feasible triples.

3.4 Probability of Collision - Car 2

Among the feasible triples, Car 1 would have to find the triple that has the least probability of collision with both

Car 2 and Car 3. Let $\delta = \frac{1}{\omega_1}\arccos(1 - \frac{L\omega_1}{v_1})$. If Car 1 decides to overtake, it is the time taken by Car 1 to move from $y_1(k) = d/2$ to $y_1(k) = d/2 + L$ given that the action of Car 1 is v_1, ω_1. It is used in the following sections to determine if a collision has occurred. We now define the collision random variable and discuss a numerical approach to find the probability of collision.

Definition 3.3. Car 1 is said to collide with Car 2 if:

$$\mathcal{C}_2 = 1 \text{ if} : \exists\, k \ni \begin{cases} x_2(k) - L \leq x_1(k) \leq x_2(k) + L, & (1) \\ \dfrac{d}{2} \leq y_1(k) \leq \dfrac{d}{2} + L. & (2) \end{cases}$$

Note that $\mathcal{C}_2$ is a function of (v_1, ω_1, t_1). Given X_2 and a triple (v_1, ω_1, t_1), the objective is now to find time intervals or instances where both the conditions hold. Since y coordinates of the three cars are known precisely at all times, the intervals where (2) holds (if Car 1 decides to overtake) are known with certainty. The intervals are : $[0, \tau]$, $[\tau, \tau + \delta]$ and $[\tau + 4\Delta + t_1 - \delta, \tau + 4\Delta + t_1]$. Since X_2 is random, the intervals where (1) is satisfied are random. Let e be the true value of error in position estimation at τ and w be the true value of error in velocity estimation at $\tau - 1$. Consider the interval $[0, \tau]$. Define:

$$\psi_1(s, e) = [\hat{x}_2(s) + e] - x_1(s).$$

$\psi_1(s, e)$ is the difference in the position of Car 2 and position of Car 1 at time s. Thus for (1) and (2) to hold in the considered time interval, it suffices to find

$$\mathcal{A}_1(e) = \{s \in [0, \tau] : -L \leq \psi_1(s, e) \leq L\}.$$

Clearly, $\mathcal{A}_1(e) \neq \emptyset$ implies $\mathcal{C}_1(e) = 1$, where $\mathcal{C}_1(e)$ is collision variable for a given value of error. Consider the time interval $[\tau, \tau + \delta]$. For a given X_2 and same triple $(v_1, \omega_1, t_1) \in S_{X_2}$, define:

$$\psi_2(s) = x_1(\tau) + \int_0^s v_1 \cos(\omega_1 t) = x_1(\tau) + \frac{v_1}{\omega_1}\sin(\omega_1 s),$$
$$\psi_3(s, e, w) = (\hat{x}_2(\tau) + e) + (\hat{v}_2(\tau - 1) - w)(s).$$

$\psi_2(s)$ is the position of Car 1 at time $\tau + s$ and $\psi_3(s, e, w)$ is the predicted position of Car 2 at time $\tau + s$. $\psi_2(s)$ is found using continuous time version of the *Dubin's* model. Thus for (1) and (2) to hold in the considered time interval, it suffices to find

$$\mathcal{A}_2(e, w) = \{s \in [0, \delta] : -L \leq \psi_2(s) - \psi_3(s, e, w) \leq L\}.$$

Consider the time interval, $[\tau + 4\Delta + t_1 - \delta, \tau + 4\Delta + t_1]$. For the given X_2, triple $(v_1, \omega_1, t_1) \in S_{X_2}$,

$$\psi_4(s) = x_1(\tau) + \int_0^{2\Delta} v_1 \cos(\omega_1 t)dt + v_1 t_1 + \int_0^{\Delta + s} v_1 \cos(\omega_1 t)dt = x_1(\tau) + \frac{v_1}{\omega_1}\sin(2\omega_1\Delta) + v_1 t_1 + \frac{v_1}{\omega_1}\sin(\omega_1(\Delta + s)),$$
$$\psi_5(s, e, w) = (\hat{x}_2(\tau) + e) + (\hat{v}_2(\tau - 1) - w)(3\Delta + t_1 + s).$$

