and

$$B_n \geq \sqrt{n}\left(\frac{\delta_1\gamma_1 S_1(z-1)}{(\delta_1\gamma_1 + zS_1)M}\right)^{1/2}$$

is obtained. Moreover, when $1 < \frac{n}{M(1+\frac{zS_1}{\delta_1\gamma_1})}$ it is seen that

$$n+1 < n\left(1 + \frac{1}{M\left(1+\frac{zS_1}{\delta_1\gamma_1}\right)}\right) = n\left(1 + \frac{\delta_1\gamma_1}{M(\delta_1\gamma_1 + zS_1)}\right).$$

Finally, we reach the following:

$$\sup_x |P(s_n < x) - \phi(x)| \leq \frac{A_\delta \sum_{k=0}^{n} E|Y_{nk}|^{2+\delta}}{B_n^{2+\delta}} \leq \frac{A_\delta (n+1) S_0}{B_n^{2+\delta}} \leq \frac{A_\delta n\left(1+\frac{\delta_1\gamma_1}{M(\delta_1\gamma_1+zS_1)}\right)S_0}{n^{1+\frac{\delta}{2}}\left(\frac{\delta_1\gamma_1 S_1(z-1)}{(\delta_1\gamma_1+zS_1)M}\right)^{1+\frac{\delta}{2}}} = cn^{-\delta/2}.$$

The proof of the case $n < n_0$ is trivial.

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Decentralized Control of Sequentially Minimum Phase Systems

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***Abstract*—Fundamental limitations in decentralized control of systems with multivariable zeros are considered. It is shown that arbitrary bandwidth can be obtained with a stable block-diagonal controller, if certain subsystems of the open-loop system fail to have zeros in the right half-plane and a high-frequency condition holds. Implications on control structure design and sequential loop-closuring methods are discussed.**

***Index Terms*—Decentralized control, multivariable zeros, performance limitations, sensitivity minimization.**

I. INTRODUCTION

Industry faces a huge number of interacting control loops. During the last three decades a variety of multivariable control design methods have been developed. Most of these are based on the assumption of a centralized control structure. However, for most industrial plants it is impossible to implement a centralized controller. Start-up schemes, identification experiments, and communication nets are only some issues that are considerably harder to face with centralized controllers than with decentralized controllers. Decentralized control is the absolutely dominating structure in practice.

It is natural to look for fundamental limitations in a control system. In particular, this is a motivation for decentralized systems, because there is a great lack of theoretical results supporting control design methods for these systems. There exist formulas for performance limitations for centralized control systems. Extending results of Bode [1], implications of right half-plane (RHP) poles, and zeros on achievable closed-loop performance for these systems are shown in [2]–[6]. For example, it is proved that for multivariable systems with no RHP zeros, the sensitivity function can be made arbitrarily small with a centralized controller.

Our main contribution is to connect multivariable zeros to closed-loop performance for decentralized systems. Performance is measured through a weighted sensitivity function [2], [7]. *Sequentially minimum phase* is introduced as when the top left submatrices of the open-loop system are minimum phase. It is then shown that if an open-loop system is sequentially minimum phase and a condition on the relative degree of the subsystems holds, then the sensitivity can be arbitrarily reduced with a diagonal controller. An earlier sufficient condition for sensitivity reduction via decentralized control was proved in [8]. Their analysis was limited to systems diagonal at high frequencies, but other assumptions were weaker. Results on achievable performance for decentralized systems were also given in [9].

The outline of the paper is as follows. Notation and some preliminary results are given in Section II. In Section III a new condition is presented for sensitivity reduction in systems with no RHP zeros under decentralized control. For systems with RHP zeros an upper bound on the performance loss due to decentralization is shown in Section IV. Results on the connection between sequential control

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design and multivariable zeros are presented in Section V. The concluding remarks in Section VI cover connections to relative gain array (RGA) analysis. Omitted proofs are given in [10], where also some further discussions are found. Part of this work has been presented as [11].

