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A piecewise-constant congestion taxing policy for repeated routing games ☆,☆☆

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ABSTRACT

In this paper, we consider repeated routing games with piecewise-constant congestion taxing in which a central planner sets and announces the congestion taxes for fixed windows of time in advance. Specifically, congestion taxes are calculated using marginal congestion pricing based on the flow of the vehicles on each road prior to the beginning of the taxing window (and, hence, there is a time-varying delay in setting the congestion taxes). We motivate the piecewise-constant taxing policy by that users or drivers may dislike fast-changing prices and that they also prefer prior knowledge of the prices. We prove for this model that the multiplicative update rule and the discretized replicator dynamics converge to a socially optimal flow when using vanishing step sizes. Considering that the algorithm cannot adapt itself to a changing environment when using vanishing step sizes, we propose adopting constant step sizes in this case. Then, however, we can only prove the convergence of the dynamics to a neighborhood of the socially optimal flow (with the size of the neighbourhood being of the order of the selected step size). The results are illustrated on a nonlinear version of Pigou's famous routing game.

1. Introduction

1.1. Motivation

In 1963, William S. Vickrey, a Nobel Laureate in Economics and well-known for the development and analysis of the second-price auction, started his paper on resource allocation in transportation with the "proposition that in no other major area are pricing practices so irrational, so out of date, and so conductive to waste as in urban transportation" (Vickrey, 1963). He argued the waste is caused by the "absence of off-peak differentials" and the "underpricing of some modes relative to the others". He went on to say that "the pricing of the use of urban streets is all but nonexistent". In Vickrey (1992), he subsequently proposed that we use a "detection and billing method" based on "electronic identifier units carried in each

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vehicle, which would activate recording devices in or on the road." These recordings should be used to charge the vehicles as closely as possible to the marginal social cost[1] of each trip in terms of the impacts on others (Vickrey, 1992).

We have come a long way since 1963 in the implementation of congestion pricing. The local government in Stockholm implemented a congestion taxing[2] system in August of 2007 after a seven-months trial period. The influence of the congestion taxes over the trial period was later studied in Eliasson et al. (2009), which showed improvements in travel times in addition to several other positive economic and environmental effects. Other behavioural aspects and influences of the congestion taxing in Stockholm was discussed in Karlström and Franklin (2009), Winslott-Hiselius et al. (2009), Eliasson and Mattsson (2006), and Börjesson et al. (2012). Several other cities, such as London, San Francisco, and Singapore, have also implemented congestions taxing schemes (Leape, 2006; Santos, 2005; Frick et al., 1996). However, there are several issues that still need to be addressed in congestion taxing systems. For instance, as the study in Börjesson et al. (2012) notes "[s]ince the traffic flow increases due to external factors, primarily increasing population in the county, the charge must, however, increase to keep the traffic flow at the present level". More importantly, the implemented fixed tolls do not react to temporary traffic changes and are designed based on the average behaviour of the travellers (i.e., they do not follow the marginal social cost of each trip in terms of the impacts on other vehicles as suggested in Vickrey (1992)).

To avoid this problem, dynamic congestion pricing techniques have been employed. For instance, in San Diego I-15 High-Occupancy Toll (HOT) Lanes, the single-occupancy vehicles must pay tolls that varies dynamically with the level of congestion over that lane[3] (Federal Highway Adminstration). The tolls are updated every six minutes and may vary in 25-cents increments, and can go as high as eight dollars. Generally, this scheme was considered to be fair and effective by the public and the media (Federal Highway Adminstration). This is thought to be because the scheme offered all travellers on I-15 a choice whether to pay for the use of the express lanes as an alternative to being stuck in a traffic jam in free[4] lanes (Federal Highway Adminstration; Lindsey and Verhoef, 2000). However, to achieve a socially optimal flow, we need to impose the congestion taxes on all the vehicles and over all the lanes (based on the marginal social cost of the trip) (Vickrey, 1992). Imposing taxes on all the lanes is certainly controversial or, to say the least, cumbersome to understand for the drivers as they need to calculate and respond to time-varying congestion taxes at the same time as driving. Therefore, it is desirable to devise a slowly-varying or piecewise-constant congestion charges for the roads (that are announced well in advance so that the drivers can respond to them properly).

1.2. Previous studies

Using routing games for understanding the drivers' behaviour and interplay dates back to the work of Wardrop (Wardrop, 1952), where the notion of equilibrium for the routing games was introduced. Due to this pioneering work, the equilibrium is widely known as the Wardrop equilibrium in the transportation literature (Smith, 1979; Haurie and Marcotte, 1985; Braess and Koch, 1979). However, authors in different communities sometimes use other names for this equilibrium, e.g., user-optimizing flow (Dafermos, 1972; Braess and Koch, 1979), Wardrop first principle (Smith, 1979), and Nash[5] equilibrium (Roughgarden and Tardos, 2002; Krichene et al., 2012). In Beckmann et al. (1956), it was shown that the problem of finding a Wardrop equilibrium can be cast as an optimization problem under a mild condition (which happens to be a convex one if the latencies are non-decreasing functions of the flow over each road). For a survey of results in routing games, see Roughgarden (2007).

It is widely known that, in general, the equilibria in non-cooperative games are inefficient (Roughgarden and Tardos, 2004; Roughgarden and Tardos, 2002; Dubey, 1986; Correa and Stier-Moses, 2010), i.e., they do not necessarily minimize the social cost function.[6] Therefore, many studies have been dedicated to bounding the price of anarchy (i.e., the worst-case ratio of the social cost of the Nash equilibrium over the social cost of the optimal flow) (Koutsoupias and Papadimitriou, 1999; Papadimitriou, 2001). For instance, in routing games with linear latency functions, it was proved that the price of anarchy is upper-bounded by 4/3 (Roughgarden and Tardos, 2002). The authors of Roughgarden and Tardos (2002) also showed that for general continuous and non-decreasing latency functions the total latency at the equilibrium is no more than the total latency incurred by the optimal flow for routing twice as much traffic between the source–destination pairs. The price of anarchy for a wide-ranging family of latency functions was subsequently captured in Roughgarden (2003).

Due to this inherent inefficiency of the Nash equilibrium, there have been several studies in reducing the inefficiency through devising appropriate congestion taxes on the edges of the transportation network (see Pigou, 1932; Yang and Huang, 2005; Yang and Huang, 2004; Yang and Huang, 1998; Beckmann et al., 1956; Hoefer et al., 2008 among other studies) and through rerouting a fixed percentage of the flow (see Stackelberg routing in Korilis et al. (1997), Kaporis and Spirakis

[1] Economists have for long suggested that marginal congestion prices (which are traffic-flow-dependent taxes) can result in socially optimal traffic flows (Pigou, 1932).

[2] We use the terms "toll", "congestion tax", "congestion price", and "congestion charge" interchangeably throughout the paper.

[3] Note that the dynamic congestion pricing was part of the second phase of the project also known as I-15 FasTrak.

[4] The term "free" is used here in the sense that the congestion tax is equal to zero.

[5] The term Nash equilibrium might be slightly confusing as, in the game theory literature, it is reserved primarily for games with finitely-many players (Haurie and Marcotte, 1985) opposed to the routing game where we deal with a continuum of players when modelling the traffic flow. Nevertheless, the notion is fairly common in the computer science literature (Roughgarden and Tardos, 2002; Krichene et al., 2012).

[6] This observation is true for a utilitarian social cost function (i.e., summation of the individual cost functions of all the players) as well as a Rawlsian social cost function (i.e., the worst-case cost function of the players).

(2006), Roughgarden (2004), Yang et al. (2007)). In an early work, Pigou, a prominent English economist and best known for his contributions to welfare economics, proposed marginal taxes[7] (i.e., each driver is charged by the marginal increase in cost caused by her multiplied by the amount of the traffic that suffers from this increase) as a way of achieving the socially optimal flow (Pigou, 1932). This claim was revisited and mathematically proved in Beckmann et al. (1956). The idea was extended to multi-class traffic in Dafermos (1973). The authors of Engelson and Lindberg (2006) designed fixed and marginal congestion taxes for static routing games in which the drivers have different value of time.

Static routing games and Wardrop equilibria are idealized models for traffic assignment in transportation networks. A study in Hall (1983) suggests that the main idealizations are that (*i*) the players are omniscient (i.e., the cost functions over the edges, the demand over various destination–source pairs, and other characteristics of the game, are universally known or, even stronger, they are common knowledge[8]) and (*ii*) the players are rational (i.e., the can correctly reason to find the optimal decision without any mistakes). To weaken such idealizations, various studies have proposed other notions of equilibrium. For instance, quantal[9] response equilibrium weakens these idealizations by assuming that the players only have access to noisy measurements of their utilities or cost functions (McKelvey and Palfrey, 1995), where the noise can model the players lack of knowledge and/or experience in their decision making. The authors of McKelvey and Palfrey (1995) mention that (*i*) in static scenarios, the players might use questionnaires, experiments, and approximations to construct noisy estimates of the cost functions and (*ii*) in repeated scenarios, the players may use their observations (from earlier stages) to estimate the cost functions, which results in noisy estimates of the actual costs (with the noise variance decreasing as the players play more and gain more experience).[10] Using quantal response equilibrium, we can model the cases where costly errors are unlikely and a players can afford to make mistakes that are not ruinous. The quantal response equilibrium has shown to match experimental results, specifically, that with gaining more experience players act more in line with the rational expectation equilibrium (McKelvey and Palfrey, 1995). These results were subsequently generalized to a dynamic setting in Mckelvey and Palfrey (1998). Other models for including uncertainties in decision making have been introduced in Rosenthal (1989) and Beja (1992). These studies did not focus on learning, but they have still inspired follow-up studies on learning in repeated games).

Repeated games have been an interest of the economists as a way to enforce cooperation through repetition and reputation[11] (Axelrod, 1984; Mertens, 1986). Repeated games in combination with various learning dynamics, such as best response dynamics (Ellison, 1993; Hopkins, 1999), fictitious play (Brown, 1951; Monderer and Shapley, 1996), logit-response dynamics (Alós-Ferrer and Netzer, 2010), no regret learning (Arora et al., 2012), and Cournot dynamics (Seade, 1980; Hahn, 1962), have introduced useful approaches for extracting Nash equilibria in games using distributed dynamics. Several studies have compared the predictions from the dynamics in repeated games with experimental data to investigate their practicality (Morgan et al., 2009; Erev and Roth, 1998; Mookherjee and Sopher, 1997; Camerer et al., 2002; Sarin and Vahid, 2001). For instance, the authors of Erev and Roth (1998) investigate experimental data from several repeated matrix games (with unique mixed strategy equilibria) repeated over one hundred periods. They show that one-parameter reinforcement learning performs very well; however, the predictions can be significantly improved by allowing richer models or considering fictitious play. Experimental data on constant-sum matrix games were studied in Mookherjee and Sopher (1997), where it was shown that dynamics from the experiments are best captured by the multiplicative update rule (an update rule in which the probability of selecting an action is proportional to the exponential of the average received payoff over the past or, equivalently, inversely proportional to the exponential of the average cost). In Mookherjee and Sopher (1997), the authors use the term “quantal response learning” instead of the multiplicative update rule since their proposed learning scheme is motivated by a family of quantal response equilibria called the logit-equilibria (McKelvey and Palfrey, 1995). Later, richer dynamics were introduced to capture the behaviour portrayed in a large set of experimental data (Camerer et al., 2002). These dynamics contain most of the learning rules in the repeated games literature as special cases and, hence, can result in better fitnesses if finely tuned to the data.

Repeated routing games,[12] as a special class of repeated games, have attracted attention recently (Blum et al., 2007; Blum et al., 2010; Krichene et al., 2014). For instance, in Blum et al. (2010), the authors studied no-regret learning (i.e., the difference between the average latency caused by the online decisions and the average latency for the best fixed decision in hindsight grows very slowly). They also proved the convergence of a subsequence of the flows to a neighbourhood of the Wardrop equilibrium. The convergence result was further strengthened to the whole sequence of flows in Krichene et al. (2014). Repeated

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[7] In the economics literature, marginal taxes have a long history in achieving a socially optimal solution when there are negative externalities (i.e., actions of a player has negative effect on the welfare of the other players in the environment) (Buchanan and Stubblebine, 1962).

[8] In the game theory literature, the term “common knowledge” refers to an event that everyone knows about, everyone knows that everyone knows about it, and so on Aumann (1976) and Milgrom (1981). For instance, the knowledge that green traffic light means that the drivers may pass through the intersection should be common knowledge between the drivers in a society otherwise they cannot make any decision in a distributed manner regarding how to pass an intersection (Fagin et al., 1995).

[9] According to McKelvey and Palfrey (1995), the term “quantal” is borrowed from the statistical literature, where in quantal choice theory the players are rational while only relying on noisy estimates of costs and payoffs.

[10] They call this learning-by-doing; however, they not study the convergence aspects of such models in repeated games. Subsequent studies have used this idea to propose learning dynamics in repeated games, e.g., (Mookherjee and Sopher, 1997).