$\psi_4(s)$ is the position of Car 1 at time $\tau + 3\Delta + t_1 + s$ and $\psi_5(s, e, w)$ is the predicted position of Car 2 at the same time. Here again, to verify if (1) and (2) hold,

$$\mathcal{A}_3(e, w) = \{s \in [\Delta - \delta, \Delta] : -L \leq \psi_4(s) - \psi_5(s, e, w) \leq L\}.$$

For the fixed value of e and w, the collision variable is :

$$\mathcal{C}_2(e, w) = 1_{\mathcal{A}_2 \cup \mathcal{A}_3 \neq \emptyset}.$$

Since (v_1, ω_1, t_1) does not affect $\mathcal{A}_1$, collision resulting in $\mathcal{A}_1$ is defined separately. It is difficult to find closed form expression for $\mathcal{C}_2(e, w)$. We find it numerically by considering finite domains for e and W. The procedure to construct domains for e and w is explained in the next section. The joint density of $e_2(\tau)$ and $W_3(\tau - 1)$ is found as follows:

$$f_2(e_2(\tau) = e, W_3(\tau - 1) = w) = f(e_2(\tau) = e | W_3(\tau - 1) = w) \times f(W_3(\tau - 1) = w).$$

Before we get to the conditional distribution we first find, $\mathbb{E}[e_2(\tau)W_3(\tau - 1)]$.

$$e_2(\tau) = (1 - G(\tau))(e_2(\tau - 1) + W_3(\tau - 1)) - G(\tau)W_1(\tau),$$
$$\mathbb{E}[e_2(\tau)W_3(\tau - 1)] = (1 - G(\tau))\mathbb{E}[e_2(\tau - 1)W_3(\tau - 1)] + (1 - G(\tau))\mathbb{E}[(W_3(\tau - 1))^2] - G(\tau)\mathbb{E}[W_1(\tau)W_3(\tau - 1)]$$
$$= (1 - G(\tau))\sigma_3^2(\tau - 1),$$

the last equality follows from independence of $e_2(\tau - 1)$, $W_3(\tau - 1)$ and $W_1(\tau)$. Since $e_2(\tau)$ and $W_3(\tau - 1)$ are Gaussian random variables, the conditional distribution is also Gaussian with mean:

$$\mathbb{E}[e_2(\tau)|W_3(\tau - 1)] = \mathbb{E}[e_2(\tau)] + \frac{\mathbb{E}[e_2(\tau)W_3(\tau - 1)]}{\mathbb{E}[(W_3(\tau))^2]} \times [W_3(\tau - 1) - \mathbb{E}[W_3(\tau - 1)]] = (1 - G(\tau))[W_3(\tau - 1)],$$

and variance:

$$\mathbb{E}[(e_2(\tau) - \mathbb{E}[e_2(\tau)|W_3(\tau - 1))^2] = \mathbb{E}[(e_2(\tau))^2] - \mathbb{E}[e_2(\tau)((1 - G(\tau))W_3(\tau - 1)] = P(\tau) - ((1 - G(\tau))^2\sigma_3^2(\tau - 1).$$

Thus,

$$f_2(e_2(\tau) = e, W_3(\tau - 1) = w) = \mathcal{N}(0, \sigma_3^2(\tau - 1)) \times \mathcal{N}((1 - G(\tau))w, P(\tau) - ((1 - G(\tau))^2\sigma_3^2(\tau - 1)).$$

Thus in the above discussion, for fixed X_2 and (v_1, ω_1, t_1), the collision variable, $\mathcal{C}_2(e, w)$, for Car 1 colliding with Car 2 has been defined. The joint distribution between the error in position and error in velocity, $f_2(e_2(\tau), W_3(\tau - 1))$ has been found. The approximate probability of Car 1 colliding with Car 2 is found numerically by integrating $\mathcal{C}_2(e, w)$ and $f_2(e_2(\tau), W_3(\tau - 1))$. The procedure for the same is discussed in section 4.