II. Preliminaries

Let the square transfer matrix G represent a system with an equal number of inputs u_j and outputs y_i.[1] The elements of G are denoted G_{ij}, $i, j = 1, \cdots, m$ and can be scalar transfer functions as well as transfer matrices. We only consider proper G with full normal rank [7]. For the top left submatrix of G, the notation

$$G_k := \begin{bmatrix} G_{11} & \cdots & G_{1k} \\ \vdots & & \vdots \\ G_{k1} & \cdots & G_{kk} \end{bmatrix}$$

is used, and the first $k-1$ elements of the last row and column of this matrix are denoted

$$\begin{aligned} L_k &:= [G_{k1} \quad \cdots \quad G_{k,k-1}] \\ R_k &:= [G_{1k} \quad \cdots \quad G_{k-1,k}] \end{aligned} \tag{1}$$

respectively. We consider a block diagonal control law $u = -Cy$, where $C = \operatorname{diag}\{C_1, \cdots, C_m\}$ and C_i is a transfer matrix of dimension one or higher, corresponding to the size of G_{ii}.

Our main result concerns stable systems. Therefore, recall that a stable open-loop system G remains stable after interconnection with feedback controller C, if and only if $C(I+GC)^{-1}$ is stable and the closed-loop system is well-posed, that is, $I + C(\infty)G(\infty)$ is nonsingular [7, p. 119]. The sensitivity function is defined as $S := (I+GC)^{-1}$ and for the subsystems we use the notation $S_k := (I + G_k\overline{C}_k)^{-1}$, where $\overline{C}_k := \operatorname{diag}\{C_1, \cdots, C_k\}$. We only need the simplest definition of a multivariable RHP zero.

Definition 1: An *RHP zero* of a stable transfer matrix G is a point z in the closed RHP for which rank $G(z)$ is smaller than the normal rank of G.

If a transfer matrix does not have any RHP zeros, it is called minimum phase and otherwise nonminimum phase. The norm $\|A\|$ of a matrix A is its largest singular value and for transfer matrices we define $\|G\|_\infty := \sup_{\operatorname{Re} s \geq 0} \|G(s)\|$.

Frequency-weighted sensitivity functions are widely used in practice; for example, loop-shaping is often done based on shaping the sensitivity and complementary sensitivity functions [2], [7]. In control design, the weights are chosen to reflect frequency contents in, for example, disturbances and perturbations. Closed-loop performance limitations have been quantified in terms of weighted sensitivity functions in [5], [6], and [8]. This will also be the framework for our analysis.

Recall the Youla parameterization [12].

Lemma 1: Let G be a stable transfer matrix. All proper stabilizing controllers are given as

$$C = (I - QG)^{-1}Q = Q(I - GQ)^{-1}$$

where Q is a proper stable transfer matrix.

The following lemma is a slight variation of [5, Corollary 6.2].

Lemma 2: Consider a stable transfer matrix G with no RHP zeros and a strictly proper stable transfer function W with no RHP zeros. For every $\varepsilon > 0$ there exists a strictly proper stabilizing and stable (centralized) controller C such that

$$\|W(I+GC)^{-1}\|_\infty < \varepsilon$$

and $\|W^{-1}C\|_\infty$ is bounded.

[1] It is straightforward to show that the main result in this paper holds also for nonsquare systems with suitable modification of the notation.

Proof: Let d be a positive integer such that $[s^d W(s)G(s)]^{-1}$ is proper. Consider

$$\hat{C}(s) = \frac{G^{-1}(s)}{(1+\tau s)^d - 1}$$

where $\tau > 0$ is chosen such that

$$\|W(I+G\hat{C})^{-1}\|_\infty = \left\| W(s)\frac{(1+\tau s)^d - 1}{(1+\tau s)^d} \right\|_\infty < \varepsilon.$$

The closed-loop system has all poles in $-\tau$, and $\hat{C}$ has all poles uniformly distributed on a circle intersecting the origin and $-2/\tau$. In order to get a stable controller let

$$C(s) = \frac{G^{-1}(s)}{(1+\tau s)^d - 1 + \delta}.$$

For $\delta > 0$ sufficiently small, it follows by continuity that the closed-loop system is stable

$$\|W(I+GC)^{-1}\|_\infty = \left\| W(s)\frac{(1+\tau s)^d - 1 + \delta}{(1+\tau s)^d + \delta} \right\|_\infty < \varepsilon$$

and that C has all poles in the open left half-plane. The proof is complete because $W^{-1}C$ is stable and proper. ■

Lemma 2 should be considered together with the lower bound on sensitivity reduction given as in [5, Th. 4], which is restated next.

