Limit Cycles With Chattering in Relay Feedback Systems

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***Abstract*—Relay feedback has a large variety of applications in control engineering. Several interesting phenomena occur in simple relay systems. In this paper, scalar linear systems with relay feedback are analyzed. It is shown that a limit cycle where part of the limit cycle consists of fast relay switchings can occur. This chattering is analyzed in detail and conditions for approximating it by a sliding mode are derived. A result on existence of limit cycles with chattering is given, and it is shown that the limit cycles can have arbitrarily many relay switchings each period. Limit cycles with regular sliding modes are also discussed. Examples illustrate the results.**

***Index Terms*—Discontinuous control, hybrid systems, nonlinear dynamics, oscillations, relay control, sliding modes.**

I. Introduction

RELAYS are common in control systems. They are used both for mode switching and as models for physical phenomena such as mechanical friction. Relay control is the oldest control principle but is still the most applicable. An early reference to on–off control is [1] (as pointed out in [2]), in which Hawkin studied temperature control and noticed that the relay controller caused oscillations. Simple mechanical and electromechanical systems were an early motivation for studying models with relay feedback [3], [4]. Other applications were in aerospace [5], [6]. A self-oscillation adaptive system, which has a relay with adjustable amplitude in the feedback loop, was tested in several American aircrafts in the 1950s [7].

Recently, there has been renewed interest in relay feedback systems due to a variety of new applications. Automatic tuning of proportional-integral-derivative (PID) controllers using relay feedback is based on the observation that if the controller is replaced by a relay, there will often be a stable oscillation in the process output [8]. The frequency and amplitude of this oscillation can be used to determine PID controller parameters similar to the classical approach by Ziegler and Nichols [9]. Another application of relay feedback is also in the design of variable-structure systems [10]. The high-gain of the relay makes it possible to design a control system that is robust to parameter variations and disturbances. Hybrid control systems have both continuous-time and discrete-event dynamics. An interesting class of hybrid systems are switched control systems [11], in which a relay feedback system is the simplest member. Switched controllers have a richer structure than regular smooth controllers and can, therefore, often give better control performance. There exist, however, no unified approach today to design switched controllers. An interesting application of relay feedback is in the design of delta–sigma modulators in signal processing [12], [13]. Delta–sigma modulators have replaced standard analog-to-digital (AD) and digital-to-analog (DA) converters in many applications, because they are often simpler to implement. The basic setup of a delta–sigma modulator is a filter in a feedback loop with a quantizer, which can be modeled as a relay. Modeling of quantization errors in digital control is another motivation to study relay feedback [14].

Limit cycles and sliding modes are two important phenomena that can occur in relay feedback systems. Research on both these issues was very active in the former Soviet Union during the 1950s and 1960s. Major contributions to the work on oscillations can be found in [3] and [4] (see also [15] on the describing function method to analyze these oscillations). While building a mathematical framework for sliding modes, interesting properties of differential equations with discontinuous right-hand sides were found. Uniqueness and existence of solutions and smooth dependency on initial conditions of a solution (all well known to hold for smooth differential equations) could easily be violated by a nonsmooth system. This was a topic for discussion in which, for example, Filippov and Neimark took part in at the first International Federation of Automatic Control (IFAC) congress [16]. A standard reference on the concept of solution to nonsmooth systems is Filippov's monograph [17]. Utkin's definition of sliding modes based on equivalent control is discussed in [10]. A regular (first-order) sliding mode is a part of a trajectory on a surface of dimension $n-1$, where n is the system order. Higher order sliding modes belongs to a surface of lower dimension. These sliding modes have many interesting properties, which, for example, can be exploited for control design [18].

A linear system with relay feedback can show several interesting phenomena. Local analysis of limit cycles are given in [19]. Few results exist on global stability of limit cycles in higher order system, but a recent contribution is given in [20]. The response of a linear system with relay feedback can be complicated. Cook showed that a low-order linear system can have a

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response that is extremely sensitive to the initial condition [21]. It was shown in [22] and [23] that there exist trajectories having arbitrarily fast relay switchings even if an exact sliding mode is not part of the trajectory. A necessary and sufficient condition for this is that the first nonvanishing Markov parameter of the linear part of the system is positive. It was shown by Anosov [24] that the pole excess is important for the stability of the origin in relay feedback systems. Systems with pole excess three or higher are unstable. From a similar discussion, it is possible to conclude that only systems with pole excess two can have a trajectory with multiple fast relay switchings [22].

The main contribution of this paper is to give conditions for existence of a new type of limit cycle. If the linear part of the relay feedback system has pole excess one and certain other conditions are fulfilled, then the system has a limit cycle with sliding mode. Because the sliding mode is exact, it is easy to analyze this system. If the linear system has pole excess two, there exist limit cycles with arbitrarily many relay switchings each period. In this case the map that describes one period of the limit cycle is quite complicated. A simulated example of such a limit cycle was first shown in [22]. The fast relay switchings give rise to (what we call) *chattering* or fast oscillations in the state variables (cf. [18] and [25]). An important step in being able to analyze the limit cycle is to approximate the chattering by a second-order sliding mode. An accurate formula is derived in this paper for how the chattering evolves. It shows that the chattering may be attracted to a second-order sliding mode depending on the system parameters. To study the limit cycle with chattering it is shown to be sufficient to study a second-order sliding mode instead of the complicated map describing the chattering trajectory. The main result of the paper (Theorem 3 in Section IV) gives sufficient conditions for the existence of a limit cycle with chattering. The technique of analyzing chattering by sliding mode approximation is related to averaging in perturbation theory [26], [27].

The paper is organized as follows. Notation is introduced in Section II. Sliding modes and limit cycles are defined. Section III describes the phenomena of fast relay switchings that we call chattering. It is proved that chattering takes place close to a second-order sliding set. An accurate formula for the evolution of the chattering is also derived. By using this result, it is possible in Section IV to prove the main theorem of the paper. It states that there exist limit cycles with chattering. These limit cycles can have arbitrarily many relay switchings each period. An example of a chattering limit cycle is also given. The paper is concluded in Section V.

II. Preliminaries

A. Notation

Consider a linear time-invariant system with relay feedback. The linear system has scalar input u and scalar output y and it is described by the minimal state-space representation

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$. Let $G(s) = b(s)/a(s) = C(sI - A)^{-1}B$ denote the transfer function of the system. The relay feedback is defined by

$$u(t) = -\operatorname{sgn} y(t) \in \begin{cases}\{-1\}, & y(t) > 0\\ [-1, 1], & y(t) = 0\\ \{1\}, & y(t) < 0.\end{cases} \tag{2}$$

Note that the relay does not have hysteresis. The switching plane is denoted $\mathcal{S} = \{x: Cx = 0\}$.

An absolutely continuous function $x: [0, \infty) \to \mathbb{R}^n$ is called a trajectory or a solution of (1) and (2) if it satisfies (1) and (2) almost everywhere. Note that a differential equation with discontinuous right-hand sides may have nonunique trajectories; see [17]. A limit cycle $\mathcal{L} \subset \mathbb{R}^n$ in this paper denotes the set of values attained by a periodic trajectory that is isolated and not an equilibrium [27]. The limit cycle $\mathcal{L}$ is *symmetric* if for every $x \in \mathcal{L}$ it is also true that $-x \in \mathcal{L}$. Let the Euclidean distance from a point x to a limit cycle $\mathcal{L}$ be denoted $d_{\mathcal{L}}(x)$. A limit cycle is then stable if for each $\epsilon > 0$ there exists $\delta > 0$ such that $d_{\mathcal{L}}(x(0)) < \delta$ implies that $d_{\mathcal{L}}\big(x(t)\big) < \epsilon$ for all $t > 0$.

