Kalman Filtering Over Gilbert–Elliott Channels: Stability Conditions and Critical Curve

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***Abstract*—This paper investigates the stability of Kalman filtering over Gilbert–Elliott channels where random packet drops follow a time-homogeneous two-state Markov chain whose state transition is determined by a pair of failure and recovery rates. First of all, we establish a relaxed condition guaranteeing peak-covariance stability described by an inequality in terms of the spectral radius of the system matrix and transition probabilities of the Markov chain. We further show that the condition can be interpreted using a linear matrix inequality feasibility problem. Next, we prove that the peak-covariance stability implies mean-square stability, if the system matrix has no defective eigenvalues on the unit circle. This connection between the two stability notions holds for any random packet drop process. We prove that there exists a critical curve in the failure-recovery rate plane, below which the Kalman filter is mean-square stable and no longer mean-square stable above. Finally, a lower bound for this critical failure rate is obtained making use of the relationship we establish between the two stability criteria, based on an approximate relaxation of the system matrix.**

***Index Terms*—Estimation, Kalman filtering, Markov processes, stability, stochastic systems.**

I. Introduction

A. Background and Related Works

WIRELESS communications are being widely used nowadays in sensor networks and networked control systems. New challenges accompany the considerable advantages wireless communications offer in these applications, one of which is how channel fading and congestion influence the performance of estimation and control. In the past decade, this fundamental question has inspired various significant results focusing on the interface of control and communication and has become a central theme in the study of networked sensor and control systems [2]–[4].

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Early works on networked control systems assumed that sensors, controllers, actuators, and estimators communicate with each other over a finite-capacity digital channel, e.g., [2] and [5]–[13], with the majority of contributions focused on one or both finding the minimum channel capacity or data rate needed for stabilizing the closed-loop system, and constructing optimal encoder–decoder pairs to improve system performance. At the same time, motivated by the fact that packets are the fundamental information carrier in most modern data networks [3], many results on control or filtering with random packet dropouts appeared.

State estimation, based on collecting measurements of the system output from sensors deployed in the field, is embedded in many networked control applications and is often implemented recursively using a Kalman filter [14], [15]. Clearly, channel randomness leads to that the characterization of performance is not straightforward. A burst of interest in the problem of the stability of Kalman filtering with intermittent measurements has arisen after the pioneering work [16], where Sinopoli *et al.* modeled the statistics of intermittent observations by an independent and identically distributed (i.i.d.) Bernoulli random process and studied how packet losses affect the state estimation. Tremendous research has been devoted to stability analysis of Kalman filtering or the closed-loop control systems over i.i.d. packet lossy packet networks in [17]–[22].

To capture the temporal correlation of realistic communication channels, the Markovian packet loss model has been introduced to partially address this problem. Since the Gilbert–Elliott channel model [23], [24], a classical two-state time-homogeneous Markov channel model, has been widely applied to represent wireless channels and networks in industrial applications [25]–[27], the problem of networked control over Gilbert–Elliott channels has drawn considerable attention. Huang and Dey [28], [29] considered the stability of Kalman filtering with Markovian packet losses. To aid the analysis, they introduced the notation of peak covariance, defined by the expected prediction error covariance at the time instances when the channel just recovers from failed transmissions, as an evaluation of estimation performance deterioration. The peak-covariance stability (PCS) can be studied by lifting the original systems at stopping times. Sufficient conditions for the

PCS were proposed for general vector systems, and a necessary and sufficient condition for scalar systems. The existing literature [28]–[31] has however made restrictive assumptions on the plant dynamics and the communication channel. We show by numerical examples that existing conditions for PCS only apply to relative reliable channels with low failure rate. Moreover, existing results rely on calculating an infinite sum of matrix norms for checking stability conditions.

The PCS describes covariance stability at random stopping times, whereas the mean-square stability (MSS) describes covariance stability at deterministic times. In the literature, the main focus is on MSS rather than PCS. This is partly because the definition of the former is more natural and practically useful than that of the latter. However, it is difficult to analyze MSS over Gilbert–Elliott channels directly through the random Riccati equation, due to temporal correlation between the packet losses. If we could build clearer connections relating these two stability notions, the MSS can be conveniently studied through PCS. The relationship between the MSS and the PCS was preliminarily discussed [29]. Improvements to these results can be found in [30] and [31]. Particularly in [32], by investigating the estimation error covariance matrices at each packet reception, necessary and sufficient conditions for the MSS were derived for second-order systems and some certain classes of higher order systems. Although it was proved that with i.i.d. packet losses the PCS is equivalent to the MSS for scalar systems and systems that are one-step observable [29], [31], for vector systems with more general packet drop processes, this relationship is unclear.