Proposition 1: Consider a stable transfer matrix G with RHP zeros in z_i, $i = 1, \cdots, \ell$, and a proper stable transfer function W with no RHP zeros. Then for every proper stabilizing controller C

$$\|W(I+GC)^{-1}\|_\infty \geq \max_{i \in \{1, \cdots, \ell\}} |W(z_i)|.$$

Proposition 1 provides a lower bound for decentralized control of systems with RHP zeros. No controller can give a tight feedback if an RHP zero of G is located in a heavily weighted part of the RHP.

III. Sequentially Minimum Phase

This section is devoted to a new theorem on minimization of the sensitivity function under decentralized control. The theorem is proved using sequential control design. It turns out that certain submatrices of G should be minimum phase.

Definition 2: A stable transfer function matrix G is *sequentially minimum phase* if $G_1, \cdots, G_m$ have full normal rank and no RHP zeros.

Under the assumption that G_{k-1}, $k \in \{2, \cdots, m\}$, has no RHP zeros and W is a proper stable transfer function with no RHP zeros, introduce the scalar $\phi_k(W) \in [0, \infty]$ as $\phi_k(W) := \|W^{-1} L_k G_{k-1}^{-1}\|_\infty$, where L_k is given by (1).

Example 1: The transfer matrix

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{(s+2)^2} & \dfrac{1}{(s+1)^2} \end{bmatrix}$$

is sequentially minimum phase because $G_1(s) = (s+1)^{-1}$ and $G_2(s) = G(s)$ have no RHP zeros. Furthermore, $\phi_2(W)$ is bounded for all weighting functions of relative degree less than two because

$$\phi_2(W) = \|W^{-1}G_{21}G_{11}^{-1}\|_\infty = \left\| W^{-1}(s)\frac{s+1}{(s+2)^2} \right\|_\infty < \infty.$$

A symmetric definition of $\phi_k(W)$ including R_k instead of L_k arises in a natural way, if the input sensitivity function $S_i = (I+CG)^{-1}$ is studied instead of the output sensitivity function $S_o = (I+GC)^{-1}$; see [2] and [7] for interpretations of S_i and S_o. Next we state our main result.

Theorem 1: Consider a stable transfer matrix G and a strictly proper stable transfer function W with no RHP zeros. If G is sequentially minimum phase and $\phi_k(W)$ is bounded for $k = 2, \cdots, m$, then for every $\varepsilon > 0$ there exists a strictly proper stabilizing and stable controller $C = \mathrm{diag}\{C_1, \cdots, C_m\}$ such that

$$\|W(I + GC)^{-1}\|_\infty < \varepsilon.$$

Proof: See the Appendix. ■

Remark 1: A similar statement for systems being diagonal at high frequencies is proved in [8]. Then there are no requirements on the zeros of $G_1, \cdots, G_{m-1}$ or on $\phi_k(W)$. The system in Example 1 satisfies the assumptions of Theorem 1 but is not ultimately diagonally dominant [8]. Decentralized two-by-two controllers that minimize $\|S_1(i\omega)\|$ are considered in [9].

Remark 2: Note that if G_k for $k < m$ has an RHP zero, then after permutation of inputs and outputs (the new) $G_1, \cdots, G_m$ do not necessarily have any RHP zeros. An obvious algorithm for control structure design can be derived, where the inputs and outputs are permuted until a suitable sequence $G_1, \cdots, G_m$ is found. During the search, the structure of the controller may change in the sense that the dimensions of $C_1, \cdots, C_m$ may vary, and thus the number of blocks m. A centralized controller corresponds to $m = 1$, in which case Theorem 1 corresponds to Lemma 2 in Section II and [5, Corollary 6.2].

Remark 3: The necessary conditions for sensitivity reduction are important. The reason why in our case the condition on $\phi_k(W)$ enters the analysis is the approach of sequential design. For example, it is obvious that we cannot do a sequential design by first closing the $u_1 - y_1$ loop for the system

$$G = \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix}.$$

The scalar $\phi_2(W)$ is not bounded for G.