B. Sliding Modes

A *sliding mode* is the part of a trajectory that belongs to the switching plane: $x(t)$ is a sliding mode for $t \in (t_1, t_2)$ with $t_2 > t_1 > 0$, if $Cx(t) = 0$ for all $t \in (t_1, t_2)$. Sliding modes are treated thoroughly in [17]. Let $r \in \{1, \dots, n-1\}$ be the pole excess of $G(s)$, so that $CA^{r-1}B \neq 0$ but $CA^kB = 0$ for $k = 0, \dots, r-2$. Then, the set

$$S_r = \{x: Cx = CAx = \cdots = CA^{r-1}x = 0\}$$

is called the rth-order sliding set (cf. [18]). A sliding mode that belongs to an rth-order sliding set is an rth-order sliding mode. We will in particular study first- and second-order sliding modes and the corresponding sets $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{S}_2$.

There is an important distinction between first- and second-order sliding modes for (1) and (2). If $CB > 0$ then a trajectory with initial condition close to the set $\{x \in \mathcal{S}_1: |CAx| < CB\}$ will have a sliding mode. Such first-order sliding modes may even be part of a limit cycle, as we will see Section II-C. If instead $CB = 0$ and $CAB > 0$, then the set of initial conditions that gives a (second-order) sliding mode is of measure zero. What will happen then is that a trajectory with an initial condition close to $\{x \in \mathcal{S}_2: |CA^2x| < CAB\}$ will wind around the second-order sliding set. This phenomena give rise to a large number of relay switchings and is therefore named chattering. Chattering is described in detail in Section III. In Section IV, it is shown that also chattering can be part of a limit cycle. Existence of fast relay switchings and their connection to limit cycles are discussed in [22]. From the analysis therein, it follows that if the linear part of the relay feedback system has pole excess greater than two, then there exists no limit cycle with a large number of fast relay switchings.

C. Limit Cycles With Sliding Modes

This paper investigates complex chattering limit cycles. In order to present the mechanism generating them, we briefly recall the simple case when the limit cycle of the relay feedback system has an exact (first-order) sliding mode. See [23] for further discussion and proofs.

Consider the relay feedback system (1) and (2) with state-space representation (A, B_1, C), where

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$
$$C = [1 \quad 0 \quad \dots \quad 0].$$

Note that $CB_1 = 1 > 0$. This implies that there is a subset of the first-order sliding set that is attractive in the sense that the set of initial conditions that gives a sliding mode is of positive measure.

There are several ways to derive the sliding mode for a differential equation with discontinuous right-hand side [17]. For linear systems with relay feedback, they all agree. System (1) and (2) with parameterization (A, B_1, C) has sliding set equal to $\mathcal{S} = \{x: Cx = 0\} = \{x: x_1 = 0\}$. The equivalent control [10] is $u_{\rm eq} = -CAx/CB_1 = -x_2$. Applying this to (1) gives the first-order sliding mode as $x_1 = 0$ together with the solution to

$$\dot{w}(t) = F_1 w(t)$$
$$w = [x_2 \quad \dots \quad x_n]^T$$

where

$$F_1 = \begin{bmatrix} -b_1 & 1 & 0 & \dots & 0 \\ -b_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -b_{n-2} & 0 & 0 & & 1 \\ -b_{n-1} & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{3}$$

Hence, we have the well-known fact that the sliding mode is stable in the sense that $w \to 0$ as $t \to \infty$, if all zeros of $G(s) = C(sI - A)^{-1}B_1$ are in the open left-half plane.

Local stability of limit cycles with first-order sliding modes can be straightforwardly analyzed by studying a Poincaré map that consists of two parts: one part corresponding to the trajectory being strictly on one side of $\mathcal{S}$ and one (sliding mode) part corresponding to the trajectory belonging to S. A limit cycle with first-order sliding modes is stable if all eigenvalues of

$$\mathcal{W}_1 = P_1 \left(I - \frac{F_1 \nu e_1^T}{e_1^T F_1 \nu} \right) e^{F_1 t_{\rm sl}} P_2 \left(I - \frac{(Ax^1 - B_1)C}{C(Ax^1 - B_1)} \right) \times e^{At_{\rm ns}} P_3^T \tag{4}$$

are in the open unit disc. The limit cycle is unstable if at least one eigenvalue is outside the unit disc. Here, P_1 denotes the projection $P_1(x_2, \dots, x_n)^T = (x_3, \dots, x_n)^T$, P_2 the projection $P_2 x = (x_2, \dots, x_n)^T$, P_3 the projection $P_3 x = (x_3, \dots, x_n)^T$, and e_1 the unit column vector of length $n-1$ with unity in the first position. Moreover, $x^0 = (0, 1, z_0^T)^T = x(0)$ denotes the initial point of the (nonsliding) part of the limit cycle outside $\mathcal{S}$, $x^1 = x(t_{\rm ns})$ the final point of this part, $x^2 = x(t_{\rm ns} + t_{\rm sl}) = -x^0$ the final point of the sliding mode part, and $\nu = \left(1, z_0^T\right)^T$. Sliding limit cycles are further analyzed in [28], where it is shown that limit cycles with several first-order sliding segments exist and can be analyzed similarly as previously shown.

III. Chattering

If $CB = 0$ and $CAB > 0$, then the set of initial conditions that gives a second-order sliding mode is of measure zero. Instead trajectories close to $\{x \in \mathcal{S}_2: |CA^2 x| < CAB\}$ may give rise to chattering. In this section, a detailed analysis of the chattering is given and a formula is proved that shows that in many cases the chattering can be approximated by a second-order sliding mode. In Section IV, this result is used to show existence of limit cycles with chattering. "Chattering" discussed here should not be mixed up with fast relay switchings occurring in systems with relay imperfections such as hysteresis. The system description here is exact. Chattering is a trajectory with a finite number of relay switchings close to a second-order sliding mode.

Consider the relay feedback system (1) and (2) with state-space representation (A, B_2, C), where

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0 \\ 1 \\ b_1 \\ \vdots \\ b_{n-2} \end{bmatrix}$$
$$C = [1 \quad 0 \quad \dots \quad 0].$$

Note that this parameterization corresponds to a linear system with pole excess two, such that $CB_2 = 0$ and $CAB_2 = 1$. Since $CAB_2 > 0$, trajectories close to the set $\{x \in \mathcal{S}_2: |x_3| < 1\}$ will give fast relay switchings [22]. Due to the choice of parameterization, the fast behavior takes place in the variables x_1 and x_2. Therefore, they are called the *chattering variables*, as opposed to the *nonchattering variables* $x_3, \dots, x_n$.