There are some other works studying distribution of error covariance matrices. Essentially, the probabilistic characteristics of the prediction error covariance are fully captured by its probability distribution function. Motivated by this, Shi *et al.* [33] studied Kalman filtering with random packet losses from a probabilistic perspective where the performance metric was defined using the error covariance matrix distribution function, instead of the mean. Mo and Sinopoli [34] studied the decay rate of the estimation error covariance matrix and derived the critical arrival probability for nondegenerate systems based on the decay rate. Weak convergence of Kalman filtering with packet losses, i.e., that error covariance matrix converges to a limit distribution, were investigated in [35]–[37] for i.i.d., semi-Markov, and Markov drop models, respectively.

B. Contributions and Paper Organization

In this paper, we focus on the PCS and MSS of Kalman filtering with Markovian packet losses. We first derive relaxed and explicit PCS conditions. Then, we establish a result indicating that PCS implies MSS under quite general settings. We eventually make use of these results to obtain MSS criteria. The contributions of this paper are summarized as follows.

1) A relaxed condition guaranteeing PCS is obtained described by an inequality in terms of the spectral radius of the system matrix and transition probabilities of the Markov chain, rather than an infinite sum of matrix norms as in [28]–[31]. We show that the condition can be recast as a linear matrix inequality (LMI) feasibility problem. These conditions are theoretically and numerically shown to be less conservative than those in the literature.
2) We prove that PCS implies MSS if the system matrix has no defective eigenvalues on the unit circle. Remarkably enough this implication holds for any random packet drop process that allows PCS to be defined. This result bridges two stability criteria in the literature and offers a tool for studying MSS of the Kalman filter through its PCS. Note that MSS was previously studied using quite different methods such as analyzing the boundness of the expectation of a kind of randomized observability Gramians over a stationary random packet loss process to establish the equivalence between stability in stopping times and stability in sampling times [32], and characterizing the decay rate of the prediction covariance's tail distribution for so-called nondegenerate systems [34].
3) We prove that there is a critical $p-q$ curve, with p being the failure rate and q being the recovery rate of the Gilbert–Elliott channel, below which the expected prediction error covariance matrices are uniformly bounded and unbounded above. The existence of a critical curve for Markovian packet losses is an extension of that of the critical packet loss rate subject to i.i.d. packet losses in [16]. However, the proof method of [16] does not apply to Markovian packet losses. In this paper, the critical curve is proved via a novel *coupling* argument, and to the best of our knowledge, this is the first time phase transition is established for Kalman filtering over Markovian channels. Finally, we present a lower bound for the critical failure rate, making use of the relationship between the two stability criteria we established. This lower bound holds without relying on the restriction that the system matrix has no defective eigenvalues on the unit circle. In other words, we obtain an MSS condition for *general* linear time-invariant (LTI) systems under Markovian packet drops.

We believe these results add to the fundamental understanding of Kalman filtering under random packet drops.

The remainder of this paper is organized as follows. Section II presents the problem setup. Section III focuses on the PCS. Section IV studies the relationship between the peak-covariance and mean-square stabilities, the critical $p-q$ curve, and presents a sufficient condition for MSS of general LTI systems. Section V demonstrates the effectiveness of our approach compared with the literature using two numerical examples. Finally, Section VI concludes this paper.

Notations: $\mathbb{N}$ is the set of positive integers. $\mathbb{S}_+^n$ is the set of n by n positive semidefinite matrices over the complex field. For a matrix X, $\sigma(X)$ denotes the spectrum of X and λ_X denotes the eigenvalue of X that has the largest magnitude. X^*, X', and $\overline{X}$ are the Hermitian conjugate, transpose, and complex conjugate of X, respectively. Moreover, $\|\cdot\|$ means the 2-norm of a vector or the induced 2-norm of a matrix. $\otimes$ is the Kronecker product of two matrices. The indicator function of a subset $\mathcal{A} \subset \Omega$ is a function $1_{\mathcal{A}} : \Omega \to \{0,1\}$, where $1_{\mathcal{A}}(\omega) = 1$

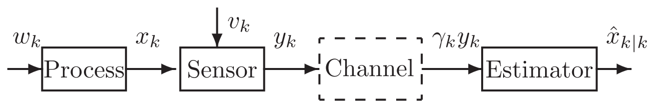

Fig. 1. Estimation over an erasure channel.

if $\omega \in \mathcal{A}$, otherwise $1_{\mathcal{A}}(\omega) = 0$. For random variables, $\sigma(\cdot)$ is the σ-algebra generated by the variables.