If a system fulfills the assumptions in Theorem 1, theoretically a decentralized controller can give arbitrarily tight control. In practice, however, the region in which the model is accurate gives the performance limitations. Hence, fulfilled assumptions imply that effort should be put into investigations of nonlinearities, such as actuator limitations and unmodeled high-frequency dynamics.

IV. Right Half-Plane Zeros

It is well-known that RHP zeros impose restrictions on the achievable closed-loop performance. Proposition 1 in Section II gave an interpretation of these restrictions in achievable sensitivity reduction. This section presents a result on how close to the estimate for centralized control systems in Proposition 1 we can get with a decentralized design.

Consider a partially closed system having the first $k - 1$ loops closed and the last $m - k + 1$ loops open. Let the controller be $\overline{C}_{k-1} = \mathrm{diag}\{C_1, \cdots, C_{k-1}\}$ and suppose it stabilizes G_{k-1}. Introduce $H_k = H_k(C_1, \cdots, C_{k-1})$ as the transfer matrix between u_k and y_k for this partially closed system. We define $H_1 := G_{11}$ and for $k = 2, \cdots, m$ it follows that

$$H_k = G_{kk} - L_k \overline{C}_{k-1} S_{k-1} R_k^T. \tag{2}$$

Note that $\overline{C}_{k-1} S_{k-1}$ is stable because the partially closed system is stable, and thus H_k is stable if G is stable. It is easy to show that if G_{k-1} is nonsingular, then $H_k = G_{kk} - L_k G_{k-1}^{-1}(I - S_{k-1})R_k^T$ for $k = 2, \cdots, m$. We also use the notation

$$\hat{H}_k := G_{kk} - L_k G_{k-1}^{-1} R_k^T. \tag{3}$$

Note that $\hat{H}_k$ is not necessarily proper and that $\hat{H}_k$ does not depend on the controller C.

Next we combine Proposition 1 with the idea of Theorem 1 to state a result that gives an upper bound on the minimal weighted sensitivity for a decentralized control system with open-loop RHP zeros.

Theorem 2: Consider a stable transfer matrix G and a strictly proper stable transfer function W with no RHP zeros. If G_{m-1} is sequentially minimum phase, $\phi_k(W)$ is bounded for $k = 2, \cdots, m$, and C_m is strictly proper and stabilizes $\hat{H}_m$ with $\|W^{-1}C_m\|_\infty$ bounded, then for every $\delta > 0$ there exists a strictly proper stabilizing controller $C = \mathrm{diag}\{C_1, \cdots, C_m\}$ such that

$$\|W(I + GC)^{-1}\|_\infty < \|W(I + \hat{H}_m C_m)^{-1}\|_\infty \times (1 + \phi_m(W)\|W\|_\infty) + \delta.$$

Proof: The proof is similar to the proof of Theorem 1; see [10]. ■

Remark 4: Lemma 3 in Section V implies that $\hat{H}_m$ has the same RHP zeros as G. The limitations imposed by $\hat{H}_m$ are in this sense similar to the limitations faced at a centralized control design for G. Theorem 2 gives a connection between sensitivity reduction using decentralized and centralized control for some open-loop systems that have RHP zeros.

Remark 5: If $L_m = 0$, which for example holds when G is upper triangular, then $\|WS\|_\infty < \|W(I + \hat{H}_m C_m)^{-1}\|_\infty + \delta$. Decentralization imposes, of course, no extra limitations on the sensitivity reduction in this case.

V. Zeros and Sequential Loop-Closure

Closing one control loop at a time is for many practical reasons the dominating way of designing control systems in industry. There exist, however, only few systematic design methods based on such a sequential loop-closure [13], [14]. From a theoretical point of view, this kind of approach has several limitations compared to an approach with all loops closed simultaneously. Nevertheless, it is interesting to quantify the fundamental properties of the sequential method. In this section results on the connection between sequential loop-closure design and multivariable zeros are derived.

A key result for sequentially closed loops is the simple fact that if $C_k(I + H_k C_k)^{-1}$, with H_k defined in (2) and G stable, is stable for all $k = 1, \cdots, m$, then $C = \mathrm{diag}\{C_1, \cdots, C_m\}$ stabilizes G; see [10]. The single condition that $C_k(I + H_k C_k)^{-1}$ is stable does not imply that the whole closed-loop system is stable after k loops closed. The opposite is, of course, true. If the system is stable after k loops closed, then $C_k(I + H_k C_k)^{-1}$ is stable because

$$[0 \quad I]\overline{C}_k(I + G_k \overline{C}_k)^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = C_k(I + H_k C_k)^{-1}.$$

The following result is a slight generalization of [15, Th. 5.2.7].