3.5 Probability of Collision - Car 3

In this section, we discuss a procedure similar to the previous section to find the probability of Car 1 colliding with Car 3.

Definition 3.4. Car 1 is said to collide with Car 3:

$$\mathcal{C}_3 = 1 \text{ if} : \exists\, k \ni \begin{cases} x_3(k) - L \leq x_1(k) \leq x_3(k) + L, & (3) \\ \dfrac{3d}{2} - L \leq y_1(k) \leq \dfrac{3d}{2}. & (4) \end{cases}$$

$\mathcal{C}_3$ is a function of (v_1, ω_1, t_1). Given X_3 and $(v_1, \omega_1, t_1) \in S_{X_3}$, if Car 1 decides to overtake, the time intervals where (4) holds are: $[\tau + 2\Delta - \delta, \tau + 2\Delta]$, $[\tau + 2\Delta, \tau + 2\Delta + t_1]$ and $[\tau + 2\Delta + t_1, \tau + 2\Delta + t_1 + \delta]$. Let e be the true value of error in position estimate of Car 3 at τ and w is true value of error in velocity estimate of Car 3 at $\tau - 1$. Consider $[\tau + 2\Delta - \delta, \tau + 2\Delta]$,

$$\phi_1(s) = x_1(\tau) + \int_0^{\Delta+s} v_1 \cos(\omega_1 t)dt = x_1(\tau) + \frac{v_1}{\omega_1}\sin(\omega_1(\Delta+s)),$$
$$\phi_2(s,e,w) = (\hat{x}_3(\tau)+e) + (\hat{v}_3(\tau-1)-w)(\Delta+s).$$

At time $\tau+\Delta+s$, $\phi_1(s)$ is the position of Car 1 while $\phi_2(s,e,w)$ is the predicted position of Car 3. To verify if (3) and (4) hold, we find :

$$\mathcal{B}_1(e,W) = \{s \in [\Delta-\delta, \Delta] : -L \leq \phi_1(s) - \phi_2(s,e,w) \leq L\}.$$

Consider the interval $[\tau+2\Delta, \tau+2\Delta+t_1]$,

$$\phi_3(s) = x_1(\tau) + \frac{v_1}{\omega_1}\sin(2\omega_1\Delta) + v_1 s,$$
$$\phi_4(s,e,w) = (\hat{x}_3(\tau)+e) + (\hat{v}_3(\tau-1)-w)(2\Delta+s).$$

At time $\tau+2\Delta+s$, $\phi_3(s)$ is the position of Car 1 while $\phi_4(s,e,w)$ is the predicted position of Car 3. To verify the collision definition, we find :

$$\mathcal{B}_2(e,w) = \{s \in [0,t_1] : -L \leq \phi_3(s) - \phi_4(s,e,w) \leq L\}.$$

For the third interval, $[\tau+2\Delta+t_1, \tau+2\Delta+t_1+\delta]$,

$$\phi_5(s) = x_1(\tau) + \frac{v_1}{\omega_1}\sin(2\omega_1\Delta) + v_1 t_1 + \frac{v_1}{\omega_1}\sin(\omega_1(s)),$$
$$\phi_6(s,e,w) = (\hat{x}_3(\tau)+e) + (\hat{v}_3(\tau-1)-w)(2\Delta+t_1+s).$$

$\phi_5(s)$ is the position of Car 1 at time $\tau+2\Delta+t_1+s$ while the $\phi_6(s,e,w)$ is the predicted position of Car 3 at the same time. To verify if (3) and (4) hold, we find :

$$\mathcal{B}_3(e,w) = \{s \in [0,\delta] : -L \leq \psi_5(s) - \psi_6(s,e,w) \leq L\}.$$

For the fixed values of e and W, the collision variable is:

$$\mathcal{C}_3(e,w) = 1_{\mathcal{B}_1(e,W)\cup\mathcal{B}_2(e,W)\cup\mathcal{B}_3(e,W)\neq\emptyset}.$$

The joint density of $e_3(\tau)$ and $W_4(\tau-1)$ is given by :

$$f_3(e_3(\tau)=e, W_4(\tau-1)=w) = \mathcal{N}(0,\sigma_4^2(\tau-1))\times \mathcal{N}((1-H(\tau))w, R(\tau) - ((1-H(\tau))^2\sigma_4^2(\tau-1))$$

The probability of Car 1 colliding with Car 3, is found numerically by integrating $\mathcal{C}_3(e,w)$ and the above density. The same procedure is followed for every $(v_1,\omega_1,t_1)) \in S_{X_3}$ to obtain $\mathbb{P}(\mathcal{C}_3(v_1,\omega_1,t_1))$.