The second-order sliding mode can be derived similarly to the first-order sliding mode in Section II. It is given by $x_1 = x_2 = 0$ and the solution of

$$\dot{w}(t) = F_2 w(t)$$
$$w = [x_3 \quad \dots \quad x_n]^T$$

where

$$F_2 = \begin{bmatrix} -b_1 & 1 & 0 & \dots & 0 \\ -b_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -b_{n-3} & 0 & 0 & & 1 \\ -b_{n-2} & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{5}$$

A trajectory starting at a point $x(0)$ with $x_1(0)=0$, $x_2(0)\neq 0$ sufficiently small, and $|x_3(0)|<1$ will wind around the set $\mathcal{S}_2$. This follows from Theorem 1 given next, which states a first-order approximation for the amplitude of the chattering.

Theorem 1: Consider (1) and (2) with parameterization (A, B_2, C) and order $n\geq 3$ over a time interval $[0,T]$, $T>0$. Let the initial state be $x(0)=\left(0, x_2(0), x_3(0), \ldots, x_n(0)\right)^T$, where $x_2(0)$ is a variable but $x_3(0),\ldots,x_n(0)$ are fixed. Let the switching times be denoted by $t_k\in[0,T]$, $k\geq 1$, i.e., let t_k be the time instances such that $x_1(t_k)=0$. If $|x_3(t)|<1$ for all $t\in[0,T]$, then the chattering variable x_1 satisfies

$$\frac{1}{|x_2(0)|}\max_{t\in[0,T]}|x_1(t)|\rightarrow 0, \text{ as } |x_2(0)|\rightarrow 0 \tag{6}$$

and the envelope of the peaks of the chattering variable x_2 is given by

$$x_2(t_k)=(-1)^k x_2(0)\exp\left[-(a_1-b_1)\frac{t_k}{3}\right] \times\left(\frac{1-x_3^2(t_k)}{1-x_3^2(0)}\right)^{1/3}+\epsilon_1(x_2(0);t_k) \tag{7}$$

where $\epsilon_1(x_2(0);t_k)/x_2(0)\rightarrow 0$ as $x_2(0)\rightarrow 0$ uniformly for all k.

Proof: See the Appendix. ■

Remark 1: The nonchattering variables $x_{\text{nc}}=(x_3,\ldots,x_n)^T$ are close to the corresponding sliding mode $w(t)$. This follows from that the solution of a linear system depends continuously on the initial data. Hence

$$\begin{aligned} x_{\text{nc}}(t) &= w(t)+\epsilon_2(x_2(0);t)\\ \dot{w}(t) &= F_2 w(t)\end{aligned}$$

for $t\in[0,T]$, where $\epsilon_2(x_2(0);t)/x_2(0)\rightarrow 0$ as $x_2(0)\rightarrow 0$ and F_2 is given by (5).

Remark 2: The variable $x_3(t)$ is almost constant over $[0,T]$ compared to the chattering variable $x_2(t)$. Therefore, (7) gives that the sign of a_1-b_1 determines if the chattering in x_2 has an increasing or decreasing amplitude.

The following result is a formula for the number of switchings on a chattering trajectory.

Theorem 2: Given the assumptions of Theorem 1, the number of switchings on the interval $[0,\widetilde{T}]$ is equal to

$$K=\frac{1}{|x_2(0)|}\left[\frac{1}{2}\left(1-x_3^2(0)\right)^{1/3}\int_0^{\widetilde{T}}\exp\left[(a_1-b_1)\frac{t}{3}\right] \times\left(1-x_3^2(t)\right)^{2/3}dt+\epsilon_3(x_2(0);\widetilde{T})\right] \tag{8}$$

where $\epsilon_3(x_2(0);\widetilde{T})\rightarrow 0$ as $x_2(0)\rightarrow 0$ uniformly for $\widetilde{T}\in[0,T]$.

Proof: See the Appendix. ■

Remark 3: Equation (7) captures the behavior of chattering quite well. Consider a chattering solution that starts with x_1 and x_2 small and $|x_3|$ smaller than one. Since x_2 changes rapidly in comparison with x_3, (7) tells that x_2 oscillates with exponentially decreasing amplitude if $a_1>b_1$. The length of the switching intervals will decrease as x_2 decreases. As $|x_3|$ approaches one, however, it follows from (8) that the intervals between the switchings increase again. Note that (7) and (8) are not proved for $|x_3(t)|\rightarrow 1$ and that the expressions are singular for $|x_3(0)|=1$. This case needs further research.

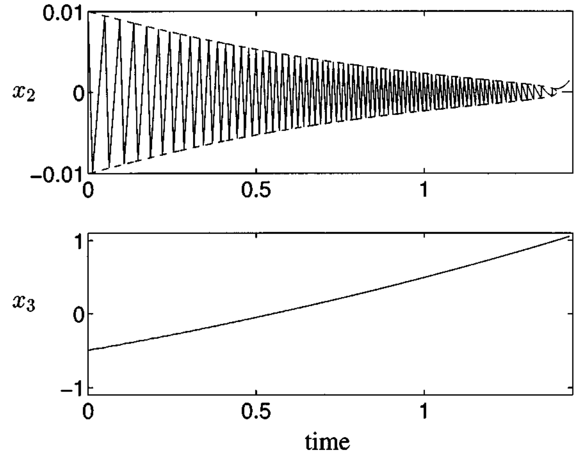

Fig. 1. Chattering for a fourth-order system (solid) together with accurate envelope estimate from Theorem 1 (dashed). Note that the chattering ends when x_3 becomes greater than one. Furthermore, as predicted, the length of the switching intervals decreases until x_3 becomes close to one and then the intervals increase.

Example 1: Consider

$$G(s)=\frac{(s-\zeta)^2}{(s+1)^4}$$

with state-space representation

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-4 & 1 & 0 & 0\\ -6 & 0 & 1 & 0\\ -4 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\end{bmatrix}x(t)+\begin{bmatrix}0\\ 1\\ -2\zeta\\ \zeta^2\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}x(t)\end{aligned}$$

and relay feedback. Let $\zeta=0.2$. Fig. 1 shows a simulation of the system starting in $x(0)=(0, 0.01, -0.5, 1.0)$ (solid line) together with the continuous estimate of the envelope of x_2 obtained from Theorem 1 (dashed lines). We see that the estimate from the theorem is accurate. The chattering ends when x_3 becomes greater than one. Note that the switching periods increase close to the end point of the chattering, as was mentioned in Remark 3. The estimated number of switchings from Theorem 2 is $K=151$, while the true number is 152.

IV. Limit Cycles With Chattering

In this section, the main result of the paper is presented. We will show that limit cycles with chattering can be analyzed, similar to limit cycles with first-order sliding modes, by using Theorem 1. This will lead to conditions for existence of chattering limit cycles. A chattering limit cycle consists of two symmetric half-periods, each of them has one chattering part (which has a finite, typically large, number of relay switchings) and one nonchattering part.

To prove existence of a chattering limit cycle, we need to confirm that the chattering is sufficiently close to a second-order sliding mode. The analysis of chattering in Section III showed that the chattering variable x_2 can be approximated to a high accuracy by a product of one time-dependent factor and one factor depending only on the nonchattering variable x_3, see (7) of Theorem 1. The variables $x_3, \ldots, x_n$ are almost constant compared to the chattering variables x_1 and x_2, so the second factor of (7) is almost constant during chattering. If the first factor, which is an exponential function in t, is decreasing, then there is contraction in the chattering variable x_2. It then follows from (6) that there is also a contraction in the chattering variable x_1. During the chattering, the variables $x_3, \ldots, x_n$ can be approximated by the differential equation for the sliding mode with an accuracy proportional to the amplitude of x_2. If also this differential equation gives a contraction, then the two contractions form a contracting mapping for the full system. Such a system has a limit cycle containing one chattering part and one nonchattering part. This is formulated in the following theorem.