Lemma 3: Consider a transfer matrix G and let $k \in \{2, \cdots, m\}$. If loops 1 to $k - 1$ are closed such that $S_{k-1}(s_0) = 0$ for some $s_0 \in \mathbb{C}$ and $G_{k-1}(s_0)$ is nonsingular, then

$$\det H_k(s_0) = \frac{\det G_k(s_0)}{\det G_{k-1}(s_0)}.$$

Proof: See [10]. ■

Lemma 3 relates zeros of the subsystem G_k to zeros in loop k. Hence, if all loops but one have tight control, the achievable performance in that loop will be given by the zeros of G. This consequence was exposed in Theorem 2. A result similar to Lemma 3 holds even if we only know that $S_{k-1}(s_0)$ is small. Let $k \in \{2, \cdots, m\}$ and $s_0 \in \mathbb{C}$. If $G_k(s_0)$ is nonsingular and loops 1 to $k-1$ are closed such that $\|S_{k-1}(s_0)\| \cdot \|G_k(s_0)\| \cdot \|G_k^{-1}(s_0)\| < 1$, then

$$\|H_k^{-1}(s_0)\| < \frac{\|G_k^{-1}(s_0)\|}{1 - \|S_{k-1}(s_0)\| \cdot \|G_k(s_0)\| \cdot \|G_k^{-1}(s_0)\|}.$$

See [10] for a proof. Hence, neither G_k loses rank in s_0, nor does H_k, provided that the feedback of the subsystem G_{k-1} is sufficiently tight and G_k is bounded. Note that the assumption $\|S_{k-1}\| \cdot \|G_k\| \cdot \|G_k^{-1}\| < 1$ is equivalent to that of $\|S_{k-1}\| < 1/\kappa(G_k)$, where $\kappa(G_k) := \|G_k\| \cdot \|G_k^{-1}\|$ is the condition number, well-known as a measure of how close a matrix is to singularity. The condition number of the open-loop system $\kappa(G)$ is suggested for plant assessment and for choosing input–output pairing in [16].

VI. Conclusions

New results on performance limitations of decentralized control systems have been presented. *Sequentially minimum phase* was introduced for the case where the top left submatrices of the open-loop system are minimum phase. The main theorem states that for stable systems any bandwidth is achievable with decentralized control, provided that the system is sequentially minimum phase and a condition on the relative degree of the subsystems holds. The zeros of $G_1, \cdots, G_{m-1}$ can be seen as the cost of choosing a certain control structure, and, hence, give suggestions for solutions to the control structure design problem. There exist only a few systematic methods to compare decentralized and centralized control structures. Our result suggests that the zeros of the subsystems of G should be considered. Another recent method is given in [17]. RHP zeros of open-loop subsystems also set constraints for stabilization of unstable plants [18].

The transfer matrices H_k and $\hat{H}_k$ arising in the preceding analysis have connections to the RGA. The RGA was introduced by Bristol [19] and is today a standard tool for interaction analysis in chemical process control [16]. For simplicity, consider a system with two inputs and two outputs. Then the dynamic RGA is represented by the transfer function $\lambda := G_{11}G_{22}/(G_{11}G_{22} - G_{12}G_{21})$. It follows from (3) that $\lambda = G_{22}/\hat{H}_2$. Hence, the RGA can be interpreted as the fraction between G_{22} and H_2 under infinitely tight feedback in loop one. Theorem 1 provides a sufficient condition for applicability of RGA analysis. Note, however, that Proposition 1 suggests that if there exist RHP zeros close to the imaginary axis, the RGA analysis might be less appropriate.

Appendix

Theorem 1 is proved in this Appendix. Notations and results from Sections IV and V as well as the following two lemmas are used in the proof.