3.6 Decision

If $S_{X_2} \cap S_{X_3} = \emptyset$, then $\mathcal{D} = 0$. Given the probability of collision with Car 2 and Car 3 for the admissible triples, Car 1 does an exhaustive search in $S_{X_2} \cap S_{X_3}$ to find the triple for which the maximum of the probability of collision with Car 2 and the probability of collision with Car 3 is minimized.

$$P^* = \min_{\{v_1,\omega_1,t_1 \in S_{X_2}\cap S_{X_3}\}} \max[\mathbb{P}(\mathcal{C}_2(v_1,\omega_1,t_1)), \mathbb{P}(\mathcal{C}_3(v_1,\omega_1,t_1))].$$
$$(v_1^*,\omega_1^*,t_1^*) = \arg\min_{\{v_1,\omega_1,t_1 \in S_{X_2}\cap S_{X_3}\}} \max[\mathbb{P}(\mathcal{C}_2(v_1,\omega_1,t_1)), \mathbb{P}(\mathcal{C}_3(v_1,\omega_1,t_1))].$$

Given a threshold level T which is a function of the cost incurred for collision (C)and the reward for overtaking (R) (example : $T = \frac{R}{C+R}$),

$$\mathcal{D} = 1, \text{ if } P^* \leq T$$
$$= 0, \text{ otherwise.}$$

$\mathcal{D} = 1$ corresponds to overtaking.

3.7 Car 1 Action

In this section we discuss the control algorithm for Car 1. The detailed algorithm is presented in algorithm 1. The summary is as follows: from $k = 0$ to $\tau - 1$, $v_1(k)$ is fixed and $\omega(k) = 0$. At τ, Car 1 first obtains both the feasible sets. Then for the feasible triples, it obtains the probability of collision. Using the probability of collision, it finds the optimal decision. If the optimal decision is to overtake, then $v_1(k)$ is set to v_1^*, while $\omega(k)$ is changed from time to time as described in the algorithm. As the estimate of the velocity could be poor, a conservative policy for Car 1 would be to change his velocity to half the estimated velocity of Car 2, when the optimal decision is to trail.

Algorithm 1 Control Algorithm

function CONTROL $(x_1, \hat{x}_2, \hat{x}_3, v_1, \hat{v}_2, \hat{v}_3, P_k, R_k, i)$ ▷ i denotes the iteration number
  if $(N \times L \leq \hat{x}_2 - x_1 \leq (N+1) \times L \wedge DecisionComplete = -1)$ **then**
    $DecisionComplete \leftarrow 0$
    $D_{It} \leftarrow i$;
  if $DecisionComplete = -1$ **then**
    $v_1(i) \leftarrow v,\ v : v \in \mathcal{V}$,
    $\omega_1(i) \leftarrow 0$
  if $DecisionComplete = 0$ **then**
    Obatin $S_{X_2} \cap S_{X_3}$
    Obatin $\mathbb{P}(\mathcal{C}_2(v_1,\omega_1,t_1)), \mathbb{P}(\mathcal{C}_3(v_1,\omega_1,t_1))$
    Obtain $\mathcal{D}, (v_1^*, \omega_1^*, t_1^*)$
    Find Δ^*
    $DecisionComplete \leftarrow 1$
  if $(\mathcal{D} = 1 \wedge DecisionComplete = 1)$ **then**
    $Delta_c = \Delta^*/timestep,\ t_{1,c} = t_1^*/timestep$
    $v_1(i) = v_1^*$
    if $0 \leq i - D_{It} < Delta_c$ **then**
      $\omega_1(i) = \omega_1^*$
    else if $Delta_c \leq i - D_{It} < 2Delta_c$ **then**
      $\omega_1(i) = -\omega_1^*$
    else if $2Delta_c \leq i - D_{It} < 2Delta_c + t_{1,c}$ **then**
      $\omega_1(i) = 0$
    else if $2Delta_c + t_{1,c} \leq i - D_{It} < 3Delta_c + t_{1,c}$ **then**
      $\omega_1(i) = -\omega_1^*$
    else if $3Delta_c + t_{1,c} \leq i - D_{It} < 4Delta_c + t_{1,c}$ **then**
      $\omega_1(i) = \omega_1^*$
    else if $i - D_{It} \geq 4Delta_c + t_{1,c}$ **then**
      $\omega_1(i) = 0$
  else if $(\mathcal{D} = 0 \wedge DecisionComplete = 1)$ **then**
    $V2est \leftarrow \frac{\sum_{j=0}^{i-1} \hat{v}_2(j)}{i}$
    $v_1(i) \leftarrow v,\ v : v \in \mathcal{V}, v \leq V2est/2$
    $\omega_1(i) \leftarrow 0$