Theorem 3: Consider (1) and (2) with $n \geq 4$ and let $b(s)/a(s) = C(sI - A)^{-1}B_2$. Assume $b(s) = \epsilon^{n-2}\bar{b}(s/\epsilon)$ with $\bar{b}(s) = s^{n-2} + \bar{b}_1 s^{n-3} + \cdots + \bar{b}_{n-2}$ and let $\bar{F}_2$ be defined as F_2 in (5) but with b_k replaced by $\bar{b}_k$. If the following conditions hold:

1) matrix A is Hurwitz and the eigenvalue of A with largest real part is unique;
2) $\bar{b}_{n-2} > 0$;
3) the solution of

$$\begin{aligned}\dot{\bar{w}}(t) &= \bar{F}_2\bar{w}(t)\\ \bar{w}(0) &= (1, \bar{b}_1, \ldots, \bar{b}_{n-3})^T\end{aligned}$$

reaches the hyperplane $\bar{w}_1 = -1$ at $t = \bar{\tau} > 0$, it holds that $|\bar{w}_1(t)| < 1$ for $t \in (0, \bar{\tau})$, and $\bar{w}_2(\bar{\tau}) < -\bar{b}_1$;
4) $e_1^T e^{At} e_4 > 0$ for all $t > 0$, where $e_1 = (1, 0, \ldots, 0)^T$ and $e_4 = (0, 0, 0, 1, 0, \ldots, 0)^T$.

Then there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$, (1) and (2) have a symmetric limit cycle with chattering.

Proof: See the Appendix. ∎

Remark 4: The number of relay switchings each period can be made arbitrarily large by choosing $\epsilon > 0$ sufficiently small. This follows from that a second-order sliding mode for the system is long if the unstable zeros of $b(s)$ are close to the origin (i.e., if ϵ is small). Therefore, the number of fast relay switchings each period of a chattering limit cycle increases as the distance to the origin for the unstable zeros decreases. Note also that if the unstable zeros are close to the origin then $a_1 - b_1 > 0$, because $b_1 = \epsilon\bar{b}_1$ and A is Hurwitz. It then follows from Theorem 1 that the variables x_1 and x_2 have decaying amplitudes during the chattering. Hence, the chattering brings the trajectory close to the second-order sliding set.

Remark 5: The location of the zeros of $b(s)$ has a nice geometric interpretation. First note that the assumptions $\bar{b}_{n-2} > 0$ and A Hurwitz imply positive steady-state gain of $G(s) = C(sI - A)^{-1}B_2$, i.e., $G(0) = b_{n-2}/a_n > 0$. The stable equilibrium point of $\dot{x} = Ax - B_2$ is equal to $\hat{x} = A^{-1}B_2$. Hence, $G(0) > 0$ gives that $C\hat{x} < 0$. Therefore, a relay switching is guaranteed to occur for any trajectory with $x(0) = x_0$ such that $Cx_0 > 0$. It is easy to see that $\hat{x}$ belongs to the hyperplane $\{x: CA^2x - CAB_2 = 0\} = \{x: x_3 = 1\}$. A Taylor expansion of $G(s)$ shows that $CA^{-1}B_2$ is small, if all zeros of $b(s)$ are close to the origin (compared to the zeros of $a(s)$), i.e., if ϵ is small. The trajectory of $\dot{x} = Ax - B_2$ will thus approach a point close to $(x_1, x_2, x_3) = (0, 0, 1)$. The assumption of Theorem 1 is thus fulfilled if ϵ is sufficiently small.

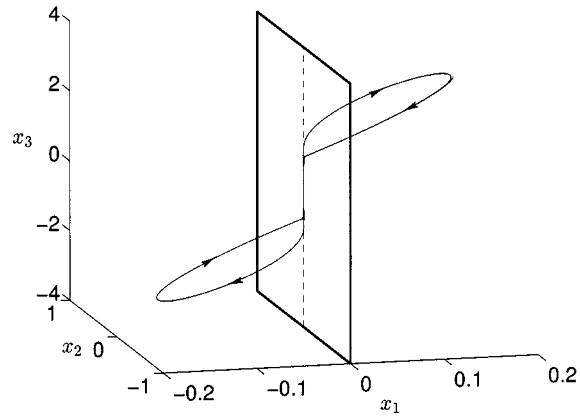

Fig. 2. Limit cycle with chattering for a system in Example 2. The dashed line is the second-order sliding set $\mathcal{S}_2$.

Remark 6: The Jacobian $\mathcal{W}_2$ of the Poincaré map consisting of one part outside $\mathcal{S}$ and one (exact) second-order sliding mode part is given by

$$\mathcal{W}_2 = \bar{P}_1\left(I - \frac{F_2\nu\bar{e}_1^T}{\bar{e}_1^T F_2\nu}\right)e^{F_2 t_{\rm sl}}\bar{P}_2\left(I - \frac{(Ax^1 - B_2)C}{C(Ax^1 - B_2)}\right) \times e^{At_{\rm ns}}\bar{P}_3^T$$

where the notation is similar to (4).

Remark 7: A ball with center in $x(0) = (0, 0, 1, \bar{b}_1, \ldots, \bar{b}_{n-3})^T$ and radius proportional to ϵ^{n-3} is invariant under the system dynamics, as is shown in the proof of the theorem. Note that although this ball captures the recurrence of the limit cycle (and although the ball can be made arbitrarily small), it does not follow that the limit cycle is stable.

The key condition of Theorem 3 is that the zeros of $G(s)$ should be close to the origin. The other four conditions are, for example, always fulfilled in the following fourth-order case.

Proposition 1: Suppose the dimension of (1) and (2) is $n = 4$. If all poles of $G(s) = C(sI - A)^{-1}B_2$ are real and stable, all zeros are real and unstable, and $G(0) > 0$, then Conditions 1)–4) of Theorem 3 are satisfied.

Proof: See the Appendix. ∎

The following example illustrates a chattering limit cycle.