Lemma 4: Let $k \in \{2, \cdots, m\}$ and suppose $I + H_kC_k$ is nonsingular. Then

$$S_k = \begin{bmatrix} S_{k-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} S_{k-1}R_k^T C_k \\ -I \end{bmatrix} \times (I + H_kC_k)^{-1}[L_k\overline{C}_{k-1}S_{k-1} \quad -I].$$

Proof: The proof follows from the definition of S_k; see [10]. ∎

Lemma 5: Consider a stable transfer matrix G_k and a strictly proper stable transfer function W with no RHP zeros. Assume G_k is sequentially minimum phase, $\phi_\ell(W)$ is bounded for $\ell = 2, \cdots, k$, and that $\overline{C}_{k-1}$ stabilizes G_{k-1}. Let C_k be given as $C_k = (I - Q\hat{H}_k)^{-1}Q$ with Q proper and stable, $\hat{H}_k$ be defined by (3), and $\|W^{-1}C_k\|_\infty$ be bounded. If $\|WS_{k-1}\|_\infty$ is sufficiently small, then $\overline{C}_k$ stabilizes G_k and

$$\begin{aligned}\|WS_k\|_\infty \le\; & \|WS_{k-1}\|_\infty \\ & + (1 + \|WS_{k-1}\|_\infty \cdot \|G\|_\infty \cdot \|W^{-1}C_k\|_\infty) \\ & \times \|W(I + \hat{H}_kC_k)^{-1}\|_\infty \\ & \times (1 - \phi_k(W)\|WS_{k-1}\|_\infty \cdot \|Q\|_\infty)^{-1} \\ & \times [1 + \phi_k(W)(\|W\|_\infty + \|WS_{k-1}\|_\infty)].\end{aligned}$$

Proof: We start by showing closed-loop stability. Note that $H_k - \hat{H}_k = L_kG_{k-1}^{-1}S_{k-1}R_k^T$ is stable and that

$$\begin{aligned}\|L_kG_{k-1}^{-1}S_{k-1}R_k^T\|_\infty &= \|W^{-1}L_kG_{k-1}^{-1}WS_{k-1}R_k^T\|_\infty \\ &\le \phi_k(W) \cdot \|WS_{k-1}\|_\infty \cdot \|G\|_\infty < \infty.\end{aligned}$$

Because H_k is proper, this gives that $\hat{H}_k$ is proper. Hence, $C_k(I + H_kC_k)^{-1} = (Q^{-1} + H_k - \hat{H}_k)^{-1}$ is stable for all $\|WS_{k-1}\|_\infty$ sufficiently small because $Q = C_k(I + \hat{H}_kC_k)^{-1}$ is stable by the assumptions. This gives closed-loop stability.[2]

From Lemma 4 we have that

$$\begin{aligned}\|WS_k\|_\infty \le \|WS_{k-1}\|_\infty &+ \left(1 + \left\|S_{k-1}R_k^TC_k\right\|_\infty\right) \\ &\times \|W(I + H_kC_k)^{-1}\|_\infty \\ &\times \left(1 + \left\|L_k\overline{C}_{k-1}S_{k-1}\right\|_\infty\right).\end{aligned} \tag{4}$$

Each of the right-hand side expressions of (4) is estimated next. First

$$\|S_{k-1}R_k^TC_k\|_\infty \le \|WS_{k-1}\|_\infty \cdot \|G\|_\infty \cdot \|W^{-1}C_k\|_\infty.$$

Second

$$\begin{aligned}&\|W(I + H_kC_k)^{-1}\|_\infty \\ &\quad = \|W(I - \hat{H}_kQ)(I - (\hat{H}_k - H_k)Q)^{-1}\|_\infty \\ &\quad \le W(I + \hat{H}_kC_k)^{-1}\|_\infty \\ &\quad \times (1 - \|L_kG_{k-1}^{-1}S_{k-1}R_k^TQ\|_\infty)^{-1} \\ &\quad \le \|W(I + \hat{H}_kC_k)^{-1}\|_\infty \\ &\quad \times (1 - \phi_k(W) \cdot \|WS_{k-1}\|_\infty \cdot \|Q\|_\infty)^{-1}\end{aligned}$$

if $\|WS_{k-1}\|_\infty$ is sufficiently small. Finally, for the last expression of (4) we have

$$\begin{aligned}&\|L_k\overline{C}_{k-1}S_{k-1}\|_\infty \\ &\quad \le \|W^{-1}L_kG_{k-1}^{-1}\|_\infty \cdot \|WG_{k-1}\overline{C}_{k-1}S_{k-1}\|_\infty \\ &\quad = \phi_k(W)\|W(I - S_{k-1})\|_\infty \\ &\quad \le \phi_k(W)(\|W\|_\infty + \|WS_{k-1}\|_\infty).\end{aligned}$$