4. SIMULATION RESULTS

4.1 Setup

We first describe the simulation setup. The lane width, d was chosen as 3.7 m. The safety distance, L was chosen as 1.5 m. M and $\bar{M}$ were set to 2. The simulation time step was taken to be 0.01 second. The initial position, velocities and variances where chosen as in Table 1. The velocity of

Parameter	Value	Variance
$x_1(0)$	0	0.001
$x_2(0)$	50 m	0.001
$x_3(0)$	275 m	0.001
$v_1(0)$	17 m/s	-
$v_2(0)$	10 m/s	-
$v_3(0)$	-9 m/s	-

Table 1. Initial position and velocity

Car 2 and Car 3 remain fixed. In practice, it is known that it is not possible to measure the relative position and velocity with the same accuracy. In the simulations, the variance of the noise in position and velocity measurements were chosen as follows: if k was even, then $\sigma_1(k) = \sigma_2(k) = c_1$ and $\sigma_3(k) = \sigma_4(k) = 10c_1$. If k was odd, then $\sigma_1(k) = \sigma_2(k) = 10c_1$ and $\sigma_3(k) = \sigma_4(k) = c_1$. The admissible velocities considered were :{V-Min, V-Min + step-size, V-Min + 2 × step-size, ..., V-Max}. Similarly, admissible sets were generated for angular velocity and time in alternate lane. The lower bound, upper bound and the step sizes for all three sets has been tabulated in Table 2. The finite sets for error in position estimate and velocity

Parameter	Maximum	minimum	step-size
v_1	30 m/s	8 m/s	0.5
ω_1	0.7 rad/s	0.1 rad/s	0.2
t_1	10 s	2 s	0.5

Table 2. Admissible velocity, angular velocity and time

estimate were generated as follows. Let $\sigma_p = \sqrt{P(\tau)}$ and $\sigma_v = \sigma_3(\tau - 1)$. Let $N1 = \lceil \frac{6\sigma_p}{\text{step-size}} \rceil$ and $M1 = \lceil \frac{6\sigma_v}{\text{step-size}} \rceil$. Then the finite domain for e, error is position estimate was $\mathcal{E} = \{-3\sigma_p, -3\sigma_p + \text{step-size}, \ldots, -3\sigma_p + N1 \times \text{step-size}\}$. The finite domain for w, error is velocity estimate was $\mathcal{U} = \{-3\sigma_v, -3\sigma_v + \text{step-size}, \ldots, -3\sigma_v + M1 \times \text{step-size}\}$. The step-size for the generation of these two sets was set to 0.2. For every triple $(v_1, \omega_1, t_1) \in S_{X_2}$, $\mathbb{P}(\mathcal{C}_2(v_1, \omega_1, t_1))$ was calculated as follows. First, for each pair $(e, w) \in \mathcal{E} \times \mathcal{U}$, the collision variable $\mathcal{C}_2(e, w)$ was found. Then, the probability of collision with Car 2 was approximated as:

$$\mathbb{P}(\mathcal{C}_2(v_1, \omega_1, t_1)) = \sum_{\{e,w\} \in \mathcal{E} \times \mathcal{U}} \mathcal{C}_2(e, w) f_2(e, w).$$