[11] This follows from the so-called Folk Theorem, which specifies that every feasible and individually-rational action profile of a static game is achievable as an equilibrium of its corresponding repeated game (Friedman, 1971; Mertens, 1986; Aumann and Shapley, 1994; Rubinstein, 1994).

[12] In repeated routing games, the drivers select their path on each iteration based on what they have observed so far in the game. The iterations can be dubbed into days, which motivates a setup in which a fixed group of drivers compete, on a daily basis, for the resources offered by the transportation network. Alternatively, repeated routing games can also be considered as an efficient methods for extracting a Wardop equilibrium using distributed dynamics.

routing games are in close connection with evolutionary game theory (Weibull, 1997; Sandholm, 2001; Sandholm, 2012; Friedman, 1991), in which users adopt simple update rules motivated by biological systems and evolutionary observations, e.g., the users meet with a given probability with other users and replicate their behaviour if it results in a better utility. For instance, the authors in Fischer and Vöcking (2004) introduced an ordinary differential equation for the evolution of the flows (motivated by population dynamics in evolutionary game theory) and studied its convergence. A wide range of dynamics frequently used in evolutionary game theory is surveyed in Hofbauer and Sigmund (2003). Other day-to-day traffic flow assignment models have also been proposed and studied extensively. Examples of these dynamics are the simplex gravity flow dynamics (Smith, 1983), proportional-switch adjustment (Smith, 1984; Smith and Wisten, 1995), network tâtonnement process (Friesz et al., 1994), and projected dynamical system (Zhang and Nagurney, 1996). In Yang and Zhang (2009), it was shown that these dynamics are a subset of rational behaviour adjustment processes, which implies that stationarity of link flows is equivalent to Wardrop equilibrium. Link-based day-to-day traffic dynamics were subsequently introduced in Guo et al. (2015) and He et al. (2010). To the best of our knowledge, none of the mentioned studies propose a piece-wise constant scheme for setting congestion taxes in repeated routing games to achieve a socially optimal flow.

In the transportation literature, when the demand and/or the cost functions are unknown, trial-and-error methods have been used for setting the congestion charges (Yang et al., 2010; Yang et al., 2004; Zhao and Kockelman, 2006; Wang and Yang, 2012; Zhou et al., 2015; Han and Yang, 2009). For instance, a trial-and-error implementation of marginal-cost pricing on road networks when the demand functions are unknown was introduced in Yang et al. (2004). An iterative two-stage approach with an adaptive step size to update the link tolls based on the observed link flows and given flow restraint levels was developed in Yang et al. (2010) to find effective link tolls that reduce flows to below a desirable target level. A modified bisection method for the trial-and-error implementation of tradable credit scheme (to achieve a revenue-neutral congestion pricing system) was proposed in Wang and Yang (2012). A combination of trial-and-error congestion pricing schemes was studied in Zhou et al. (2015) to both consider the minimization of the total system cost and address the capacity constraints. Although extremely powerful in setting and adapting the congestion taxes with minimal knowledge of the demand and based on recent changes in the transportation network, these studies do not consider the day-to-day traffic assignment aspect of the problem.

1.3. Contributions

In this paper, we consider a repeated routing game in which a group of drivers uses the transportation network on a daily basis. We specifically follow two widely-recognized discrete-time dynamics, namely, the multiplicative update rule (Kleinberg et al., 2009; Cominetti et al., 2010) and the discretized replicator dynamics (Fischer and Vöcking, 2004; Cominetti et al., 2010), for updating the flows on various paths. We assume that the central planner sets the congestion taxes for wide windows of iterations in advance and announce the taxes publicly for those days. This would amount to piecewise-constant congestion taxes as the tolls stay constant for a number of days (e.g., a week, a month). Fig. 1 shows an illustrative example of such piecewise-constant congestion taxes when the congestion taxes gets updated on a weekly basis. Our interest in this scheme is motivated by the facts that the drivers (*i*) dislike fast-changing prices (as it is cumbersome to be updated with the latest information to make educated decisions) and (*ii*) want prior knowledge of the prices (so that they can make informed decisions prior to the trip). The proposed congestion taxes are calculated using marginal congestion prices based on the flow of the vehicles on each road prior to the beginning of the taxing window (and, therefore, the tolls are not really marginal congestion prices since they do not reflect the actual flow of cars). Firstly, we show that the introduced dynamics (for the repeated routing game) are very similar to each other and, therefore, we can analyze both of them together. Then, we prove that, for the proposed piecewise-constant congestion taxes, both dynamics converge to a socially optimal decision if their step sizes is set to be of the order of $1/k$ for iteration k. Unfortunately, the shrinking step size renders the algorithms impractical for the cases where the parameters of the routing game (e.g., the demands) vary over time since the proposed dynamics cannot adapt fast enough, especially, after a long time, because the step size becomes very small. Following this observation, we propose using a constant step size. Doing so, we realize that we can only converge to a

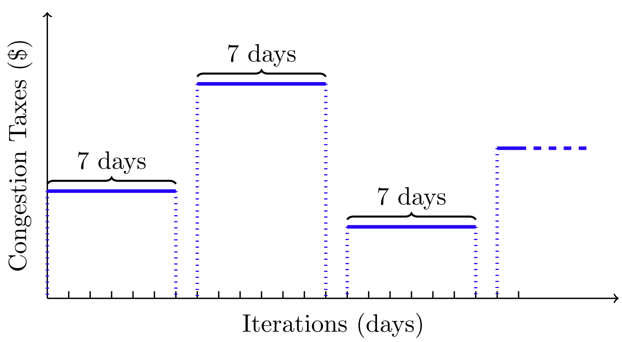

Fig. 1. An illustrative example of the piecewise-constant congestion taxing policy.

neighborhood of the socially optimal flow, with its size being proportional to the selected step size. This is an interesting trade-off because so long as the step size is large the algorithm can adapt rapidly to the changes in the routing game; however, the solution can potentially be far from the socially optimal flow. Although other dynamics have been introduced in Camerer et al. (2002) to capture specific behaviours in experimental data, we prefer using the mentioned dynamics because of three main reasons. Firstly, the proposed dynamics, specifically, the multiplicative update rule, have been proved very successful for predicting people's behaviour when playing various games (Mookherjee and Sopher, 1997). Secondly, the multiplicative update rule and replicator dynamics are, respectively, rooted in the logit-equilibria (from quantal response equilibrium theory) and biological traits in nature, which have been repeatedly tested on various theoretical and experimental setups with successful results and observations over the past. Thirdly, the proposed dynamics are simple enough to allow us to achieve meaningful theoretical results without overly-technical and cumbersome proofs. However, as an avenue for future research, we certainly suggest investigating also richer set of dynamics, such as the one introduced in Camerer et al. (2002).

1.4. Organization

The rest of the paper is organized as follows. In Section 2, we introduce our notations for routing game and review some results in this area. We present our results in the repeated routing game in Section 3. Numerical examples are presented in Section 4. Finally, we finish the paper with the conclusions and avenues for future research in Section 5.

2. Routing game

In what follows, we use notations $\mathbb{R}$ and $\mathbb{N}$ to denote the sets of real and integer numbers, respectively. We also define $\mathbb{R}_{\geqslant(>)a} = \{x \in \mathbb{R} | x \geqslant (>) a\}$ for any $a \in \mathbb{R}$. Furthermore, let $[\![K]\!] = \{1, \ldots, K\}$ for any $K \in \mathbb{N}$.

We model the transportation network with a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which $\mathcal{V}$ denotes the nodes in the network (e.g., intersections) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edges in the network (e.g., roads). We assume that the graph can admit parallel edges. We are also provided with a set of source–destination nodes $\{(s_k, d_k)\}_{k\in[\![K]\!]}, K \in \mathbb{N}$, where each pair $(s_k, d_k), k \in [\![K]\!]$, should transfer a total flow of $F_k \in \mathbb{R}_{>0}$. The assumption that $F_k \neq 0$ is without loss of generality as, otherwise, we can remove the source–destination nodes with zero flow from the problem without changing the underlying routing game. Let $\mathcal{P}_k$ denote the set of all directed paths that connect the source s_k to destination d_k for any $k \in [\![K]\!]$, where a directed path from node s_k to node d_k is an ordered sequence of edges $((i_j, i_{j+1}))_{j=1}^{n} \in \mathcal{E}^n$ such that $i_1 = s_k$ and $i_{n+1} = d_k$. Moreover, let us define the set of all paths as $\mathcal{P} = \cup_{k\in[\![K]\!]}\mathcal{P}_k$.

Example 1 (*Transporation Network*). Let us consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Fig. 2, where the solid black arrows show the edges in the transportation network. We have $K = 3$ source–destination nodes $(s_1, t_1) = (0, 1), (s_2, t_2) = (7, 3)$, and $(s_3, t_3) = (0, 8)$. The corresponding paths for these source–destination nodes are

$$\begin{aligned}
\mathcal{P}_1 &= \{((0,1)), ((0,4),(4,5),(5,1)), ((0,4),(4,6),(6,1))\},\\
\mathcal{P}_2 &= \{((7,2),(2,3)), ((7,2),(2,4),(4,5),(5,3)), ((7,2),(2,4),(4,6),(6,3))\},\\
\mathcal{P}_3 &= \{((0,1),(1,3),(3,8)), ((0,4),(4,5),(5,3),(3,8)), ((0,4),(4,6),(6,3),(3,8)), ((0,4),(4,5),(5,1),(1,3),(3,8)),\\
&\qquad ((0,4),(4,6),(6,1),(1,3),(3,8))\}.
\end{aligned}$$

In Fig. 2, each source–destination pair is portrayed in a separate color.[13] Furthermore, the set of paths for each source–destination pair is illustrated with dashed color lines. □

We use the notation $f_p \in \mathbb{R}_{\geqslant 0}$ to denote the flow of vehicles on a path $p \in \mathcal{P}$. In addition, we may define the aggregate flow vector as $f = (f_p)_{p\in\mathcal{P}} \in \mathbb{R}^{|\mathcal{P}|}$. A flow vector is called feasible if $\sum_{p\in\mathcal{P}_k} f_p = F_k$ for all $k \in [\![K]\!]$. Let us denote the set of all such feasible flows with $\mathcal{F}((F_k)_{k\in[\![K]\!]})$. When the source–destination flows $(F_k)_{k\in[\![K]\!]}$ can be deduced from the context or are irrelevant to the discussion, with slight abuse of notation, we shorten the notation to $\mathcal{F}$. For any aggregate vector of path flows $(f_p)_{p\in\mathcal{P}}$, we can define edge flows[14] $\phi_e = \sum_{p\in\mathcal{P}:e\in p} f_p \in \mathbb{R}_{\geqslant 0}$ for all $e \in \mathcal{E}$. We make the following standing assumption.

Assumption 1. $\mathcal{F}((F_k)_{k\in[\![K]\!]}) \neq \emptyset$.

A necessary and sufficient condition for guaranteeing the satisfaction of Assumption 1 is to ensure that $\mathcal{P}_k \neq \emptyset$ for all $k \in [\![K]\!]$.

[13] For interpretation of color in 'Fig. 2', the reader is referred to the web version of this article.

[14] We use the aggregate vector of edge flows $\phi = (\phi)_{e\in\mathcal{E}}$ and the aggregate vector of path flows $f = (f_p)_{p\in\mathcal{P}}$ interchangeably as there is a one-to-one correspondence between them (i.e., given one, we can determine the other one uniquely).

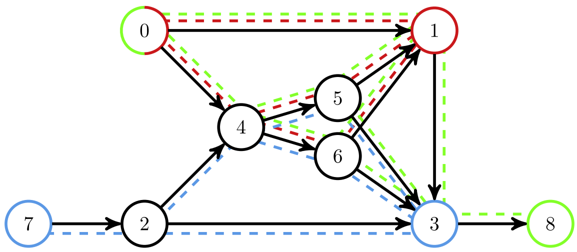

Fig. 2. An example of a transportation network.

A vehicle that travels along the edge $e \in \mathcal{E}$ observes a cost (e.g., latency) of $\tilde{\ell}_e(\phi_e)$ with a given mapping $\tilde{\ell}_e : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$. Hence, a vehicle that uses the path $p \in \mathcal{P}$ observes a total cost of $\ell_p(f) = \sum_{e \in p} \tilde{\ell}_e(\phi_e)$. Notice that we use $\tilde{\ell}_e$ and ℓ_p for, respectively, denoting the cost of using the edge $e \in \mathcal{E}$ and the path $p \in \mathcal{P}$.

Example 2 (*Bureau of Public Roads's Edge Latency Functions (Singh and Dowling, 2002)*). A widely used model for the edge cost functions is the Bureau of Public Roads model for the delay on each edge

$$\tilde{\ell}_e(\phi_e) = \frac{d_e}{v_e^{\max}} \left[1 + 0.15 \left(\frac{\phi_e}{c_e} \right)^4 \right],$$

where $d_e \in \mathbb{R}_{\geqslant 0}$ is length of the road, $v_e^{\max} \in \mathbb{R}_{\geqslant 0}$ is the speed limit (e.g., 50 km/h in most of the inner-city Stockholm), and c_e is the capacity of the road (e.g., approximately 2000 vehicles per hour multiplied by the number of lanes (Roess and McShane, 1987)). $\square$

In this formulation, each player is an infinitesimal amount of flow that tries to minimize its cost by selecting its path. Now, we define the Wardrop equilibrium for the introduced routing game.