∎

Proof of Theorem 1: We prove by mathematical induction that for every ε_ℓ, $\ell \in \{1, \cdots, m\}$, there exists a strictly proper stabilizing and stable controller $\overline{C}_\ell = \mathrm{diag}\{C_1, \cdots, C_\ell\}$ such that $\|W(I + G_\ell\overline{C}_\ell)^{-1}\|_\infty < \varepsilon_\ell$. Lemma 2 gives that this is true for $\ell = 1$. Suppose it holds also for $\ell = 2, \cdots, k-1$. From the assumptions and Lemma 3 it follows that $\hat{H}_k$ has no RHP zeros. Lemma 2 gives that for every $\delta_k > 0$ there exists a strictly proper and stable C_k such that $C_k(I + \hat{H}_kC_k)^{-1}$ is stable, $\|W^{-1}C_k\|_\infty$ is bounded, and $\|W(I + \hat{H}_kC_k)^{-1}\|_\infty < \delta_k$. Hence, by first choosing $\delta_k > 0$ and then $\varepsilon_{k-1} > 0$ sufficiently small, we obtain from Lemma 5 that for every $\varepsilon_k > 0$ there exists a stabilizing and stable controller $\overline{C}_k$ such that $\|WS_k\|_\infty < \varepsilon_k$. The induction completes the proof.

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[2] A crucial point here and in the remaining part of the proof is that G_{k-1} has no RHP zeros. If G_{k-1} has an RHP zero, then there does not exist any stabilizing controller $\overline{C}_{k-1}$ such that $\|WS_{k-1}\|_\infty$ is arbitrarily small; see Proposition 1.

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Compensation of the RLS Algorithm for Output Nonlinearities

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***Abstract*—It is shown how the recursive least squares (RLS) algorithm can be modified to compensate for *a priori* known errors of linearity in the output measurement. A novel signal model is used for this purpose. Only the nonlinear effects are modeled by an output error model, and much of the output measurements are used directly in the regression vector. The main benefit with this approach is that the advantages of the RLS, like quick initial convergence for infinite impulse response (IIR) models, can be retained for *small* linearity errors. At the same time *the output nonlinearity is allowed to be noninvertible*. This can be important to treat, for example, small deadzones and also to avoid the amplification of additive measurement disturbances. Such amplification can result from inversion of the output nonlinearity. Simulations illustrate the performance of the algorithm.**

***Index Terms*— Least squares, nonlinear systems, recursive identification, Wiener model.**

I. INTRODUCTION

The recursive least squares (RLS) algorithm is a well-understood standard tool in adaptive control, signal processing, and recursive system identification. The advantages include quick initial convergence, availability of fast algorithms for efficient numerical implementation, and well-behaved global convergence properties without local minima for infinite impulse response (IIR) models. All this is well known [1]. There are also disadvantages, as compared to output error methods, when considering the extension to simple nonlinear systems. Since the regression vector contains the output *measurements* of an assumed linear system, it is for example not straightforward to extend the equation error-based RLS algorithm to the case where a nonlinear sensor affects the measured output signal. This is, however, a straightforward task when output error methods are used.

In the general nonlinear case, identification is an extremely difficult and problem-dependent task. There are some tools of general validity though, including Volterra and Wiener series-based methods [2] or methods based on combinations of numerical integration and optimization; see [3] and the references therein. For less complicated nonlinear systems, like the Wiener system consisting of linear dynamics followed by a static nonlinearity, the task of designing identification algorithms is significantly simpler than in the general case. Offline algorithms for Wiener system identification based on Volterra series expansions are available [4]. Recently, subspace identification methodology has been applied to this problem; see [5]. Algorithms for recursive identification and adaptive filtering of Wiener-type systems have also been designed and analyzed for a number of different scenarios [6]–[9]. These online methods are all of output error type since the output signal of the linear dynamic block is not measurable. The signal is therefore generated from

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