II. Kalman Filtering With Markovian Packet Losses

Consider an LTI system

$$x_{k+1} = Ax_k + w_k \tag{1a}$$

$$y_k = Cx_k + v_k \tag{1b}$$

where $A \in \mathbb{R}^{n\times n}$ is the system matrix and $C \in \mathbb{R}^{m\times n}$ is the observation matrix, $x_k \in \mathbb{R}^n$ is the process state vector and $y_k \in \mathbb{R}^m$ is the observation vector, $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are zero-mean Gaussian random vectors with autocovariance $\mathbb{E}[w_k w_j{}'] = \delta_{kj} Q$ $(Q \geq 0)$, $\mathbb{E}[v_k v_j{}'] = \delta_{kj} R$ $(R > 0)$, $\mathbb{E}[w_k v_j{}'] = 0\ \forall j, k$. Here, δ_{kj} is the Kronecker delta function with $\delta_{kj} = 1$ if $k = j$ and $\delta_{kj} = 0$ otherwise. The initial state x_0 is a zero-mean Gaussian random vector that is uncorrelated with w_k and v_k and has covariance $\Sigma_0 \geq 0$. We assume that (C, A) is detectable and $(A, Q^{1/2})$ is stabilizable. By applying a similarity transformation, the unstable and stable modes of the considered LTI system can be decoupled. An open-loop prediction of the stable mode always has a bounded estimation error covariance, therefore, this mode does not play any key role in the stability issues considered in this paper. Without loss of generality, we assume that (A1) *All the eigenvalues of A have magnitudes no less than 1.* Certainly A is nonsingular.

We consider an estimation scheme illustrated in Fig. 1, where the raw measurements of the sensor $\{y_k\}_{k\in\mathbb{N}}$ are transmitted to the estimator via an erasure communication channel over which packets may be dropped randomly. Denote by $\gamma_k \in \{0, 1\}$ the arrival of y_k at time k: y_k arrives errorfree at the estimator if $\gamma_k = 1$; otherwise $\gamma_k = 0$. Whether γ_k takes value 0 or 1 is assumed to be known by the receiver at time k. Define $\mathcal{F}_k$ as the filtration generated by all the measurements received by the estimator up to time k, i.e., $\mathcal{F}_k \triangleq \sigma\,(\gamma_t y_t, \gamma_t; 1 \leq t \leq k)$ and $\mathcal{F} = \sigma\,(\cup_{k=1}^{\infty} \mathcal{F}_k)$. We will use a triple $(\Omega, \mathcal{F}, \mathbb{P})$ to denote the probability space capturing all the randomness in the model.

To describe the temporal correlation of realistic communication channels, we assume the Gilbert–Elliott channel [23], [24], where the packet loss process is a time-homogeneous two-state Markov chain. To be precise, $\{\gamma_k\}_{k\in\mathbb{N}}$ is the state of the Markov chain with initial condition, without loss of generality, $\gamma_1 = 1$. The transition probability matrix for the Gilbert–Elliott channel is given by

$$\mathbf{P} = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix} \tag{2}$$

where $p \triangleq \mathbb{P}(\gamma_{k+1} = 0 | \gamma_k = 1)$ is called the failure rate, and $q \triangleq \mathbb{P}(\gamma_{k+1} = 1 | \gamma_k = 0)$ the recovery rate. Assume that (A2) *The failure and recovery rates satisfy* $p, q \in (0, 1)$.