Definition 1 (*Wardrop Equilibrium*). An aggregate vector of path flows $f = (f_p)_{p \in \mathcal{P}}$ is a Wardrop equilibrium for the routing game if, for all $k \in [\![K]\!], f_p > 0$ for a path $p \in \mathcal{P}_k$ implies that $\ell_p(f) \leqslant \ell_{p'}(f)$ for all $p' \in \mathcal{P}_k$.

This definition implies that for each source–destination pair $(s_k, d_k), k \in [\![K]\!]$, all the paths with a nonzero flow (i.e., the utilized paths) have equal latencies and the rest (i.e., the paths with a zero flow) have a larger (or equal) latency. Throughout this paper, we make the following assumptions.

Assumption 2. For each $e \in \mathcal{E}$, the mapping $\tilde{\ell}_e : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is (*i*) twice continuously differentiable, (*ii*) convex, and (*iii*) non-decreasing.

Theorem 1 (*Wardrop Equilibrium (Beckmann et al., 1956; Roughgarden, 2007)*). *The path flows $f^* = (f_p^*)_{p \in \mathcal{P}}$, with their corresponding edge flows $\phi^* = (\phi_e^*)_{e \in \mathcal{E}}$, constitute a Wardrop equilibrium if and only if*

$$\begin{aligned}
(f^*, \phi^*) \in \underset{(f,\phi) \in \mathbb{R}_{\geqslant 0}^{|\mathcal{P}|} \times \mathbb{R}_{\geqslant 0}^{|\mathcal{E}|}}{\operatorname{argmin}} \quad & \sum_{e \in \mathcal{E}} \int_0^{\phi_e} \tilde{\ell}_e(\xi) \mathrm{d}\xi, \\
\text{s.t.} \quad & \phi_e = \sum_{p \in \mathcal{P}: e \in p} f_p, \\
& F_k = \sum_{p \in \mathcal{P}} f_p.
\end{aligned}$$

Therefore, the problem of finding a Nash equilibrium boils down to solving a convex optimization[15] problem because, under Assumption 2 (*i*)–(*ii*), the routing game is a potential game (i.e., it admits a potential function whose minimizers are the equilibria of the game).

We can now define the social cost function

$$C(f) = \sum_{p \in \mathcal{P}} f_p \ell_p(f) = \sum_{e \in \mathcal{E}} \phi_e \tilde{\ell}_e(\phi_e),$$

where the second equality can be proved following simple algebraic manipulations (Roughgarden, 2007).

[15] Notice that Assumption 2 (*iii*) guarantees that the cost function of the optimization problem in Theorem 1 is indeed convex.

Definition 2 (*Socially Optimal Flow*). An aggregate vector of path flows $f = (f_p)_{p\in\mathcal{P}}$ is a socially optimal flow for the routing game if $f \in \arg\min_{f'\in\mathcal{F}} C(f')$.

It is a widely-known fact that the Wardrop equilibria can be inefficient (i.e., the social cost of the Wardrop equilibrium is larger than the social cost of a socially optimal flow) (Roughgarden and Tardos, 2002). Due to this, there have been several studies in reducing the inefficiency gap (Roughgarden, 2007). In the remainder of this section, we will discuss the effect of imposing tolls on the edges of the graph $\mathcal{G}$ to reduce the inefficiency of the Wardrop equilibria.

Let us assume that a driver must pay a toll $\tilde{\tau}_e(\phi_e)$, with $\tilde{\tau}_e : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, for using the edge $e \in \mathcal{E}$, where (as stated earlier) $\phi_e = \sum_{p\in\mathcal{P}:e\in p} f_p$ is the flow on edge $e \in \mathcal{E}$. Therefore, a vehicle that is using path $p \in \mathcal{P}_k$ endures a total cost of $\ell_p(f) + \tau_p(f)$, where $\tau_p(f)$ is total amount of money that this vehicle must pay for using path p and can be calculated as $\tau_p(f) = \sum_{e\in p} \tilde{\tau}_e(\phi_e)$. Hence, the definition of the equilibrium should be slightly modified to account for the tolls.

Definition 3 (*Wardrop Equilibrium with Tolls*). An aggregate vector of path flows $f = (f_p)_{p\in\mathcal{P}}$ is a Wardrop equilibrium for the routing game with tolls if, for all $k \in [\![K]\!], f_p > 0$ for a path $p \in \mathcal{P}_k$ implies that $\ell_p(f) + \tau_p(f) \leqslant \ell_{p'}(f) + \tau_{p'}(f)$ for all $p' \in \mathcal{P}_k$.

In Pigou (1932), Pigou suggested marginal cost taxes

$$\tilde{\tau}_e(\phi_e) = \phi_e \frac{\mathrm{d}\tilde{\ell}_e(\phi_e)}{\mathrm{d}\phi_e}, \tag{1}$$

as a powerful way for reducing the inherent inefficiency of the equilibria in routing games. These tolls are called marginal cost taxes since they correspond to the marginal increase in cost caused by adding one user to the edge $\mathrm{d}\tilde{\ell}_e(\phi_e)/\mathrm{d}\phi_e$ multiplied by the amount of the traffic that suffers from this increase ϕ_e. The following theorem shows the effectiveness of these tolls.

Theorem 2 (*Beckmann et al., 1956*). *Let us impose the marginal cost taxes in* (1) *for all* $e \in \mathcal{E}$*. Then, a flow* f *is a Wardrop equilibrium for the routing game with tolls if and only if it is a socially optimal flow of the routing game.*

Although extremely effective, it is difficult to implement these taxes since they are flow dependent (i.e., the drivers do not know the actual value of tolls prior to using the road and, hence, the might not be able to make an informed decision). To remove this dependency, one can use the following result.

Theorem 3 (*Engelson and Lindberg, 2006*). *Let* f^* *be a socially optimal flow. Let us impose the constant tax*

$$\tilde{\tau}_e = \left[\phi_e \frac{\mathrm{d}\tilde{\ell}_e(\phi_e)}{\mathrm{d}\phi_e}\right]_{\phi_e=\sum_{p\in\mathcal{P}:e\in p} f_p^*}, \quad \forall e \in \mathcal{E}.$$

Then, the flow f^* *is a Wardrop equilibrium for the routing game with tolls.*

Although much more convenient (since the tolls are constant), this scheme has two main problems. First, the tolls are a function of the socially optimal solution that may not be available *a priori*. Secondly, there might be other Wardrop equilibriums associated with this equilibrium that are inefficient. Hence, it would be advantageous if we could devise an online method for setting the tolls adaptively so as to recover a socially optimal flow; however, we would like the scheme to result in piecewise-constant taxes over relatively large periods of time (see Fig. 1). This way, we can guarantee that the drivers have enough time to (re)calculate their preferred route and make an informed decision.

3. Repeated routing game

Here, we assume that the routing game is played repeatedly on each day $n \in \mathbb{N}$ for an infinite horizon. On each day, the vehicles select their preferred path, which generates the flows $f[n] = (f_p[n])_{p\in\mathcal{P}}$. Then, they observe the cost associated with each path, i.e., the actual travel cost and the imposed tolls, and use this information, accompanied with their (finite) memory, to select their path on the subsequent day(s). Here, we consider two widely-known update rules, namely, multiplicative update rule and discretized replicator dynamics.

The multiplicative update rule is a no-regret learning strategy (see (Kleinberg et al., 2009) for more information on no-regret strategies), which is very common in repeated games (Littlestone and Warmuth, 1994; Freund and Schapire, 1999). In this strategy, the agents select their actions with a probability distribution inversely proportional to the exponential of the average cost that they have observed so far. Considering that there is a continuum of players in a repeated routing game, this amounts to dividing the players into various paths with portions inversely proportional to the exponential of the average cost of the paths. This results in Algorithm 1. Steps 4–5 in Algorithm 1 show that, for each source–destination pair, the total flow is split between various paths according to the accumulated costs (the weighted sum of the latencies and the congestion charges over the horizon). The division is so that less cars commute in roads that have high costs (as drivers tend to avoid congested roads or expensive ones).

Algorithm 1. Multiplicative update rule.

Require: $\{\epsilon[n]\}_{n\in\mathbb{N}}$ and $\rho_1, \rho_2 \in \mathbb{R}_{>0}$
1: Initialize $w_p[1] = 1, \forall p \in \mathcal{P}_k, \forall k \in [\![K]\!]$
2: Initialize $\tilde{\tau}_e[n'] = 0, \forall e \in \mathcal{E}, \forall n' \in [\![D]\!]$
3: **for** $n = 1, 2, \ldots$ **do**
4: Calculate the flows

$$f_p[n] = F_k \frac{w_p[n]}{\sum_{p'\in\mathcal{P}_k} w_{p'}[n]}, \forall p \in \mathcal{P}_k, \forall k \in [\![K]\!]$$

5: Update the weights

$$w_p[n+1] = w_p[n] \exp\left(-\frac{\epsilon[n]}{\rho_1+\rho_2} \sum_{e\in p} (\tilde{\ell}_e(f[n]) + \tilde{\tau}_e[n]) \right)$$

6: **if** $n \equiv 0 \,(\mathrm{mod}\, D)$ **then**
7: Set the tolls for the next D days

$$\tilde{\tau}_e[n'] = \left[\phi_e \frac{\mathrm{d}\tilde{\ell}_e(\phi_e)}{\mathrm{d}\phi_e} \right]_{\phi_e = \sum_{p\in\mathcal{P}:e\in p} f_p[n]}, \forall e \in \mathcal{E}, \forall n' \in \mathbb{N} : n' - n \in [\![D]\!]$$

8: **end if**
9: **end for**

Remark 1. Notice that, in the multiplicative update rule in Algorithm 1, for each driver to calculate the evolution of the weights $(w_p[n])_{p\in\mathcal{P}}$, she needs to have access the measurements of the costs (not necessarily the actual cost functions) at each iteration over all the available paths. Evidently, the driver can directly measure the cost of the path that she has chosen in that iteration. We assume the drivers can measure the costs of the alternative paths as well. This can be achieved by broadcasting the measurements from a central node (e.g., a traffic forecast unit in a radio station). In support of this assumption, experimental studies show that players sometimes notice and consider the costs and payoffs of the actions that they have not selected (Camerer et al., 2002), even if they are not directly fed the information.[16] An alternative approach could be to assume that drivers only update the average cost of the paths that they have travelled. This may, however, result in asymmetric update rules that are more difficult to analyze and slower to converge.

Through out the rest of the paper, we make the following assumption regarding the parameters of Algorithm 1.

Assumption 3. Parameters $\rho_1, \rho_2 \in \mathbb{R}_{>0}$ are selected so that $\ell_p(f[n]) \leqslant \rho_1$ and $\tau_p[n] \leqslant \rho_2$ for all $p \in \mathcal{P}$ and all $n \in \mathbb{N}$.

Note that finite constants ρ_1 and ρ_2 can always be found when dealing with smooth edge cost functions and upper-bounded flows. In the presented numerical algorithms, we use these constants to scale down the adjustments to the traffic flows, due to the costs of the paths in each iteration, to avoid sudden jerks in the flows. In the reminder of this section, we prove that the multiplicative update rule in Algorithm 1 converges to a socially optimal flow. To do so, first, we need to prove the following lemma.

Lemma 1. *For Algorithm 1, we have*

$$f_p[n+1] = f_p[n] + \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n] \left[\left(\sum_{p'\in\mathcal{P}_k} \frac{f_{p'}[n]}{F_k} (\ell_{p'}(f[n]) + \tau_{p'}[n]) \right) - (\ell_p(f[n]) + \tau_p[n]) \right] + \mathcal{O}(\epsilon[n]^2).$$

Proof. See Appendix A. □

The replicator equation is a deterministic monotone nonlinear dynamics which is commonly used in the evolutionary game theory (Fischer and Vöcking, 2004). The replicator dynamics is motivated by case where the agents randomly switch their actions (i.e., path selections) if the other agents have a better utility than them. Once discretized, this update rule results in Algorithm 2. Intuitively, Step 4 in Algorithm 2 reduces the flow of cars on paths that have a higher cost than the average cost (of all the paths that connect a source–destination pair) and increases the flow on paths that have a lower cost than the average.

[16] In Gallistel (1990), the author presents an experiments with pigeons in which the animals account for the forgone payoffs (i.e., the payoff of the options that they have not tried) if the information is provided. Moreover, they use "probability matching" to account for these forgone payoffs if the information is missing.

Following Lemma 1, we can easily see that discretized replicator dynamics is very similar to the multiplicative update dynamics (i.e., the discretized replicator dynamics is the linearized version of the multiplicative update rule).

Algorithm 2. Discretized replicator dynamics.