Similarly the probability of collision with Car 3 was approximated. The threshold T was set to 0.01. Three values of N was considered, $N = 8, 5, 2$. Three values of c_1 was considered, $c_1 = 1, 0.5, 0.1$. With these settings simulations were performed. P^* was found to be zero in all except one scenario, for the pair ($N = 2$,$c_1 = 0.5$). For $N = 2$,$c_1 = 0.5$, P^* was found to be 0.0945. For $N = 5$, $c_1 = 0.1$, the optimal triple was (15 m/s, 0.3 rad/s, 2 s). For that pair and rest of the simulation setup described above, the trajectory of the three cars has been plotted in Figure 3. For higher velocities of Car 1 and Car 3, it was observed that the optimal decision is to not overtake.

5. CONCLUSION AND FUTURE WORK

In this paper we studied a stochastic control approach to the car overtake problem. Based on the information available to Car 1, the probability of collision with either

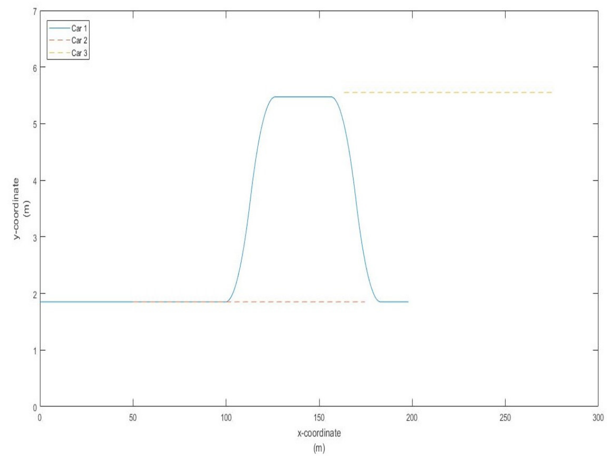

Fig. 3. Trajectory of the three cars

car was calculated as a function of the velocity, angular velocity and time spent in the alternate lane. The control action was chosen by Car 1 in such way that maximum of the probability of collision with other two cars was minimized. For the simulation settings mentioned, when Car 1 decided to overtake Car 2 at farther distance, it was observed that the probability of collision was lower. The novelty of the solution is that even though the problem involves multiple agents, the solution relies only on the probability space constructed by the agent making the decision, i.e, Car 1.

In the present formulation, Car 1 does not have the flexibility to return to its original lane while taking over. So it would be interesting to include this action into the formulation. Another question that arises is that, if Car 1 decides to return what is the "optimal" time. We could also study the effect of collaboration or noncooperation between Car 1, Car 2 and Car 3.

ACKNOWLEDGEMENTS

The authors would like to thank Fatima Alimardani (UMD), Siddharth Bansal (UMD) and Yulong Gao (KTH) for their initial simulation based study of the problem.

REFERENCES

Böhm, A., Jonsson, M., and Uhlemann, E. (2011). Adaptive cooperative awareness messaging for enhanced overtaking assistance on rural roads. In *2011 IEEE Vehicular Technology Conference (VTC Fall)*, 1–5.

Petrov, P. and Nashashibi, F. (2014). Modeling and nonlinear adaptive control for autonomous vehicle overtaking. *IEEE Transactions on Intelligent Transportation Systems*, 15(4), 1643–1656.

Sezer, V. (2017). Intelligent decision making for overtaking maneuver using mixed observable markov decision process. *Journal of Intelligent Transportation Systems*, 0(0), 1–17.

Vinel, A., Belyaev, E., Egiazarian, K., and Koucheryavy, Y. (2012). An overtaking assistance system based on joint beaconing and real-time video transmission. *IEEE Transactions on Vehicular Technology*, 61(5), 2319–2329.