The estimator computes $\hat{x}_{k|k}$, the minimum mean-squared error estimate, and $\hat{x}_{k+1|k}$, the one-step prediction, according to $\hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{F}_k]$ and $\hat{x}_{k+1|k} = \mathbb{E}[x_{k+1}|\mathcal{F}_k]$. Let $P_{k|k}$ and $P_{k+1|k}$ be the corresponding estimation and prediction error covariance matrices, i.e., $P_{k|k} = \mathbb{E}[(x_k - \hat{x}_{k|k})(\cdot)'|\mathcal{F}_k]$ and $P_{k+1|k} = \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k})(\cdot)'|\mathcal{F}_k]$. They can be computed recursively via a modified Kalman filter [16]. The recursions for $\hat{x}_{k|k}$ and $\hat{x}_{k+1|k}$ are omitted here. To study the Kalman filtering system's stability, we focus on the prediction error covariance matrix $P_{k+1|k}$, which is recursively computed as

$$\begin{aligned} P_{k+1|k} = {} & AP_{k|k-1}A' + Q \\ & - \gamma_k AP_{k|k-1}C'(CP_{k|k-1}C' + R)^{-1}CP_{k|k-1}A'. \end{aligned}$$

It can be seen that $P_{k+1|k}$ inherits the randomness of $\{\gamma_t\}_{1\leq t\leq k}$. We focus on characterizing the impact of $\{\gamma_k\}_{k\in\mathbb{N}}$ on $P_{k+1|k}$. To simplify notations in the sequel, let $P_{k+1} \triangleq P_{k+1|k}$, and define the functions h, g, h^k, and $g^k\colon \mathbb{S}_+^n \to \mathbb{S}_+^n$ as follows

$$h(X) \triangleq AXA' + Q \tag{3}$$

$$g(X) \triangleq AXA' + Q - AXC'(CXC' + R)^{-1}CXA'$$

$$h^k(X) \triangleq \underbrace{h \circ h \circ \cdots \circ h}_{k\text{times}}(X) \text{ and } g^k(X) \triangleq \underbrace{g \circ g \circ \cdots \circ g}_{k\text{times}}(X) \tag{4}$$

where $\circ$ denotes the function composition. The following stability notion is standard.

Definition 1: The Kalman filtering system with packet losses is mean-square stable if $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| < \infty$.

Define

$$\begin{aligned} \tau_1 &\triangleq \min\{k : k \in \mathbb{N}, \gamma_k = 0\} \\ \beta_1 &\triangleq \min\{k : k > \tau_1, \gamma_k = 1\} \\ &\ \ \vdots \\ \tau_j &\triangleq \min\{k : k > \beta_{j-1}, \gamma_k = 0\} \\ \beta_j &\triangleq \min\{k : k > \tau_j, \gamma_k = 1\}. \end{aligned} \tag{5}$$

It is straightforward to verify that $\{\tau_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ are two sequences of stopping times because both $\{\tau_j \leq k\}$ and $\{\beta_j \leq k\}$ are $\mathcal{F}_k$−measurable; see [38] for details. Due to the strong Markov property and the ergodic property of the Markov chain defined by (2), the sequences $\{\tau_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ have finite values $\mathbb{P}$-almost surely. Then, we can define the sojourn times at the state 1 and state 0, respectively, by τ_j^* and β_j^* $\forall j \in \mathbb{N}$ as

$$\tau_j^* \triangleq \tau_j - \beta_{j-1} \text{ and } \beta_j^* \triangleq \beta_j - \tau_j$$

where $\beta_0 = 1$ by convention. A result given by [29, Lemma 2] demonstrates that $\{\tau_k^*\}_{j\in\mathbb{N}}$ and $\{\beta_k^*\}_{j\in\mathbb{N}}$ are mutually independent and have geometric distribution. Let us denote the prediction error covariance matrix at the stopping time β_j by P_{β_j} and

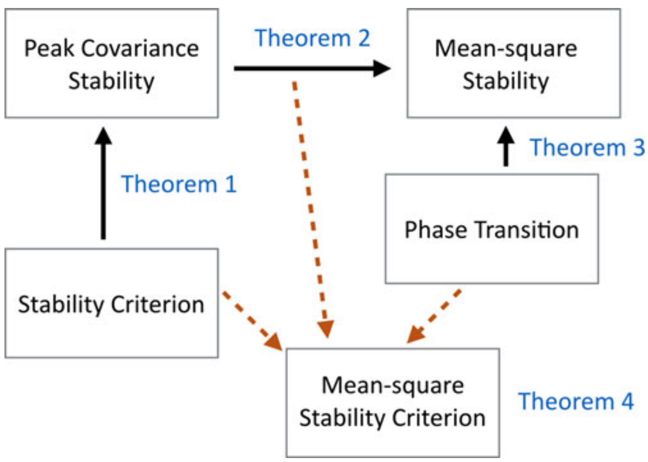

Fig. 2. Road map of this work.

call it the peak covariance[1] at β_j. We introduce the notion of PCS [29] as follows:

Definition 2: The Kalman filtering system with packet losses is said to be peak-covariance stable if $\sup_{j\in\mathbb{N}} \mathbb{E}\|P_{\beta_j}\| < \infty$.