Require: $\{\epsilon[n]\}_{n\in\mathbb{N}}$ and $\rho_1,\ \rho_2 \in \mathbb{R}_{>0}$
1: Initialize $f_p[1] = F_k/|\mathcal{P}_k|,\ \forall p \in \mathcal{P}_k,\ \forall k \in [\![K]\!]$
2: Initialize $\tilde{\tau}_e[n'] = 0,\ \forall e \in \mathcal{E},\ \forall n' \in [\![D]\!]$
3: **for** $n = 1, 2, \ldots$ **do**
4: Update the flows as

$$f_p[n+1] = f_p[n] + \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\sum_{e\in p'}(\tilde{\ell}_e(f[n]) + \tilde{\tau}_e[n])\right) - \sum_{e\in p}(\tilde{\ell}_e(f[n]) + \tilde{\tau}_e[n])\right]$$

5: **if** $n \equiv 0 \,(\mathrm{mod}\, D)$ **then**
6: Set the tolls for the next D days

$$\tilde{\tau}_e[n'] = \left[\phi_e \frac{\mathrm{d}\tilde{\ell}_e(\phi_e)}{\mathrm{d}\phi_e}\right]_{\phi_e=\sum_{p\in\mathcal{P}:e\in p} f_p[n]},\ \forall e \in \mathcal{E}, \forall n' \in \mathbb{N} : n' - n \in [\![D]\!]$$

7: **end if**
8: **end for**

To present the rest of the results, let us, for each $p \in \mathcal{P}$, define the mapping

$$\eta_p : \mathbb{R}^{|\mathcal{P}|}_{\geqslant 0} \to \mathbb{R}$$
$$f \mapsto \sum_{e\in p}\left[\phi_e \frac{\mathrm{d}\tilde{\ell}_e(\phi_e)}{\mathrm{d}\phi_e}\right]_{\phi_e=\sum_{p'\in\mathcal{P}:e\in p'} f_{p'}}.$$

Evidently, the imposed piecewise-constant tolls in Algorithms 1 and 2 can now be calculated as $\tau_p[n] = \eta_p(f[n - D_n])$, where $D_n = n - D\lfloor n/D \rfloor$.

Lemma 2. *Let us select the step size sequence* $\{\epsilon[n]\}_{n\in\mathbb{N}}$ *either as*

- $\epsilon[n] = \alpha/(n+\beta)$ *for some* $\alpha,\ \beta \in \mathbb{R}_{>0}$,
- $\epsilon[n] = \epsilon \in \mathbb{R}_{>0}$,

for all $n \in \mathbb{N}$. *Then, for both Algorithms 1 and 2, we have*

$$f_p[n+1] = f_p[n] + \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\ell_{p'}(f[n]) + \eta_{p'}(f[n])\right)\right) - \left(\ell_p(f[n]) + \eta_p(f[n])\right)\right] + \mathcal{O}(\epsilon[n]^2).$$

Proof. See Appendix B. □

This lemma shows that the time-varying delay for setting the tolls is not important so long as it is bounded (i.e., the delay does not change the update rule significantly). Before stating the main result of this paper, we prove the following simple lemma.

Lemma 3. *For all* $p \in \mathcal{P}$, *the mappings* $\ell_p(\cdot)$ *and* $\eta_p(\cdot)$ *are Lipschitz continuous over* $\mathcal{F}$.

Proof. We only present the proof for $\ell_p(\cdot)$ as the proof for $\eta_p(\cdot)$ follows the same logic. First note that, for any $f, f' \in \mathcal{F}$, we may define the mapping $\pi : \mathbb{R} \to \mathbb{R}$ as $\pi(\omega) = \ell_p(\omega f + (1-\omega) f')$ for all $\omega \in \mathbb{R}$. Mean value theorem (Rudin, 1976) shows that there exists $\tilde{\omega} \in [0, 1]$ such that

$$\pi(1) - \pi(0) = \frac{\mathrm{d}}{\mathrm{d}\omega}\pi(\omega)\Big|_{\omega=\tilde{\omega}},$$

and as a result

$$\ell_p(f) - \ell_p(f') = \pi(1) - \pi(0) = \frac{\mathrm{d}}{\mathrm{d}\omega}\pi(\omega)\Big|_{\omega=\tilde{\omega}} = \left[\frac{\partial \ell_p(\tilde{f})}{\partial \tilde{f}}\right]^\top (f - f'), \quad (2)$$

where $\tilde{f} = \tilde{\omega} f + (1-\tilde{\omega}) f' \in \mathcal{F}$ (since $\mathcal{F}$ is a convex set and $\tilde{\omega} \in [0,1]$). Therefore, using (2) and the Cauchy–Schwarz inequality (Rudin, 1976), we get

$$|\ell_p(f') - \ell_p(f)| \leqslant \left| \left[\frac{\partial \ell_p(\tilde{f})}{\partial \tilde{f}}\right]^\top (f - f') \right| \leqslant \left\| \frac{\partial \ell_p(\tilde{f})}{\partial \tilde{f}} \right\|_2 \|f - f'\|_2 \leqslant \left[\max_{\bar{f} \in \mathcal{F}} \left\| \frac{\partial \ell_p(\bar{f})}{\partial \bar{f}} \right\|_2 \right] \|f - f'\|_2,$$

where the Lipschitz constant is bounded because $\mathcal{F}$ is compact and $\ell_p(\cdot)$ is continuously differentiable (see Assumption 2). □

Now, we are ready to present the main result of the paper.

Theorem 4. *Let us select $\epsilon[n] = \alpha/(n+\beta)$ for some $\alpha, \beta \in \mathbb{R}_{>0}$ and for all $n \in \mathbb{N}$, and define $\mathcal{S} = \arg\min_{f \in \mathcal{F}} C(f)$. Then, for both Algorithms 1 and 2, we get*

$$\lim_{n \to \infty} \mathrm{dist}(\mathcal{S}, (f_p[n])_{p \in \mathcal{P}}) = 0.$$

Proof. Let us define the sequence $\{t[n]\}_{n \in \mathbb{N}}$ such that $t[0] = 0$ and $t[n+1] - t[n] = \epsilon[n]$ for all $n \in \mathbb{N}$. For all $t \in \mathbb{R}_{\geqslant 0}$, we may define

$$\bar{f}_p(t) = f_p[n] + (f_p[n+1] - f_p[n]) \frac{t - t[n]}{t[n+1] - t[n]}.$$

This is a first-order interpolation of the discrete-time flow sequence $(f_p[n])_{n \in \mathbb{N}}$ for all $p \in \mathcal{P}$. Moreover, for all $t \in \mathbb{R}_{\geqslant \tau}$, we may define $\tilde{f}_p^\tau(t)$ as the unique solution of the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{f}_p^\tau(t) = \frac{1}{\rho_1 + \rho_2} \tilde{f}_p^\tau(t) \left[\left(\sum_{p' \in \mathcal{P}_k} \frac{\tilde{f}_{p'}^\tau(t)}{F_k} \left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t)) \right) \right) - \left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t)) \right) \right], \ \tilde{f}^\tau(\tau) = \bar{f}(\tau), \tag{3}$$

where $\tilde{f}^\tau(t) = \left(\tilde{f}_p^\tau(t)\right)_{p \in \mathcal{P}}$ and $\bar{f}(t) = (\bar{f}_p(t))_{p \in \mathcal{P}}$. Note that Lemma 3 implies that the mapping on the right-hand side of (3) is Lipschitz continuous. Now, combining the results of Lemma 2 in this paper and Lemma 1 in Borkar (2008, Ch. 2, p. 12), specifically, from the third extension introduced in Section 2.2 of Borkar (2008, Ch. 2, p. 17), we can see that

$$\lim_{\tau \to \infty} \lim_{t \in [\tau, \tau + T]} \|\tilde{f}^\tau(t) - \bar{f}(t)\| = 0, \forall T \in \mathbb{R}_{>0}.$$

Evidently, the set of socially optimal solutions $\mathcal{S}$ is an invariant set of the ordinary differential equations in (3). This holds because if $\tilde{f}_p^\tau(t) \in \mathcal{S}$, we get

$$\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t)) = \sum_{p' \in \mathcal{P}_k} \frac{\tilde{f}_{p'}^\tau(t)}{F_k} \left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t)) \right), \quad \forall p \in \mathcal{P}.$$

To show the next step, first, we should prove that, for any $k \in [\![K]\!]$, we have $\tilde{f}_p^\tau(t)/F_k \geqslant 0$ for each $p \in \mathcal{P}_k$ and $\sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(t)/F_k = 1$. The first property that $\tilde{f}_p^\tau(t)/F_k \geqslant 0$ follows directly from the ordinary differential equation in (3). For the second property note that by definition of the initial point, we have $\sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(\tau)/F_k = 1$; see Algorithms 1 and 2 in conjunction with the definition of the interpolation for constructing $\bar{f}(t)$. Now, we can easily see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_{p \in \mathcal{P}_k} \frac{1}{F_k} \tilde{f}_p^\tau(t) \right] = \sum_{p \in \mathcal{P}_k} \frac{1}{F_k} \frac{\mathrm{d}}{\mathrm{d}t} \tilde{f}_p^\tau(t) = 0, \forall t \in \mathbb{R}_{\geqslant \tau},$$

and, as a result, $\sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(t) \Big/ F_k = \sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(\tau) \Big/ F_k = 1$ for all $t \in \mathbb{R}_{\geqslant \tau}$. Using this property of flows, we can prove that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} C(\tilde{f}_p^\tau(t)) &= \sum_{p \in \mathcal{P}} \left. \frac{\partial C(f)}{\partial f_p} \right|_{f = (\tilde{f}_p^\tau(t))_{p \in \mathcal{P}}} \frac{\mathrm{d}}{\mathrm{d}t} \tilde{f}_p^\tau(t) \\
&= \frac{1}{\rho_1 + \rho_2} \sum_{k \in [\![K]\!]} \sum_{p \in \mathcal{P}_k} \left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t)) \right) \tilde{f}_p^\tau(t) \\
&\quad \times \left[\left(\sum_{p' \in \mathcal{P}_k} \frac{\tilde{f}_{p'}^\tau(t)}{F_k} \left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t)) \right) \right) - \left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t)) \right) \right] \\
&= \frac{1}{\rho_1 + \rho_2} \sum_{k \in [\![K]\!]} F_k \left[\left(\sum_{p' \in \mathcal{P}_k} \frac{\tilde{f}_{p'}^\tau(t)}{F_k} \left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t)) \right) \right)^2 - \sum_{p \in \mathcal{P}} \frac{\tilde{f}_p^\tau(t)}{F_k} \left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t)) \right)^2 \right] \leqslant 0,
\end{aligned} \tag{4}$$

where the last inequality follows from Jensen's inequality (when using the fact that the mapping $x \mapsto x^2$ is a convex function). Because the mapping $x \mapsto x^2$ is strictly convex, the equality in (4) holds if and only if $\ell_{p''}(\tilde{f}^\tau(t)) + \eta_{p''}(\tilde{f}^\tau(t)) = \ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t))$ for any two $p', p'' \in \mathcal{P}$ such that $\tilde{f}^\tau_{p'}(t), \tilde{f}^\tau_{p''}(t) \neq 0$. This is the definition of $\mathcal{S}$. Therefore, the equality in (4) holds if and only if $\tilde{f}^\tau(t) \in \mathcal{S}$. By defining the Lyapunov function in Corollary 3 in Borkar (2008, Ch. 2, p. 15) as $V(f) = C(f) - \min_{f \in \mathcal{F}} C(f)$, we can see that $\{(f_p[n])_{p\in\mathcal{P}}\}_{n\in\mathbb{N}}$ converges to an internally chain transitive invariant set contained in $\mathcal{S}$. This concludes the proof. □

Unfortunately, the shrinking step sizes in Theorem 4 renders the algorithms impractical for the cases where the parameters of the routing game (e.g., the demands over the source–destination nodes) are time varying since the algorithm cannot adapt itself fast enough (especially, after many steps because the step size is very small). This observation motivates using a constant step size, however, the price for such a selection is that we can only converge to a neighborhood of the socially optimal flow.

Theorem 5. *Let us select $\epsilon[n] = \epsilon \in \mathbb{R}_{>0}$ for all $n \in \mathbb{N}$ and define $\mathcal{S} = \arg\min_{f\in\mathcal{F}} C(f)$. Then, for both Algorithms 1 and 2, we get*

$$\lim_{n\to\infty} \mathrm{dist}(\mathcal{S}, (f_p[n])_{p\in\mathcal{P}}) = \mathcal{O}(\epsilon).$$

Proof. The proof follows the same line of reasoning as in the proof of Theorem 4; however, it builds upon using Lemma 1 and Theorem 3 of Borkar (2008, Ch. 9, pp. 103–114). □

As long as the step size ϵ is large enough, the algorithm can adapt rapidly to the changes in the parameters of the routing game; however, the solution can potentially be far from the socially optimal flow. By reducing the step size, we can achieve a better solution (in terms of the social cost function) but the algorithm, in such case, would respond slower to the changes.

4. Numerical results

In this section, we illustrate the results on two numerical examples. The first example is based on a nonlinear version of Pigou's routing game. The second example presents a general transportation network with Bureau of Public Roads's edge latency functions.