Example 2: Consider the system in Example 1 again. The parameter $\zeta = 0.2$ gives zeros that are sufficiently close to the origin to give a limit cycle with chattering. Fig. 2 shows the limit cycle in the subspace (x_1, x_2, x_3). The fast oscillations close to $\mathcal{S}_2 = \{x: Cx = CAx = 0\}$ during the chattering are magnified in Fig. 3. Fig. 4 shows the four state variables during the chattering. In agreement with the analysis above, the chattering starts at $x_3(t) = -1$, ends at $x_3(t) = 1$, and $x_4(t)$ is almost constant. By approximating the chattering with a second-order sliding mode, it is possible to get a rough estimate of the behavior. For the example, this leads to a nonsliding time of $t_{\rm ns} = 7.5$ and a sliding time of $t_{\rm sl} = 4.3$, while for the chattering limit cycle simulations give the times 7.4 and 4.2. The Jacobian in Remark 6 is $\mathcal{W}_2 = -0.0025$. Note that the existence of the limit cycle in this example is not formally proved by Theorem

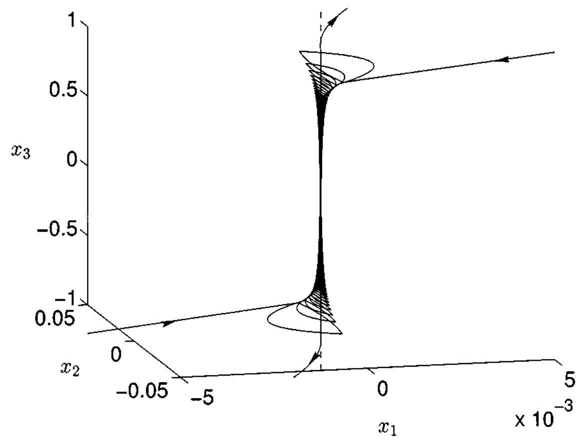

Fig. 3. A closer look on the winding around the second-order sliding set $\mathcal{S}_2$ (dashed line) for the limit cycle in Fig. 2.

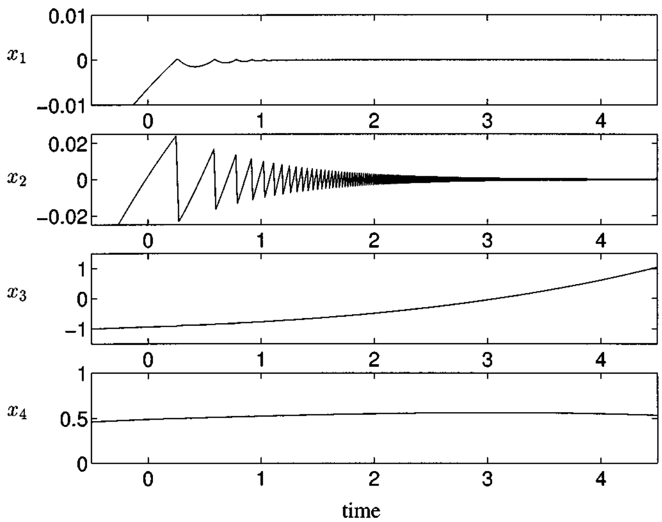

Fig. 4. Chattering part of the limit cycle in Example 2. The behavior is well described by the presented theory. Note the chattering in x_1 and x_2, how this chattering starts when $x_3 = -1$ and ends when $x_3 = 1$, and that x_4 is almost constant.

3, because the theorem does not provide a bound on how close to the origin the zeros have to be.

V. Conclusion

A large number of fast relay switchings can appear in linear systems with relay feedback if the linear part has pole excess two. This chattering was analyzed in detail in this paper and a sufficient condition for existence was derived. It was also shown that chattering may be part of a limit cycle. The limit cycle can have arbitrarily many relay switchings each period. The main result of this paper stated that the chattering in the limit cycle can be approximated by a sliding mode.

Chattering occurs in systems with pole excess two. Many consecutive fast switchings can, however, not occur in systems with higher order pole excess. This can be understood intuitively, since a system whose first nonvanishing Markov parameter M is of order k have fast behavior similar to M/s^k. A double integrator gives a periodic solution with arbitrarily short period, while higher-order integrators are unstable under relay feedback. It is shown in [22] that for systems with pole excess three or higher there exist limit cycles with only a few extra switchings each period.

Appendix

A. Proof of Theorem 1

Consider $x(0)$ with $x_1(0) = 0$, $x_2(0)$ small, and $|x_3(0)| < 1$. For $t > 0$ up to next switching instant, it holds that

$$\begin{aligned} x(t) =& e^{At}x(0) + (e^{At} - I)A^{-1}B_2u \\ =& x(0) + t\left(Ax(0) + B_2u\right) + \frac{t^2}{2}\left(A^2x(0) + AB_2u\right) \\ &+ \frac{t^3}{6}\left(A^3x(0) + A^2B_2u\right) + \kappa(t)t^4 \end{aligned}$$

where $u = \pm 1$ is constant and

$$|\kappa(t)| \le \max_{\xi\in(0,t)} \frac{\left\|e^{A\xi}A^3\left(Ax(0) + B_2u\right)\right\|}{24}.$$

Note that it follows from $CAB_2 = 1 > 0$ that there will be a next switching if $x_2(0)$ is sufficiently small. For the sake of simplicity, introduce the notation

$$\begin{aligned} \alpha_1 =& CAx(0) = x_2(0) \\ \alpha_2 =& CA^2x(0) + CAB_2u = x_3(0) + u - a_1x_2(0) \\ \approx& x_3(0) + u \\ \alpha_3 =& CA^3x(0) + CA^2B_2u \\ \approx& x_4(0) + b_1u - a_1(x_3(0) + u) \end{aligned}$$

where the last equation holds if the order $n \ge 4$. If $n = 3$, this equation and the following still holds, but with $x_4 \equiv 0$. Note that $\alpha_1u = -|\alpha_1| < 0$ and that $\alpha_1\alpha_2 < 0$. Now, assume that t is the next switching instant, i.e., $Cx(t) = x_1(t) = 0$. Then, it holds that

$$0 = x_1(t) = \alpha_1t + \frac{\alpha_2t^2}{2} + \frac{\alpha_3t^3}{6} + \mathcal{O}(t^4) \tag{9}$$

and

$$CAx(t) = x_2(t) = \alpha_1 + \alpha_2t + \alpha_3\frac{t^2}{2} + \mathcal{O}(t^3) \tag{10}$$

for small t. Introduce t_0 as an approximation of t to the accuracy of $\mathcal{O}(t^3)$ through the equation

$$\alpha_1 + \frac{\alpha_2t_0}{2} + \frac{\alpha_3t_0^2}{6} = 0. \tag{11}$$

Then, since for small β

$$\frac{1}{1+\sqrt{1-\beta}} = \frac{1}{2} + \frac{1}{8}\beta + \mathcal{O}(\beta^2)$$

we get

$$\begin{aligned} t_0 =& \frac{4|\alpha_1|}{|\alpha_2| + \sqrt{\alpha_2^2 - \frac{8\alpha_1\alpha_3}{3}}} \\ =& \frac{2|\alpha_1|}{|\alpha_2|}\left(1 + \frac{2}{3}\cdot\frac{\alpha_1\alpha_3}{\alpha_2^2}\right) + \mathcal{O}(\alpha_1^3) \end{aligned} \tag{12}$$

as $x_2(0) = \alpha_1 \to 0$. It is obvious from this expression that t_0 has the same order as α_1 as $\alpha_1 \to 0$. For this reason, the expressions $\mathcal{O}(t^k)$ and $\mathcal{O}\left(\alpha_1^k\right)$ are equivalent for every $k > 0$. In particular, from $x_1(t_0) = \alpha_1 t_0 + O\left(t_0^2\right)$ we have that $x_1(\tau) = \mathcal{O}\left(x_2^2(0)\right)$ as $x_2(0) \to 0$ for all $\tau \in [0, t_0]$, which proves (6).