4.1. Nonlinear Pigou's example

In this subsection, we study a nonlinear variant of Pigou's famous example in routing game literature (Roughgarden, 2007). Let us consider the transportation network portrayed by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Fig. 3 for some $\delta \in \mathbb{R}_{>1}$. This transportation network can be seen as an example of a network of two parallel highways, where one highway (corresponding to e_2) is older with fewer lanes, however, the other highway (corresponding to e_1) has several lanes (and, hence, always in free flow mode). Let us assume that we want to route a total of $F_1 = 1000$ vehicle/h for the only source–destination pair $(s_1, t_1) = (0, 1)$. This example is of special interest in the routing game literature as it can result in an arbitrarily large price of anarchy.

Proposition 1 (*The Price of Anarchy of the Nonlinear Pigou's Example*). *We have*

$$\frac{C(f^{\mathrm{Wardrop}})}{\min_{f\in\mathcal{F}} C(f)} = \frac{1}{1 + \left(\frac{1}{\delta+1}\right)^{(\delta+1)/\delta} - \left(\frac{1}{\delta+1}\right)^{1/\delta}} = \mathcal{O}(\delta/\ln(\delta)).$$

Proof. The unique Wardrop equilibrium is certainly equal to $\phi_{e_1} = 0$ and $\phi_{e_2} = 1000$, which results in $C(f^{\mathrm{Wardrop}}) = 1000$. Now, we can calculate the socially optimal flow by solving the optimization problem

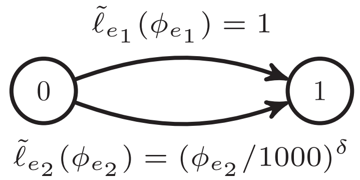

Fig. 3. A nonlinear variant of Pigou's example.

$$\min_{\phi_{e_1},\phi_{e_2}\in\mathbb{R}_{\geqslant 0}} \quad \phi_{e_1}\tilde{\ell}_{e_1}(\phi_{e_1}) + \phi_{e_2}\tilde{\ell}_{e_2}(\phi_{e_2}),$$
$$\phi_{e_1} + \phi_{e_2} = 1000,$$

which is equivalent to solving

$$\min_{0\leqslant\phi_{e_2}\leqslant 1} \quad \frac{1}{1000^{\delta}}\phi_{e_2}^{\delta+1} + 1000 - \phi_{e_2}.$$

Therefore, we can see that the socially optimal flow is equal to $\phi_{e_2} = 1000(1/(\delta+1))^{1/\delta}$ and $\phi_{e_1} = 1000-1000(1/(\delta+1))^{1/\delta}$ and, hence,

$$C\left(f^{\text{Optimal}}\right) = \frac{1}{1000^{\delta}}\left(1000\left(\frac{1}{\delta+1}\right)^{1/\delta}\right)^{\delta+1} + 1000 - 1000\left(\frac{1}{\delta+1}\right)^{1/\delta} = 1000\left(1+\left(\frac{1}{\delta+1}\right)^{(\delta+1)/\delta} - \left(\frac{1}{\delta+1}\right)^{1/\delta}\right).$$

This concludes the proof. □

4.1.1. Convergence

Let us set $\delta = 10$ and $D = 30$. Here, we use Algorithm 1 with step size $\epsilon[n] = 10/n$ for all $n \in \mathbb{N}$ and $\rho_1 = \rho_2 = 10$. We also initialize the algorithm at flows $f[0] = f[1] = 500$ vehicle/h. Fig. 4(left) and (right), respectively, show the flow of vehicles and the congestion taxes versus the iterations of multiplicative update rule in Algorithm 1. As we expect, the congestion taxes stay constant over windows of thirty iterations. To show the convergence of the algorithm, Fig. 5 illustrates the social cost

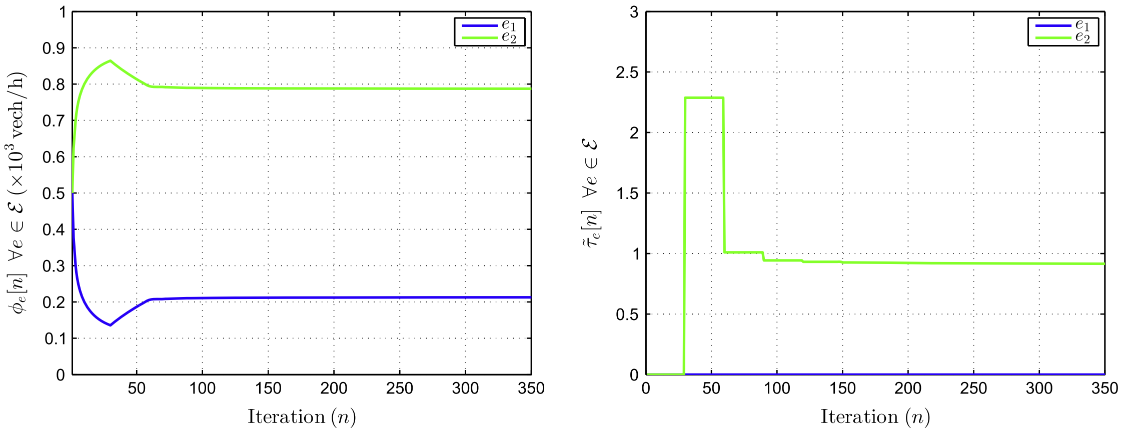

Fig. 4. The flows of vehicles (left) and the congestion taxes (right) over both roads for the Pigou's example.

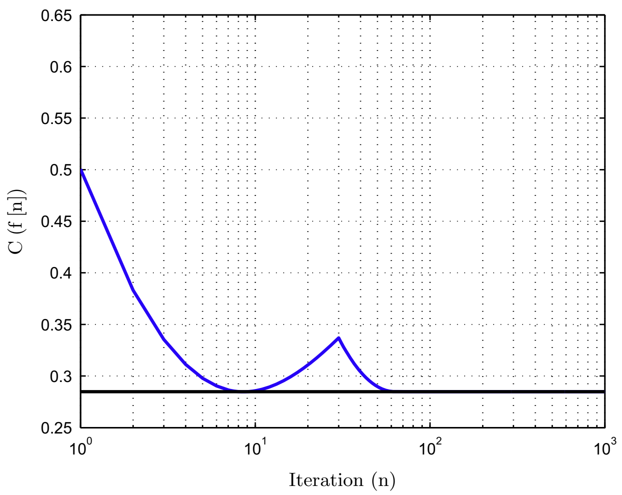

Fig. 5. The the social cost function of the flows extracted using the multiplicative update rule versus the iteration number for the Pigou's example.

function of the flows extracted using the multiplicative update rule as function of the iteration numbers. Clearly, the social cost function of the generated flows converges rapidly to that of socially optimal flow.

4.2. Bureau of public roads's edge latency functions

Let us consider the transportation network portrayed by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Fig. 2. Similar to Example 1, we assume that there are $K = 3$ source–destination nodes $(s_1, t_1) = (0, 1), (s_2, t_2) = (7, 3)$, and $(s_3, t_3) = (0, 8)$. We adopt the edge cost function introduced in Example 2 to model the delay on each edge. In this cost function, for all $e \in \mathcal{E}$, we set the speed limit as $v_e^{\max} = 70$ km/h and set the capacity of the road $c_e = 2000$ vehicle/h (as recommended in Roess and McShane (1987) for single lane roads). Moreover, the length of each road $d_e,\ e \in \mathcal{E}$, is presented in Table 1. In the reminder of this section, we use Algorithm 1.

4.2.1. Fixed demand

First, we consider the case where the total flows that need to pass through source–destination nodes are constant and equal to $F_1 = 8000$ vehicle/h, $F_2 = 3000$ vehicle/h, and $F_3 = 4000$ vehicle/h. We set $D = 30$, which means that the congestion taxes get updated monthly. Finally, let us use vanishing step sizes $\epsilon[n] = 1/n$ for all $n \in \mathbb{N}$.

Table 1
Length of roads in the transportation network employed for the numerical example in Section 4.2.

e	$(3,8)$	$(0,1)$	$(0,4)$	$(5,1)$	$(4,5)$	$(6,1)$	$(5,3)$	$(4,6)$	$(6,3)$	$(2,4)$	$(2,3)$	$(7,2)$	$(1,3)$
d_e (km)	40.81	17.22	17.68	52.77	55.15	29.16	12.19	45.14	26.16	25.13	46.47	22.1	25.22

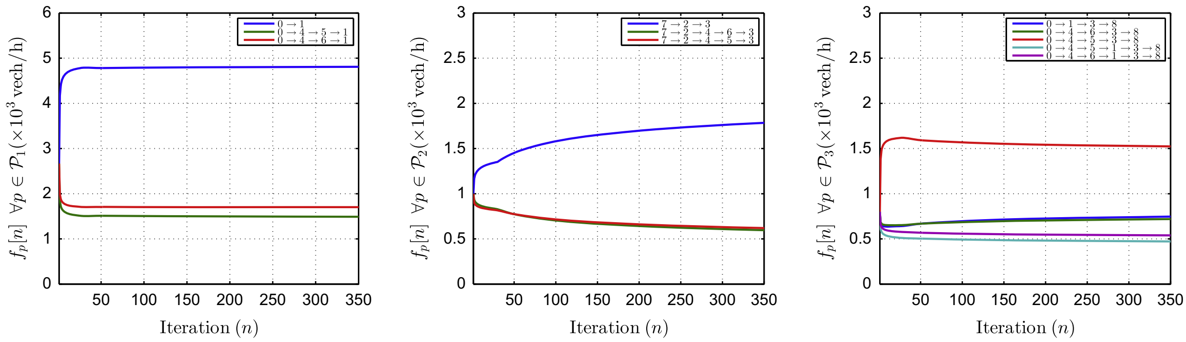

Fig. 6. The flows of vehicles over various paths for all source–destination pairs for the numerical example with the Bureau of Public Roads's edge latency functions.

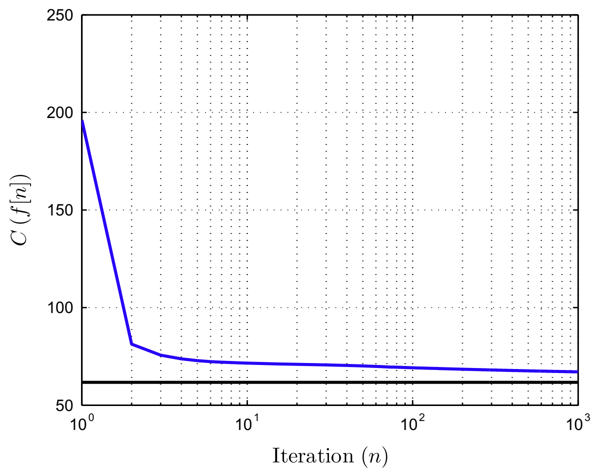

Fig. 7. The social cost function of flows $f[n]$ versus the iteration number n for the numerical example with the Bureau of Public Roads's edge latency functions.

Fig. 6 shows the flows of various paths for all the source–destination pairs. For $k = 1$, the flows settle very rapidly; however, for $k = 2, 3$, the flows adapt much slower. Fig. 7 illustrates the social cost of the flows extracted from Algorithm 1 as a function of the iteration numbers (solid blue curve) as well as the cost of the socially optimal flow (solid black curve). As we expect, the social cost of the extracted flows approaches the cost of the socially optimal flow.

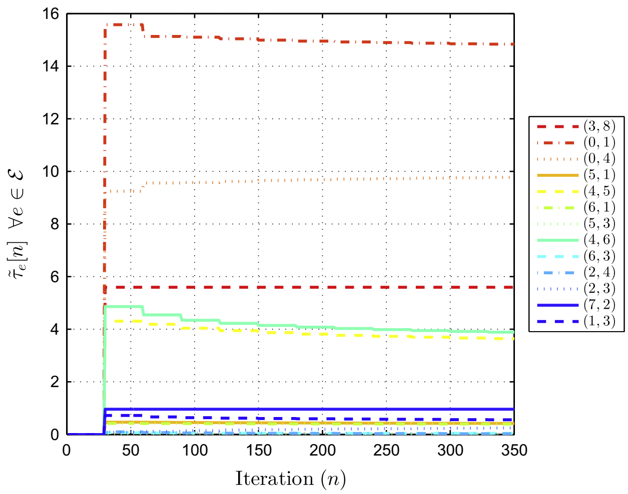

Fig. 8. The congestion taxes over various edges of the transportation network versus the iteration number for the numerical example with the Bureau of Public Roads's edge latency functions.

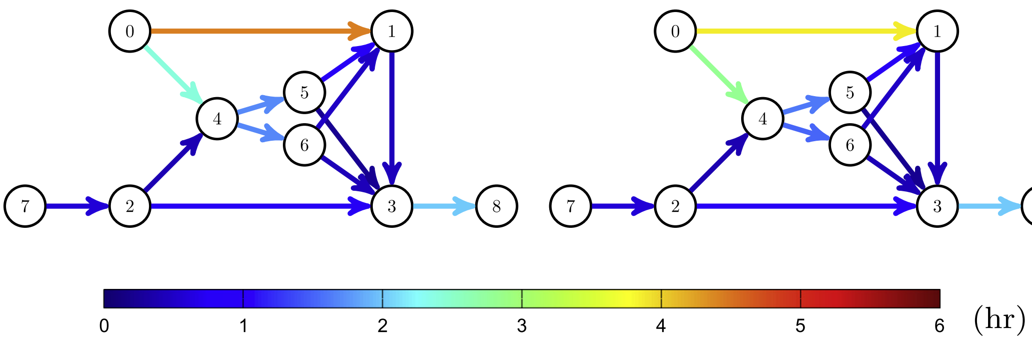

Fig. 9. The delays over the roads in the transportation network at the Wardrop equilibrium of the routing game in the absence of congestion taxes (left) and in the presence of congestion taxes (right).