In the following, it will be shown that $x_2(t_0)$ is proportional to $x_2(0)$ and the relation (7) will be derived. Let $\widetilde{\alpha}_1 = x_2(t_0)$ be the starting point for the next part of the trajectory in the chattering mode between two successive switchings. The map $\alpha_1 \mapsto \widetilde{\alpha}_1$ describes the envelope of $x_2(t)$ in the chattering mode. By substituting t with t_0 and taking into account that $\alpha_1\alpha_2 < 0$ at any switching point, we get from (10) and (11) that

$$\begin{aligned}\widetilde{\alpha}_1 =& \alpha_1 + \alpha_2 t_0 - 3\left(\alpha_1 + \frac{\alpha_2 t_0}{2}\right) + \mathcal{O}\left(\alpha_1^3\right)\\ =& -2\alpha_1 - \frac{1}{2}\alpha_2 t_0 + \mathcal{O}\left(\alpha_1^3\right).\end{aligned}$$

Then, (12) gives

$$\begin{aligned}\widetilde{\alpha}_1 =& -\alpha_1\left(1 - \frac{2}{3}\cdot\frac{\alpha_1\alpha_3}{\alpha_2^2}\right) + \mathcal{O}\left(\alpha_1^3\right)\\ =& -\alpha_1\left(1 + \frac{\alpha_3}{3\alpha_2}t + \mathcal{O}(t^2)\right)\end{aligned} \tag{13}$$

where the last equality follows from (9). The chattering variable $x_2(t)$ thus shifts sign in successive switching points. After neglecting these sign shifts, the last equation looks very similar to a one-step iteration of a numerical solution of a differential equation. Next, we show that such a differential equation exists and that it describes the envelope of $x_2(t)$ at the switching instants t_k. It is surprising that this equation can be analytically integrated.

Consider three successive switching points at the time instants 0, t, and $t + \widetilde{t}$. The relay output u has opposite sign in the intervals $(0, t)$ and $(t, t + \widetilde{t})$. This influences α_2 and α_3, so that they show a gap in two successive switching points, whereas they are close with a step of two switchings. Denote x_2 in three successive switching points by α_1, $\widetilde{\alpha}_1$, and $\widetilde{\widetilde{\alpha}}_1$. Denote by $\widetilde{\alpha}_2$ and $\widetilde{\alpha}_3$ the corresponding values for α_2 and α_3. It was previously proven that

$$\begin{aligned}\widetilde{\alpha}_1 =& -\alpha_1(1+\gamma) + \mathcal{O}\left(\alpha_1^3\right) \quad \gamma = -\frac{2}{3}\cdot\frac{\alpha_1\alpha_3}{\alpha_2^2}\\ \widetilde{\widetilde{\alpha}}_1 =& -\widetilde{\alpha}_1(1+\widetilde{\gamma}) + \mathcal{O}(\widetilde{\alpha}_1^3) \quad \widetilde{\gamma} = -\frac{2}{3}\cdot\frac{\widetilde{\alpha}_1\widetilde{\alpha}_3}{\widetilde{\alpha}_2^2}.\end{aligned}$$

Therefore, after two successive switching points

$$x_2(t+\widetilde{t}) = x_2(0)\left[1 + \gamma + \widetilde{\gamma} + \mathcal{O}((t+\widetilde{t})^2)\right].$$

Straightforward calculations using (12), (13), and $\alpha_2\widetilde{\alpha}_2 = x_3^2(0) - 1 + \mathcal{O}(\alpha_1)$ show that

$$t + \widetilde{t} = \frac{4|\alpha_1|}{1 - x_3^2(0)} + \mathcal{O}\left(\alpha_1^2\right). \tag{14}$$

Furthermore

$$\begin{aligned}\gamma + \widetilde{\gamma} =& \frac{4|\alpha_1|}{3\left(1 - x_3^2(0)\right)^2}\\ &\times\left[a_1\left(x_3^2(0) - 1\right) + b_1\left(x_3^2(0) + 1\right) - 2x_3(0)x_4(0)\right]\\ &+ \mathcal{O}\left(\alpha_1^2\right)\\ =& (t+\widetilde{t})\left[\frac{b_1 - a_1}{3} - \frac{1}{3}\cdot\frac{2x_3(0)\left(x_4(0) - b_1x_3(0)\right)}{1 - x_3^2(0)}\right]\\ &+ \mathcal{O}\left(\alpha_1^2\right).\end{aligned}$$

This gives the differential equation associated with the peak values of the chattering variable $x_2(t)$ as

$$\dot{\bar{x}}_2(t) = \bar{x}_2(t)\left[\frac{b_1 - a_1}{3} - \frac{1}{3}\cdot\frac{2\bar{x}_3(t)\left(\bar{x}_4(t) - b_1\bar{x}_3(t)\right)}{1 - \bar{x}_3^2(t)}\right]$$

where $(\bar{x}_3, \ldots, \bar{x}_n)^T = w$ is the solution to the sliding mode equation $\dot{w} = F_2 w$ with F_2 given by (5). We have

$$\dot{\bar{x}}_3(t) = \bar{x}_4(t) - b_1\bar{x}_3(t).$$

Therefore, the associated differential equation can be rewritten as

$$\frac{d}{dt}\log\left(\bar{x}_2(t)\right) = \frac{b_1 - a_1}{3} + \frac{1}{3}\cdot\frac{d}{dt}\log\left(1 - \bar{x}_3^2(t)\right).$$

Integration of this equation leads to the formula for x_2 and the proof is completed. ■

B. Proof of Theorem 2

Introduce a slower time τ associated with the number of switchings on a trajectory. The monotonous function $t = t(\tau)$ indicates the time instants of switchings for an integer argument: $t(k) = t_k$ is switching instant number k. Equation (14) in the proof of Theorem 1 states that the increments of this function can be approximated as

$$t(k+2) - t(k) = \frac{4\left|x_2(t(k))\right|}{1 - x_3^2\left(t(k)\right)} + \mathcal{O}\left(x_2^2(t(k))\right).$$

Since the increments are small as $x_2(0) \to 0$, the function $t(\tau)$ can be approximated by the solution of the differential equation

$$\frac{d}{d\tau}\bar{t}(\tau) = \frac{2|\bar{x}_2(\bar{t}(\tau))|}{1 - \bar{x}_3^2(\bar{t}(\tau))}.$$

The inverse function $\tau = \tau(t)$ satisfies

$$\frac{d}{d\bar{t}}\tau(\bar{t}) = \frac{1 - \bar{x}_3^2(\bar{t})}{2|\bar{x}_2(\bar{t})|}.$$

It remains now only to substitute $\bar{x}_2$ with the expression given in Theorem 1 and integrate over $\bar{t}$. ■

C. Proof of Theorem 3

We will show that a trajectory starting close to the second-order sliding set $\mathcal{S}_2$ has one part outside $\mathcal{S}$ and one chattering part. By application of Theorem 1 and the fixed point theorem, it will be proved that two such parts form a limit cycle.