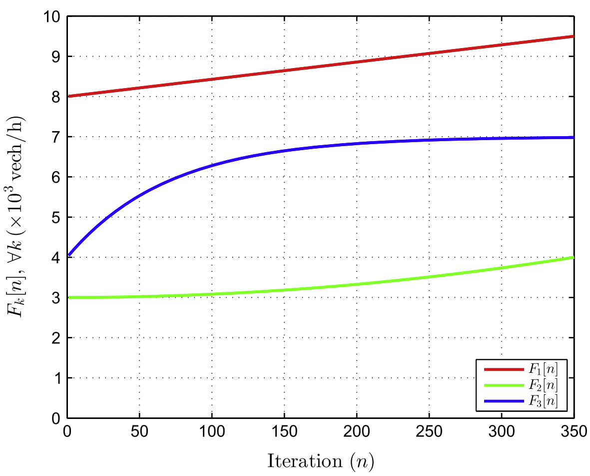

Fig. 10. The total flows for various source–destination nodes as a function of time for the numerical example with time-varying demand.

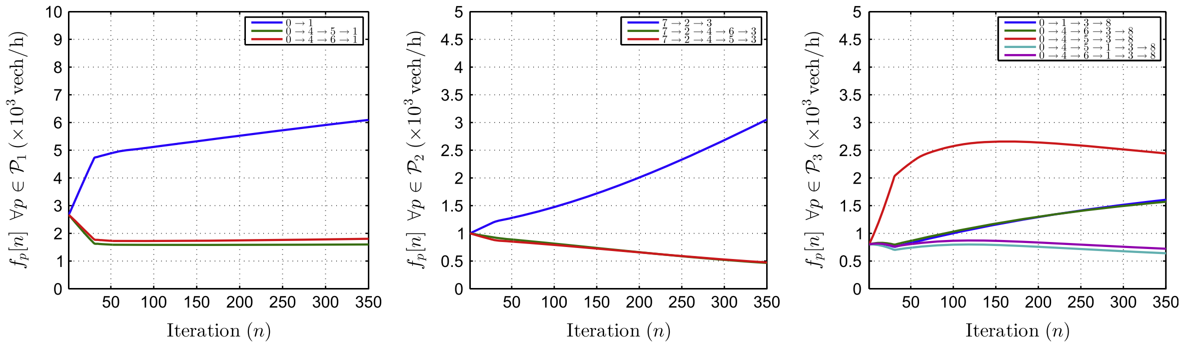

Fig. 11. The flows of vehicles over various paths for all source–destination pairs for the numerical example with time-varying demand.

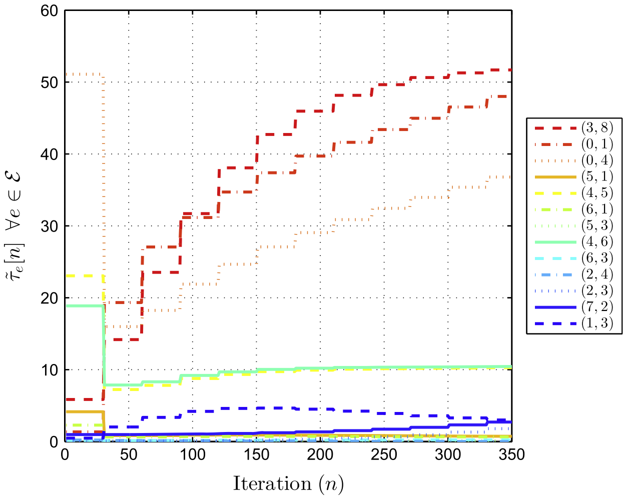

Fig. 12. The congestion taxes for the numerical example with time-varying demand.

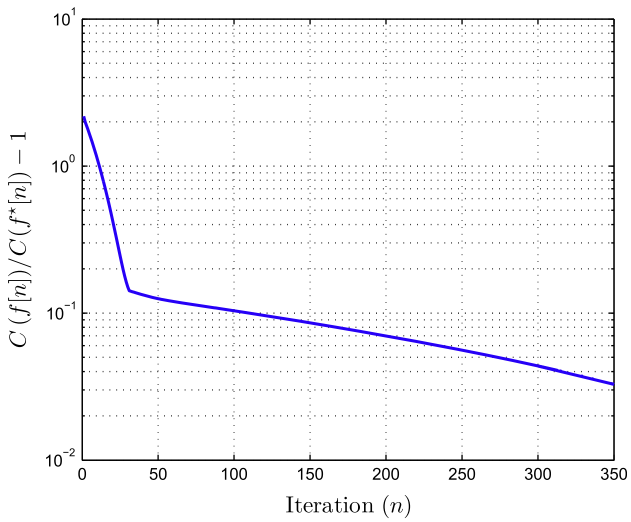

Fig. 13. A measure of the efficiency of the extracted flows $f[n]$ versus the iteration number n for the numerical example with time-varying demand.

Fig. 8 illustrates the congestion charges for various edges in the transportation network $\tilde{\tau}_e[n], e \in \mathcal{E}$, versus the iteration number n. As we expect, the drivers on highly congested roads, e.g., $(0,1)$, should pay much more to be persuaded to use less-congested alternatives (that are perhaps longer or less convenient for them).

Fig. 9(left) portrays the delays over the roads in the transportation network at the Wardrop equilibrium of the routing game *in the absence* of congestion taxes. In contrast, Fig. 9(right) illustrates the delays over the roads at the Wardrop equilibrium of the routing game *in the presence* of congestion taxes. As we expect, with imposing taxes, a portion of the flow (i.e., some of the vehicles) switch from highly congested roads, e.g., $(0,1) \in \mathcal{E}$, to slightly less congested roads, e.g., $(0,4) \in \mathcal{E}$, at the expense of taking a longer path which is now desirable because of the high level of congestion taxes over the shorter path. This behaviour improves the social cost function by 4.6% in such a simple system.

4.2.2. Time-varying demand

Here, let us consider the case where the total flows for various source–destination nodes vary with time as in Fig. 10. In this case, we use Algorithm 1 with a constant step size $\epsilon[n] = 5 \times 10^{-2}$ for all $n \in \mathbb{N}$. Fig. 11 shows the flow of vehicles over various paths for all the source–destination pairs. Furthermore, Fig. 12 illustrates the congestion taxes versus the iterations of the algorithm. Clearly, the algorithm updates these congestion taxes in response to the changes in the demand.

Now, allow us to define $f^*[n]$ to be the socially optimal flow for demands $(F_k[n])_{k=1}^3$ in Fig. 10. Fig. 13 illustrates $C(f[k])/C(f^*[n]) - 1$ as a function of the iteration numbers. Evidently, the smaller $C(f[k])/C(f^*[n]) - 1$ is, the closer the social cost of the generated flow is to the cost of the socially optimal flow. This figure clearly show that the algorithm closely follows the socially optimal flow.

5. Conclusions

We introduced repeated routing game using both multiplicative update rule and discretized replicator dynamics with vanishing and constant step-size. We devised a rule to construct piecewise-constant congestion taxes to guarantee the convergence of the flows to a socially optimal solution. Using vanishing step sizes, we proved the convergence to the set of socially optimal flows; however, using constant step sizes, we could only prove the convergence to a neighbourhood of the socially optimal flows. Future research can focus on devising piecewise-constant congestion charges policies for only a subset of the edges in the transportation network. We can also focus on the multi-class traffic to understand the influence of the drivers' value-of-time.

Appendix A. Proof of Lemma 1

Let us take a closer look at the update rule of the path flow $f_p[n], p \in \mathcal{P}$, as function of time

$$\begin{aligned} f_p[n+1] &= \frac{w_p[n+1]F_k}{\sum_{p' \in \mathcal{P}_k} w_{p'}[n+1]} = \frac{w_p[n]F_k \exp(-\epsilon[n](\ell_p(f[n]) + \tau_p[n])/(\rho_1+\rho_2))}{\sum_{p' \in \mathcal{P}_k} w_{p'}[n] \exp(-\epsilon[n](\ell_{p'}(f[n]) + \tau_{p'}[n])/(\rho_1+\rho_2))} \\ &= F_k \frac{w_p[n] \exp(-\epsilon[n](\ell_p(f[n]) + \tau_p[n])/(\rho_1+\rho_2))}{\sum_{p' \in \mathcal{P}_k} w_{p'}[n] \exp(-\epsilon[n](\ell_{p'}(f[n]) + \tau_{p'}[n])/(\rho_1+\rho_2))} \frac{F_k \Big/ \sum_{p'' \in \mathcal{P}_k} w_{p''}[n]}{F_k \Big/ \sum_{p'' \in \mathcal{P}_k} w_{p''}[n]} \\ &= F_k \frac{f_p[n] \exp(-\epsilon[n](\ell_p(f[n]) + \tau_p[n])/(\rho_1+\rho_2))}{\sum_{p' \in \mathcal{P}_k} f_{p'}[n] \exp(-\epsilon[n](\ell_{p'}(f[n]) + \tau_{p'}[n])/(\rho_1+\rho_2))}, \end{aligned}$$

where $\tau_p[n] = \sum_{e \in p} \tilde{\tau}_e[n]$ for all $p \in \mathcal{P}$. For each $p \in \mathcal{P}$, we define the function $G_p^n : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ such that

$$G_p^n(\epsilon) = F_k \frac{f_p[n] \exp(-\epsilon(\ell_p(f[n]) + \tau_p[n])/(\rho_1+\rho_2))}{\sum_{p' \in \mathcal{P}_k} f_{p'}[n] \exp(-\epsilon(\ell_{p'}(f[n]) + \tau_{p'}[n])/(\rho_1+\rho_2))}, \forall \epsilon \in \mathbb{R}_{\geqslant 0}.$$

Now, noting that $G_p^n \in \mathcal{C}^\omega$, we can use the Taylor's theorem (see Rudin (1976, p. 110)) to get

$$G_p^n(\epsilon) = G_p^n(0) + \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G_p^n(\varepsilon)\bigg|_{\varepsilon=0} \epsilon + \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} G_p^n(\varepsilon)\bigg|_{\varepsilon=\epsilon'} \epsilon^2 = f_p[n] + \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G_p^n(\varepsilon)\bigg|_{\varepsilon=0} \epsilon + \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} G_p^n(\varepsilon)\bigg|_{\varepsilon=\epsilon'} \epsilon^2,$$

for some $\epsilon' \in [0, \epsilon]$. Hence, we need to calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon} G_p^n(\epsilon) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(F_k \frac{f_p[n] \exp(-\epsilon(\ell_p(f[n]) + \tau_p[n])/(\rho_1+\rho_2))}{\sum_{p' \in \mathcal{P}_k} f_{p'}[n] \exp(-\epsilon(\ell_{p'}(f[n]) + \tau_{p'}[n])/(\rho_1+\rho_2))} \right) \\ &= F_k \frac{g_p^n(\epsilon)}{\left(\sum_{p' \in \mathcal{P}_k} f_{p'}[n] \exp(-\epsilon[\ell_{p'}(f[n]) + \tau_{p'}[n]]/(\rho_1+\rho_2))\right)^2}, \end{aligned}$$

where

$$g_p^n(\epsilon) = f_p[n]\exp\left(-\epsilon\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)\left(-\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)\left[\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right] - f_p[n]\exp\left(-\epsilon\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)\left[\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\left(-\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right].$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}G_p^n(\epsilon)\bigg|_{\epsilon=0} = F_k\frac{g_p^n(0)}{\left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\right)^2} = \frac{1}{\rho_1+\rho_2}f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\ell_{p'}(f[n])+\tau_{p'}[n]\right)\right) - \left(\ell_p(f[n])+\tau_p[n]\right)\right].$$

Furthermore, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}G_p^n(\epsilon) = F_k\frac{\xi_p^n(\epsilon)}{\left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp(-\epsilon[\ell_{p'}(f[n])+\tau_{p'}[n]]/(\rho_1+\rho_2))\right)^3},$$

where

$$\xi_p^n(\epsilon) = \left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}g_p^n(\epsilon)\right)\left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right) - 2g_p^n(\epsilon)\left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\left(-\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right).$$

Notice that

$$\begin{aligned}\left|\xi_p^n(\epsilon)\right| &\leqslant \left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}g_p^n(\epsilon)\right|\left|\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right| \\ &\quad + 2\left|g_p^n(\epsilon)\right|\left|\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\left(-\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right| \\ &\leqslant \left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}g_p^n(\epsilon)\right|F_k + 2\left|g_p^n(\epsilon)\right|F_k,\end{aligned} \tag{A.1}$$

where the second inequality follows from

$$\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2} \leqslant 1, \exp\left(-\epsilon\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right) \leqslant 1.$$

To simplify this expression, we can note that

$$\begin{aligned}|g_p^n(\epsilon)| &\leqslant f_p[n]\exp\left(-\epsilon\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)\left(\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)\left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right) \\ &\quad + f_p[n]\exp\left(-\epsilon\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)\left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\left(\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right) \\ &\leqslant 2f_p[n]\sum_{p'\in\mathcal{P}_k} f_{p'}[n] \leqslant 2F_k^2.\end{aligned} \tag{A.2}$$

We can also calculate

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}g_p^n(\epsilon) = f_p[n]\exp\left(-\epsilon\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)\left(-\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right)^2 \times \left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\right) - f_p[n]\exp\left(-\epsilon\frac{\ell_p(f[n])+\tau_p[n]}{\rho_1+\rho_2}\right) \times \left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\left(-\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)^2\right),$$

which gives

$$\left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}g_p^n(\epsilon)\right| \leqslant 2F_k^2. \tag{A.3}$$

Finally, using Jensen's inequality (because $\sum_{p'\in\mathcal{P}_k} f_{p'}[n]/F_k = 1$ and $f_{p'}[n]/F_k \geqslant 0$ for each $p' \in \mathcal{P}_k$), we get

$$\begin{aligned}\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right) &= F_k\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\exp\left(-\epsilon\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\\ &\geqslant F_k\exp\left(-\epsilon\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\frac{\ell_{p'}(f[n])+\tau_{p'}[n]}{\rho_1+\rho_2}\right)\geqslant F_k\exp(-\epsilon). \end{aligned}\tag{A.4}$$

Hence, we

$$\begin{aligned}\left|\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2}G_p^n(\epsilon)\right| &= F_k\left|\frac{\xi_p^n(\epsilon)}{\left(\sum_{p'\in\mathcal{P}_k} f_{p'}[n]\exp(-\epsilon[\ell_{p'}(f[n])+\tau_{p'}[n]]/(\rho_1+\rho_2))\right)^3}\right|\\ &\leqslant \frac{|\xi_p^n(\epsilon)|}{F_k^2\exp(-3\epsilon)}\quad \text{by(A.4)}\\ &\leqslant \frac{1}{F_k^2\exp(-3\epsilon)}\left(\left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}g_p^n(\epsilon)\right|F_k+2\left|g_p^n(\epsilon)\right|F_k\right)\quad \text{by(A.1)}\\ &\leqslant 6F_k\exp(3\epsilon)\quad \text{by(A.2)and(A.3)}\\ &\leqslant 6F_k\exp\left(3\sup_{n\in\mathbb{N}}\epsilon[n]\right).\end{aligned}$$

Therefore, we get

$$f_p[n+1] = f_p[n]+\frac{\epsilon[n]}{\rho_1+\rho_2}f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}(\ell_{p'}(f[n])+\tau_{p'}[n])\right)-(\ell_p(f[n])+\tau_p[n])\right]+\mathcal{O}(\epsilon[n]^2).$$

Appendix B. Proof of Lemma 2

Notice that

$$\eta_p(f[n+1]) = \eta_p\left((f_{p'}[n])_{p'\in\mathcal{P}}+\epsilon[n](\Delta f_{p'}[n])_{p'\in\mathcal{P}}\right),$$

where, using Lemma 1, we have

$$\Delta f_{p'}[n] = \frac{1}{\rho_1+\rho_2}f_{p'}[n]\left[\left(\sum_{p''\in\mathcal{P}_k}\frac{f_{p''}[n]}{F_k}(\ell_{p''}(f[n])+\tau_{p''}[n])\right)-(\ell_{p'}(f[n])+\tau_{p'}[n])\right]+\kappa_{p'}[n]\epsilon[n]$$

with $|\kappa_{p'}[n]| \leqslant 6\sup_{k\in[\![K]\!]}F_k\exp(3\sup_{n\in\mathbb{N}}\epsilon[n]) = \varrho\in\mathbb{R}_{\geqslant 0}$. Let us define a mapping $h_p:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$, such that

$$h_p(\epsilon) = \eta_p\left((f_{p'}[n])_{p'\in\mathcal{P}}+\epsilon(\Delta f_{p'}[n])_{p'\in\mathcal{P}}\right).$$

Again, using Taylor's theorem (see Rudin (1976, p. 110)), we get

$$h_p(\epsilon) = h_p(0)+\frac{\mathrm{d}}{\mathrm{d}\varepsilon}h_p(\varepsilon)\Big|_{\varepsilon=\epsilon'}\epsilon,$$

for some $\epsilon'\in[0,\epsilon]$. Note that

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}h_p(\epsilon)\right| &= \left|\sum_{p'\in\mathcal{P}}\Delta f_{p'}[n]\frac{\partial\eta_p(f)}{\partial f_{p'}}\Big|_{f=(f_{p'}[n])_{p'\in\mathcal{P}}+\epsilon(\Delta f_{p'}[n])_{p'\in\mathcal{P}}}\right|\\ &\leqslant \sum_{p'\in\mathcal{P}}\left|\Delta f_{p'}[n]\frac{\partial\eta_p(f)}{\partial f_{p'}}\Big|_{f=(f_{p'}[n])_{p'\in\mathcal{P}}+\epsilon(\Delta f_{p'}[n])_{p'\in\mathcal{P}}}\right|\leqslant\sum_{p'\in\mathcal{P}}|\Delta f_{p'}[n]|\times\left|\frac{\partial\eta_p(f)}{\partial f_{p'}}\Big|_{f=(f_{p'}[n])_{p'\in\mathcal{P}}+\epsilon(\Delta f_{p'}[n])_{p'\in\mathcal{P}}}\right|\\ &= \sum_{p'\in\mathcal{P}}(2F_k+\varrho\epsilon[n])\varpi\leqslant|\mathcal{P}|(2F_k+\varrho\epsilon[n])\varpi,\end{aligned}$$

where $\varpi = \sup_{p,p'\in\mathcal{P}}\sup_{f\in\mathcal{F}}\left|\partial\eta_p(f)/\partial f_{p'}\right|$. Evidently, $\varpi<\infty$ because of Assumption 2 (i) and the fact that $\mathcal{F}$ is compact set. Therefore, we get

$$\eta_p(f[n+1]) = \eta_p(f[n])+\xi_p[n]\epsilon[n],\tag{B.1}$$

where $|\xi_p[n]| \leqslant |\mathcal{P}|(2\sup_{k\in[\![K]\!]} F_k + \varrho\epsilon[n])\varpi$. This shows that

$$\begin{aligned}
|\eta_p(f[n]) - \eta_p(f[n-D_n])| &= \left|\sum_{t=0}^{D_n-1} \eta_p(f[n-t]) - \eta_p(f[n-t-1])\right| \\
&\leqslant \sum_{t=0}^{D_n-1}\left|\eta_p(f[n-t]) - \eta_p(f[n-t-1])\right| \\
&\leqslant \sum_{t=0}^{D-1}\left|\eta_p(f[n-t]) - \eta_p(f[n-t-1])\right| \quad \text{because } D_n \leqslant D, \forall n\in\mathbb{N} \\
&= \sum_{t=0}^{D-1}|\xi_p[n-t-1]|\epsilon[n-t-1] \quad \text{by} \\
&\leqslant \zeta\sum_{t=0}^{D-1}\epsilon[n-t-1],
\end{aligned} \tag{B.1}$$

where $\zeta = |\mathcal{P}|(2\sup_{k\in[\![K]\!]} F_k + \varrho\sup_{n\in\mathbb{N}}\epsilon[n])\varpi$. Using Lemma 1, we have

$$\begin{aligned}
f_p[n+1] =& f_p[n] + \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\ell_{p'}(f[n]) + \eta_{p'}(f[n])\right)\right) - \left(\ell_p(f[n]) + \eta_p(f[n])\right)\right] + \mathcal{O}(\epsilon[n]^2) \\
&+ \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\ell_{p'}(f[n]) + \eta_{p'}(f[n-D_n])\right)\right) - \left(\ell_p(f[n]) + \eta_p(f[n-D_n])\right)\right] \\
&- \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\ell_{p'}(f[n]) + \eta_{p'}(f[n])\right)\right) - \left(\ell_p(f[n]) + \eta_p(f[n])\right)\right] \\
=& f_p[n] + \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\ell_{p'}(f[n]) + \eta_{p'}(f[n])\right)\right) - \left(\ell_p(f[n]) + \eta_p(f[n])\right)\right] + \mathcal{O}(\epsilon[n]^2) \\
&+ \frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\eta_{p'}(f[n-D_n]) - \eta_{p'}(f[n])\right)\right) - \left(\eta_p(f[n-D_n]) - \eta_p(f[n])\right)\right].
\end{aligned}$$

Now, notice that

$$\begin{aligned}
&\left|\frac{\epsilon[n]}{\rho_1+\rho_2} f_p[n]\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left(\eta_{p'}(f[n-D_n]) - \eta_{p'}(f[n])\right)\right) - \left(\eta_p(f[n-D_n]) - \eta_p(f[n])\right)\right]\right| \\
&\leqslant \frac{\epsilon[n]}{\rho_1+\rho_2}\max_{k\in[\![K]\!]} F_k\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{f_{p'}[n]}{F_k}\left|\eta_{p'}(f[n-D_n]) - \eta_{p'}(f[n])\right|\right) + \left|\eta_p(f[n-D_n]) - \eta_p(f[n])\right|\right] \\
&\leqslant \left[\frac{2\zeta}{\rho_1+\rho_2}\max_{k\in[\![K]\!]} F_k\right]\epsilon[n]\sum_{t=0}^{D-1}\epsilon[n-t-1] = \mathcal{O}(\epsilon[n]^2),
\end{aligned}$$

where the last equality holds if either $\epsilon[n] = \alpha/(n+\beta)$ for $\alpha, \beta \in \mathbb{R}_{>0}$ or $\epsilon[n] = \varepsilon \in \mathbb{R}_{\geqslant 0}$.

References

Alós-Ferrer, C., Netzer, N., 2010. The logit-response dynamics. Games and Economic Behavior 68 (2), 413–427.

Arora, S., Hazan, E., Kale, S., 2012. The multiplicative weights update method: a meta-algorithm and applications. Theory of Computing 8 (1), 121–164.

Aumann, R.J., 1976. Agreeing to disagree. The Annals of Statistics, 1236–1239.

Aumann, R.J., Shapley, L.S., 1994. Long-term competition—a game-theoretic analysis. In: Megiddo, N. (Ed.), Essays in Game Theory. Springer, New York, pp. 1–15.

Axelrod, R., 1984. The Evolution of Cooperation. Basic Books, New York.

Beckmann, M., McGuire, C., Winsten, C., 1956. Studies in the Economics of Transportation. Yale University Press.

Beja, A., 1992. Imperfect equilibrium. Games and Economic Behavior 4 (1), 18–36.

Blum, A., Monsour, Y., 2007. Learning, regret minimization, and equilibria. In: Nisan, N., Roughgarden, T., Tardos, É., Vazirani, V.V. (Eds.), Algorithmic Game Theory. Cambridge University Press.

Blum, A., Even-Dar, E., Ligett, K., 2010. Routing without regret: on convergence to nash equilibria of regret-minimizing algorithms in routing games. Theory of Computing 6 (1), 179–199.

Börjesson, M., Eliasson, J., Hugosson, M.B., Brundell-Freij, K., 2012. The Stockholm congestion charges—5 years on. Effects, acceptability and lessons learnt. Transport Policy 20, 1–12.

Borkar, V.S., 2008. Stochastic Approximation: A Dynamical Systems Viewpoint. Cambridge University Press.

Braess, D., Koch, G., 1979. On the existence of equilibria in asymmetrical multiclass-user transportation networks. Transportation Science 13 (1), 56–63.

Brown, G.W., 1951. Iterative solution of games by fictitious play. In: Koopmans, T.C. (Ed.), Activity Analysis of Production and Allocation, vol. 13. John Wiley & Sons, Inc., New York, pp. 374–376.

Buchanan, J.M., Stubblebine, W.C., 1962. Externality. Economica 29 (116), 371–384.