Consider the trajectory $\bar{w}(t)$ defined in Condition 3) and let $W = -\bar{w}(\bar{\tau})$. Then it holds that $W_1 = 1$. Let $x(t)$ be a solution of (1) and (2) with

$$x(0) = (0, 0, 1, W_2\epsilon, \ldots, W_{n-2}\epsilon^{n-3})^T.$$

It follows from the assumption $\bar{w}_2(\bar{\tau}) < -\bar{b}_1$ in Condition 3) that $\dot{x}_3(0) = \epsilon(W_2 - \bar{b}_1) > 0$. With $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$, this implies that $x_1(t) > 0$ and $u(t) = -1$ for small $t > 0$. Thus, the solution can, for small $t > 0$, be written as

$$x(t) = A^{-1}B_2 + e^{At}\left(x(0) - A^{-1}B_2\right)$$

where

$$\begin{aligned} A^{-1}B_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ b_{n-3} \end{bmatrix} - \frac{b_{n-2}}{a_n}\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ \bar{b}_{n-3}\epsilon^{n-3} \end{bmatrix} - \frac{\bar{b}_{n-2}\epsilon^{n-2}}{a_n}\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} \end{aligned}$$

and

$$x(0) - A^{-1}B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ (W_2 - \bar{b}_1)\epsilon \\ \vdots \\ (W_{n-2} - \bar{b}_{n-3})\epsilon^{n-3} \end{bmatrix} + \frac{\bar{b}_{n-2}\epsilon^{n-2}}{a_n}\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}.$$

Denote by $t_{\rm nc}$ the first instant t when $x_1(t) = 0$. This instant exists because $x_1(t) \to -b_{n-2}/a_{n-2} < 0$ as $t \to \infty$ by Conditions 1) and 2). Note that

$$\begin{aligned} e^{At}\left(x(0) - A^{-1}B_2\right) =& e^{At}\left(0, 0, 0, \epsilon(W_2 - \bar{b}_1), 0, \ldots, 0\right)^T \\ &+ \mathcal{O}(\epsilon^2) \\ =& \epsilon(W_2 - \bar{b}_1)e^{At}e_4 + \mathcal{O}(\epsilon^2). \end{aligned}$$

The first entry of the first term of the right-hand side is positive for all $t > 0$, because $e_1^T e^{At} e_4 > 0$ by Condition 4). Denote by λ the eigenvalue of A with the largest real part. Then λ is real and negative by Condition 1). The corresponding eigenvector of A is

$$V = \begin{bmatrix} 1 \\ \lambda + a_1 \\ \lambda^2 + a_1\lambda + a_2 \\ \vdots \\ \lambda^{n-1} + a_1\lambda^{n-2} + \cdots + a_{n-2}\lambda + a_{n-1} \end{bmatrix}.$$

All entries V_k are positive, because of the following argument. Since $a(s)$ is a stable polynomial and $a(\lambda) = 0$, it holds that $a(s) = (s - \lambda)c(s)$ for some stable polynomial $c(s) = c_0 s^{n-1} + \cdots + c_{n-1}$. Obviously, $a_0 = c_0$, $a_k = c_k - \lambda c_{k-1}$ for $1 \le k \le n-1$, and $a_n = -\lambda c_{n-1}$. We prove by mathematical induction that $c_{k-1} = V_k$ for all $1 \le k \le n$. It then follows that V_k is positive. First, note that $c_0 = 1 = V_1$. Assume that $c_{i-1} = V_i$ for $1 \le i \le k$. Then

$$c_k = a_k + \lambda c_{k-1} = a_k + \lambda V_k = V_{k+1}.$$

Since $t_{\rm nc} \to \infty$ as $\epsilon \to 0$, it holds that $x(t_{\rm nc}) = A^{-1}B_2 + \gamma[V + o(1)]$, where $o(1) \to 0$ as $\epsilon \to 0$. The scalar γ follows from the equation $x_1(t_{\rm nc}) = 0$ and is, hence, given by $\gamma = b_{n-2}/a_n = \epsilon^{n-2}\bar{b}_{n-2}/a_n > 0$.

Define the vector-valued function $X(t) = \left(X_k(t)\right)_{k=1}^n$ by the equation

$$X(t) = x(t + t_{\rm nc}) + (0, 0, 1, b_1, \ldots, b_{n-3})^T, \qquad t > 0.$$

Then, $X(0) = \gamma\lambda[(0, 1, V_2, \ldots, V_{n-1})^T + o(1)]$. All entries $X_k(0)$, $k \ge 2$, are proportional to ϵ^{n-2} and negative for sufficiently small ϵ, since $\lambda < 0$.

For small $t > 0$, it holds that $x_1(t) < 0$ and $u = +1$. A quick jump occurs, because $x_3(t_{\rm nc}) + u \approx 2 > 0$. The motion is similar to the one described in the proof of Theorem 1. It follows that it takes the time $\Delta t_{1-} = |X_2(0)| + \mathcal{O}(X_2^2(0))$ to reach $X_1(\Delta t_{1-}) = 0$, where $X_2(0) = \gamma\lambda + o(\epsilon^{n-2})$. Therefore, since $\dot{X}(0) = AX(0) - 2B_2$, we have

$$\begin{aligned} X(\Delta t_{1-}) =& \gamma\lambda\begin{bmatrix} 0 \\ -1 \\ V_2 - 2\epsilon\bar{b}_1 \\ \vdots \\ V_{n-1} - 2\epsilon^{n-2}\bar{b}_{n-2} \end{bmatrix} + o(\epsilon^{n-2}) \\ =& \gamma\lambda\begin{bmatrix} 0 \\ -1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix} + o(\epsilon^{n-2}). \end{aligned}$$

For small ϵ, all the entries $X_k(\Delta t_{1-})$, $k \ge 3$, remain negative.

For small $t > 0$, $X_1(\Delta t_{1-} + t)$ is positive and $u = -1$. It follows from the (1) and (2) that the function $X(t)$ satisfies

$$\dot{X} = AX - (0, 0, \ldots, 0, b_{n-2})^T$$

whenever $u = -1$. The structure of the matrix A indicates that if $X_k < 0$ for $k \ge 3$ and $X_1 > 0$ then $\dot{X}_k < 0$ for $k \ge 2$. Hence, the values of $X_k(\Delta t_{1-} + t)$, $k \ge 2$, decrease, the value of X_2 becomes negative, and $X_1(t)$ reaches zero at $t = \Delta t_{1-} + \Delta t_{1+} = \Delta t_1$. All the entries of $X(t)$, $0 \le t \le \Delta t_1$, and b_{n-2} are proportional to ϵ^{n-2}. Therefore, the time length Δt_{1+} of the motion with $u = -1$ does not tend to zero as $\epsilon \to 0$, in contrast to Δt_{1-}. It is easy from the relation $\Delta t_{1-} \ll \Delta t_{1+}$ to derive that $X_k(\Delta t_1) < X_k(0)$ for $k \ge 3$. The same argument proves that these "nonchattering" variables decrease on the next switch intervals provided that $x_3(t)$ is not close to one. All conditions of Theorem 1 hold on the interval $x_3 \in [-1 + \delta, 1 - \delta]$ for any fixed $\delta > 0$. Hence, the chattering mode starts at the point

$x(t_{\rm nc})$, where we recall that $t_{\rm nc}$ is defined as the first time instant when $x_1(t) = 0$.