Camerer, C.F., Ho, T.-H., Chong, J.-K., 2002. Sophisticated experience-weighted attraction learning and strategic teaching in repeated games. Journal of Economic Theory 104 (1), 137–188.
Cominetti, R., Melo, E., Sorin, S., 2010. A payoff-based learning procedure and its application to traffic games. Games and Economic Behavior 70 (1), 71–83.
Correa, J.R., Stier-Moses, N.E., 2010. Wardrop equilibria. In: Cochran, J.J. (Ed.), Wiley Encyclopedia of Operations Research and Management Science. John Wiley & Sons, Inc..
Dafermos, S.C., 1972. The traffic assignment problem for multiclass-user transportation networks. Transportation Science 6 (1), 73–87.
Dafermos, S.C., 1973. Toll patterns for multiclass-user transportation networks. Transportation Science 7 (3), 211–223.
Dubey, P., 1986. Inefficiency of nash equilibria. Mathematics of Operations Research 11 (1), 1–8.
Eliasson, J., Mattsson, L.-G., 2006. Equity effects of congestion pricing: quantitative methodology and a case study for Stockholm. Transportation Research Part A 40 (7), 602–620.
Eliasson, J., Hultkrantz, L., Nerhagen, L., Rosqvist, L.S., 2009. The stockholm congestion–charging trial 2006: overview of effects. Transportation Research Part A 43 (3), 240–250.
Ellison, G., 1993. Learning, local interaction, and coordination. Econometrica 61 (5), 1047–1071.
Engelson, L., Lindberg, P., (2006). Congestion pricing of road networks with users having different time values. In: Lawphongpanich, S., Hearn, D., Smith, M. (Eds.), Mathematical and Computational Models for Congestion Charging, vol. 101 of Applied Optimization, 2006, pp. 81–104.
Erev, I., Roth, A.E., 1998. Predicting how people play games: reinforcement learning in experimental games with unique, mixed strategy equilibria. American Economic Review, 848–881.
Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y., 1995. Reasoning About Knowledge, vol. 4. MIT Press, Cambridge, MA.
Farokhi, F., Johansson, K.H., 2015. Using piecewise-constant congestion taxing policy in repeated routing games. In: Proceedings of the SIAM Conference on Control Theory and its Applications, in press, http://www.siam.org/meetings/ct15/.
Federal Highway Adminstration, U.S. Department of Transportation, VPP projects involving tolls: Priced lanes: High-occupancy toll (HOT) lanes. <http://ops.fhwa.dot.gov/tolling_pricing/value_pricing/projects/involving_tolls/priced_lanes/hot_lanes/ca_hotlanes_i15sd.htm>.
Federal Highway Adminstration, U.S. Department of Transportation, I-15 congestion pricing project monitoring and evaluation services: Task 13: Phase II year three overall report. <http://ops.fhwa.dot.gov/tolling_pricing/value_pricing/pubs_reports/projectreports/interst15_congestion.htm>.
Fischer, S., Vöcking, F., 2004. On the evolution of selfish routing. In: Albers, S., Radzik, T. (Eds.), Algorithms-ESA 2004, Lecture Notes in Computer Science, vol. 3221. Springer, Berlin Heidelberg, pp. 323–334.
Freund, Y., Schapire, R.E., 1999. Adaptive game playing using multiplicative weights. Games and Economic Behavior 29 (1), 79–103.
Frick, K.T., Heminger, S., Dittmar, H., 1996. Bay bridge congestion-pricing project: lessons learned to date. Transportation Research Record: Journal of the Transportation Research Board 1558 (1), 29–38.
Friedman, J.W., 1971. A non-cooperative equilibrium for supergames. The Review of Economic Studies 38 (1), 1–12.
Friedman, D., 1991. Evolutionary games in economics. Econometrica: Journal of the Econometric Society, 637–666.
Friesz, T.L., Bernstein, D., Mehta, N.J., Tobin, R.L., Ganjalizadeh, S., 1994. Day-to-day dynamic network disequilibria and idealized traveler information systems. Operations Research 42 (6), 1120–1136.
Gallistel, C.R., 1990. The Organization of Learning. The MIT Press, Cambridge, MA.
Guo, R.-Y., Yang, H., Huang, H.-J., Tan, Z., 2015. Link-based day-to-day network traffic dynamics and equilibria. Transportation Research Part B 71, 248–260.
Hahn, F.H., 1962. The stability of the cournot oligopoly solution. The Review of Economic Studies, 329–331.
Hall, R.W., 1983. Traveler route choice: travel time implications of improved information and adaptive decisions. Transportation Research Part A 17 (3), 201–214.
Han, D., Yang, H., 2009. Congestion pricing in the absence of demand functions. Transportation Research Part E 45 (1), 159–171.
Haurie, A., Marcotte, P., 1985. On the relationship between Nash–Cournot and Wardrop equilibria. Networks 15 (3), 295–308.
He, X., Guo, X., Liu, H.X., 2010. A link-based day-to-day traffic assignment model. Transportation Research Part B 44 (4), 597–608.
Hoefer, M., Olbrich, L., Skopalik, A., 2008. Taxing subnetworks. In: Papadimitriou, C., Zhang, S. (Eds.), Internet and Network Economics, Lecture Notes in Computer Science, vol. 5385. Springer, Berlin Heidelberg, pp. 286–294.
Hofbauer, J., Sigmund, K., 2003. Evolutionary game dynamics. Bulletin of the American Mathematical Society 40 (4), 479–519.
Hopkins, E., 1999. A note on best response dynamics. Games and Economic Behavior 29 (1), 138–150.
Kaporis, A.C., Spirakis, P.G., 2006. The price of optimum in Stackelberg games on arbitrary single commodity networks and latency functions. In: Proceedings of the 18th Annual ACM Symposium on Parallelism in Algorithms and Architectures, pp. 19–28.
Karlström, A., Franklin, J.P., 2009. Behavioral adjustments and equity effects of congestion pricing: analysis of morning commutes during the Stockholm Trial. Transportation Research Part A 43 (3), 283–296.
Kleinberg, R., Piliouras, G., Tardos, É., 2009. Multiplicative updates outperform generic no-regret learning in congestion games. In: Proceedings of the Forty-First Annual ACM Symposium on Theory of Computing. ACM, pp. 533–542.
Korilis, Y.A., Lazar, A.A., Orda, A., 1997. Achieving network optima using Stackelberg routing strategies. IEEE/ACM Transactions on Networking 5 (1), 161–173.
Koutsoupias, E., Papadimitriou, C., 1999. Worst-case equilibria. In: Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science, pp. 404–413.
Krichene, W., Reilly, J., Amin, S., Bayen, A., 2012. On the characterization and computation of Nash equilibria on parallel networks with horizontal queues. In: Proceedings of the IEEE 51st Annual Conference on Decision and Control, pp. 7119–7125.
Krichene, W., Drighés, B., Bayen, A.M., 2014. On the convergence of no-regret learning in selfish routing. In: Proceedings of the 31st International Conference on Machine Learning, pp. 163–171.
Leape, J., 2006. The london congestion charge. The Journal of Economic Perspectives 20 (4), 157–176.
Lindsey, C.R., Verhoef, E.T., 2000. Traffic Congestion and Congestion Pricing. Tinbergen Institute Discussion Paper 00-101/3. <http://hdl.handle.net/10419/85524>.
Littlestone, N., Warmuth, M.K., 1994. The weighted majority algorithm. Information and Computation 108 (2), 212–261.
McKelvey, R.D., Palfrey, T.R., 1995. Quantal response equilibria for normal form games. Games and Economic Behavior 10 (1), 6–38.
Mckelvey, R.D., Palfrey, T.R., 1998. Quantal response equilibria for extensive form games. Experimental Economics 1 (1), 9–41.
Mertens, J.-F., 1986. Repeated games. In: Proceedings of the International Congress of Mathematicians, pp. 1528–1577.
Milgrom, P., 1981. An axiomatic characterization of common knowledge. Econometrica: Journal of the Econometric Society, 219–222.
Monderer, D., Shapley, L.S., 1996. Potential games. Games and Economic Behavior 14 (1), 124–143.
Mookherjee, D., Sopher, B., 1997. Learning and decision costs in experimental constant sum games. Games and Economic Behavior 19 (1), 97–132.
Morgan, J., Orzen, H., Sefton, M., 2009. Network architecture and traffic flows: experiments on the Pigou–Knight–Downs and braess paradoxes. Games and Economic Behavior 66 (1), 348–372.
Papadimitriou, C., 2001. Algorithms, games, and the internet. In: Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, pp. 749–753.
Pigou, A.C., 1932. The Economics of Welfare. McMillan & Co., London.
Roess, R.P., McShane, W.R., 1987. Changing concepts of level of service in the 1985 highway capacity manual: some examples. ITE Journal 57 (5), 27–31.
Rosenthal, R.W., 1989. A bounded-rationality approach to the study of noncooperative games. International Journal of Game Theory 18 (3), 273–292.
Roughgarden, T., 2003. The price of anarchy is independent of the network topology. Journal of Computer and System Sciences 67 (2), 341–364.
Roughgarden, T., 2004. Stackelberg scheduling strategies. SIAM Journal on Computing 33 (2), 332–350.
Roughgarden, T., 2007. Routing games. In: Nisan, N., Roughgarden, T., Tardos, É., Vazirani, V.V. (Eds.), Algorithmic Game Theory. Cambridge University Press.
Roughgarden, T., Tardos, É., 2002. How bad is selfish routing? Journal of the ACM 49 (2), 236–259.

Roughgarden, T., Tardos, Éva, 2004. Bounding the inefficiency of equilibria in nonatomic congestion games. Games and Economic Behavior 47 (2), 389–403.
Rubinstein, A., 1994. Equilibrium in supergames. In: Megiddo, N. (Ed.), Essays in Game Theory. Springer, New York, pp. 17–27.
Rudin, W., 1976. Principles of Mathematical Analysis, International Series in Pure and Applied Mathematics, third ed. McGraw-Hill International.
Sandholm, W.H., 2001. Potential games with continuous player sets. Journal of Economic Theory 97 (1), 81–108.
Sandholm, W.H., 2012. Evolutionary game theory. In: Meyers, R.A. (Ed.), Computational Complexity. Springer, New York, pp. 1000–1029.
Santos, G., 2005. Urban congestion charging: a comparison between london and singapore. Transport Reviews 25 (5), 511–534.
Sarin, R., Vahid, F., 2001. Predicting how people play games: a simple dynamic model of choice. Games and Economic Behavior 34 (1), 104–122.
Seade, J., 1980. The stability of cournot revisited. Journal of Economic Theory 23 (1), 15–27.
Singh, R., Dowling, R., 2002. Improved speed-flow relationships: application to transportation planning models. In: Proceedings of the 7th TRB Conference on the Application of Transportation Planning Methods, pp. 340–349.
Smith, M.J., 1979. The existence, uniqueness and stability of traffic equilibria. Transportation Research Part B 13 (4), 295–304.
Smith, M.J., 1983. The existence and calculation of traffic equilibria. Transportation Research Part B 17 (4), 291–303.
Smith, M.J., 1984. The stability of a dynamic model of traffic assignment—an application of a method of lyapunov. Transportation Science 18 (3), 245–252.
Smith, M., Wisten, M., 1995. A continuous day-to-day traffic assignment model and the existence of a continuous dynamic user equilibrium. Annals of Operations Research 60 (1), 59–79.
Vickrey, W.S., 1963. Pricing and resource allocation in transportation and public utilities: pricing in urban and suburban transport. American Economic Review 53 (2), 452–465.
Vickrey, W.S., 1992. Principles of Efficient Congestion Pricing. <http://www.vtpi.org/vickrey.htm>.
Wang, X., Yang, H., 2012. Bisection-based trial-and-error implementation of marginal cost pricing and tradable credit scheme. Transportation Research Part B 46 (9), 1085–1096.
Wardrop, J.G., 1952. Some theoretical aspects of road traffic research. In: Proceedings of the Institute of Civil Engineers: Engineering Divisions, vol. 3, pp. 325–362.
Weibull, J., 1997. Evolutionary Game Theory. MIT Press.
Winslott-Hiselius, L., Brundell-Freij, K., Vagland, Å., Byström, C., 2009. The development of public attitudes towards the Stockholm congestion trial. Transportation Research Part A 43 (3), 269–282.
Yang, H., Huang, H.-J., 1998. Principle of marginal-cost pricing: how does it work in a general road network? Transportation Research Part A 32 (1), 45–54.
Yang, H., Huang, H.-J., 2004. The multi-class, multi-criteria traffic network equilibrium and systems optimum problem. Transportation Research Part B 38 (1), 1–15.
Yang, H., Huang, H.J., 2005. Mathematical and Economic Theory of Road Pricing. Elsevier Science.
Yang, F., Zhang, D., 2009. Day-to-day stationary link flow pattern. Transportation Research Part B 43 (1), 119–126.
Yang, H., Meng, Q., Lee, D.-H., 2004. Trial-and-error implementation of marginal-cost pricing on networks in the absence of demand functions. Transportation Research Part B 38 (6), 477–493.
Yang, H., Zhang, X., Meng, Q., 2007. Stackelberg games and multiple equilibrium behaviors on networks. Transportation Research Part B 41 (8), 841–861.
Yang, H., Xu, W., He, B.-s., Meng, Q., 2010. Road pricing for congestion control with unknown demand and cost functions. Transportation Research Part C 18 (2), 157–175.
Zhang, D., Nagurney, A., 1996. On the local and global stability of a travel route choice adjustment process. Transportation Research Part B 30 (4), 245–262.
Zhao, Y., Kockelman, K.M., 2006. On-line marginal-cost pricing across networks: incorporating heterogeneous users and stochastic equilibria. Transportation Research Part B 40 (5), 424–435.
Zhou, B., Bliemer, M., Yang, H., He, J., 2015. A trial-and-error congestion pricing scheme for networks with elastic demand and link capacity constraints. Transportation Research Part B 72, 77–92.