The trajectory $x(t)$, $t > t_{\rm nc}$, can be approximated according to Theorem 1 and Remark 1. The nonchattering part of the state $(x_3, \ldots, x_n)^T$ is close to the solution of

$$\dot{w}(t) = F_2 w(t), \qquad t > t_{\rm nc}$$
$$w(t_{\rm nc}) = (1, \epsilon \bar{b}_1, \ldots, \epsilon^{n-3}\bar{b}_{n-3})^T.$$

Make the following change of variables: $\tau = \epsilon(t - t_{\rm nc})$ and $\bar{w}_k(\tau) = w_k(t)/\epsilon^{k-1}$ for $k = 1, \ldots, n-2$. It is easy to see that the new state vector $\bar{w}(\tau)$ satisfies the equation

$$\dot{\bar{w}}(\tau) = \bar{F}_2 \bar{w}(\tau), \qquad \tau > 0$$
$$\bar{w}(0) = (1, \bar{b}_1, \ldots, \bar{b}_{n-3})^T.$$

According to Condition 3), this trajectory reaches the hyperplane $\bar{w}_1 = -1$ at time $\tau = \bar{\tau}$, i.e., $\bar{w}_1(\bar{\tau}) = -1$. The condition $|\bar{w}_1(t)| < 1$ implies that the sliding mode for a trajectory that starts in $\left(0, 0, \bar{w}(0)\right)^T$ is not broken on the interval $(0, \bar{\tau})$. This gives that the end point of the sliding mode corresponds to $\bar{w}(\bar{\tau}) = -W$ or equivalently $w(t_{\rm nc} + t_{\rm sl}) = -(W_1, \epsilon W_2, \ldots, \epsilon^{n-3} W_{n-2})^T$, where the sliding time is given by $t_{\rm sl} = \bar{\tau}/\epsilon$.

According to Theorem 1, the chattering variable $x_2(t)$ is proportional to the initial value $x_2(t_{\rm nc}) = \mathcal{O}(\epsilon^{n-2})$. Thus, starting from the point $x(0)$ the trajectory reaches the point $x(t_{\rm nc} + t_{\rm sl}) = -\left(x(0) + \mathcal{O}(\epsilon^{n-2})\right)$ by passing through the nonchattering part and the chattering part. Denote the corresponding map by ϕ_ϵ, i.e.,

$$\phi_\epsilon(x(0)) = -x(t_{\rm nc} + t_{\rm sl}).$$

Next, it will be proved that the mapping ϕ_ϵ can be defined in a neighborhood of the point $x(0)$ and that this mapping is a contraction. The existence of a symmetric limit cycle follows by the fixed-point theorem.

Let D_ϵ be the ball in the hyperplane $x_1 = 0$ with a center at $x(0)$ and with the radius ϵ^{n-3}. Consider a trajectory $\tilde{x}(t)$ starting from a point $\tilde{x}(0) \in D_\epsilon$. Similarly to the trajectory $x(t)$, the first part of $\tilde{x}(t)$ is nonchattering and lies in the set $\tilde{x}_1 > 0$. A switch to $\tilde{x}_1 < 0$ occurs at the time instant $\tilde{t}_{\rm nc}$, and $\tilde{t}_{\rm nc} \to \infty$ as $\epsilon \to 0$ uniformly in D_ϵ. Hence, it holds that

$$\tilde{x}(\tilde{t}_{\rm nc}) = (0, 0, 1, \epsilon\bar{b}_1, \ldots, \epsilon^{n-3}\bar{b}_{n-3})^T + \mathcal{O}(\epsilon^{n-2}).$$

Since the dominant parts of the values $\tilde{x}(\tilde{t}_{\rm nc})$ and $x(t_{\rm nc})$ are equal, the next chattering parts of these trajectories are close. In the normalized time $\tau = \epsilon(t - \tilde{t}_{\rm nc})$ the trajectory of the state vector $(\tilde{x}_3, \ldots, \tilde{x}_n)^T$ is close to $\bar{w}$ defined in Condition 3). In particular, there exists an instant $\tilde{t}_{\rm sl}$ where the sliding mode is broken. The vector $-\tilde{x}(\tilde{t}_{\rm nc} + \tilde{t}_{\rm sl})$ is the value of the function ϕ_ϵ on the vector $\tilde{x}(0)$. Thus, the function ϕ_ϵ is well-defined on D_ϵ. It holds that $\phi_\epsilon(\tilde{x}(0)) - x(0) = \mathcal{O}(\epsilon^{n-2})$. Therefore, the mapping ϕ_ϵ transforms the ball D_ϵ into itself for small ϵ. The mapping ϕ_ϵ is continuous, because the time interval is uniformly bounded and the vector field of the relay system generates trajectories which continuously depend on the initial states, where the latter follows from Theorem 1. Any continuous mapping of a ball into itself has a fixed point, hence, a fixed point of ϕ_ϵ exists in D_ϵ. The fixed point defines the limit cycle. This concludes the proof.

D. *Proof of Proposition 1*

Conditions 1) and 2) are obviously satisfied. To show Condition 3), first assume that $\bar{b}(s) = (s - \mu_1)(s - \mu_2)$ where $\mu_2 > \mu_1 > 0$. Then

$$\dot{\bar{w}}(t) = \begin{bmatrix} \mu_1 + \mu_2 & 1 \\ -\mu_1\mu_2 & 0 \end{bmatrix} \bar{w}(t)$$
$$\bar{w}(0) = \begin{bmatrix} 1 \\ -(\mu_1 + \mu_2) \end{bmatrix}$$

which has the solution

$$\bar{w}(t) = \frac{1}{\mu_2 - \mu_1} \begin{bmatrix} \mu_2 e^{\mu_1 t} - \mu_1 e^{\mu_2 t} \\ \mu_1^2 e^{\mu_2 t} - \mu_2^2 e^{\mu_1 t} \end{bmatrix}.$$

Hence, there exists $\bar{\tau} > 0$ such that $\bar{w}_1(\bar{\tau}) = -1$. It is easy to see that $|\bar{w}_1(t)| < 1$ for $t \in (0, \bar{\tau})$. Furthermore

$$\bar{w}_2(\bar{\tau}) = -(\mu_1 + \mu_2)\bar{w}_1(\bar{\tau}) - \frac{\mu_1\mu_2}{\mu_2 - \mu_1}\left(e^{\mu_2\bar{\tau}} - e^{\mu_1\bar{\tau}}\right)$$
$$< -(\mu_1 + \mu_2)\bar{w}_1(\bar{\tau}) = -\bar{b}_1.$$

Thus, Condition 3) holds if $\mu_1 \neq \mu_2$. A similar calculation shows that Condition 3) also holds if $\mu_1 = \mu_2$.

Finally, to check Condition 4), note that $e_1^T e^{At} e_4$ is equal to the impulse response of a system with transfer function

$$\frac{1}{a(s)} = \frac{1}{(s - \lambda_1)\cdots(s - \lambda_4)}$$

where $\lambda_i < 0$ are the eigenvalues of A. The impulse response is equal to

$$\mathcal{L}^{-1}\{a^{-1}\} = e^{-\lambda_1 t} * \cdots * e^{-\lambda_4 t} > 0$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform and $*$ convolution. This completes the proof. ■

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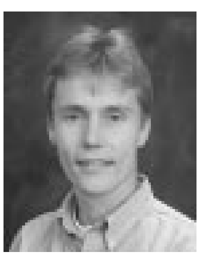

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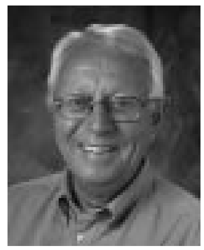

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