Remark 1: In [32], the authors defined stability in stopping times as the stability of P_k at packet reception times. Note that $\{\beta_j\}_{j\in\mathbb{N}}$, at which the peak covariance is defined, can also be treated as the stopping times defined on packet reception times. Clearly, in scalar systems, the covariance is at maximum when the channel just recovers from failed transmissions; therefore, peak covariance sequence gives an upper envelop of covariance matrices at packet reception times. For higher order systems, the relation between them is still unclear.

The results of this paper are organized as follows (cf., Fig. 2). First of all, we present a relaxed condition for PCS (Theorem 1). Then, we establish a result indicating that for general packet drop processes PCS implies MSS under mild conditions (Theorem 2). We continue to show that MSS inherits a sharp phase transition reflected by a critical $p - q$ curve (Theorem 3). We finally manage to combine all these results and conclude an MSS condition for general LTI systems (Theorem 4).

III. Peak-Covariance Stability

In this section, we study the PCS [29] of the Kalman filter. (9) shown at the bottom of this page.

We introduce the observability index of the pair (C, A).

[1]The definition of peak covariance was first introduced in [29], where the term "peak" was attributed to the fact that for an unstable scalar system P_k monotonically increases to reach a local maximum at time β_j. This maximum property does not necessarily hold for the multidimensional case.

Definition 3: The observability index $\mathsf{I_o}$ is defined as the smallest integer such that $[C', A'C', \ldots, (A^{\mathsf{I_o}-1})'C']'$ has rank n. If $\mathsf{I_o} = 1$, the system (C, A) is called one-step observable.

We also introduce the operator $\mathcal{L}_K : \mathbb{S}_+^n \to \mathbb{S}_+^n$ defined as

$$\mathcal{L}_K(X) = p\sum_{i=1}^{\mathsf{I_o}-1}(1-p)^{i-1}(A^i + K^{(i)}C^{(i)})^*\Phi_X(A^i + K^{(i)}C^{(i)}) \tag{6}$$

where Φ_X is the positive definite solution of the Lyapunov equation $(1-q)A'\Phi_X A + qA'XA = \Phi_X$ with $|\lambda_A|^2(1-q) < 1$, and $K = [K^{(1)}, \ldots, K^{(\mathsf{I_o}-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions. It can be easily shown that $\mathcal{L}_K(X)$ is linear and nondecreasing in the positive semidefinite cone.

We have the following result.

Theorem 1: Suppose $|\lambda_A|^2(1-q) < 1$. If any of the following conditions hold.

i) $\exists\, K \triangleq [K^{(1)}, \ldots, K^{(\mathsf{I_o}-1)}]$, where $K^{(i)}$s are matrices with compatible dimensions, such that $|\lambda_{H(K)}| < 1$, where

$$\begin{aligned} H(K) = {} & qp\Big[(A\otimes A)^{-1} - (1-q)I\Big]^{-1} \\ & \cdot\sum_{i=1}^{\mathsf{I_o}-1}\overline{(A^i + K^{(i)}C^{(i)})} \\ & \otimes (A^i + K^{(i)}C^{(i)})(1-p)^{i-1}. \end{aligned} \tag{7}$$

ii) There exists $K \triangleq [K^{(1)}, \ldots, K^{(\mathsf{I_o}-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions such that $\lim_{k\to\infty}\mathcal{L}_K^k(X) = 0$ for any $X \in \mathbb{S}_+^n$.

iii) There exist $K \triangleq [K^{(1)}, \ldots, K^{(\mathsf{I_o}-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions and $P > 0$ such that $\mathcal{L}_K(P) < P$.

iv) There exist $F_1, \ldots, F_{\mathsf{I_o}-1}, X > 0, Y > 0$ such that

$$\begin{bmatrix} Y & \sqrt{1-q}A'Y & \sqrt{q}A'X \\ \sqrt{1-q}YA & Y & 0 \\ \sqrt{q}XA & 0 & X \end{bmatrix} \geq 0 \tag{8}$$

and $\Psi > 0$ where Ψ is given in (9).

Then, $\sup_{j\geq 1}\mathbb{E}\|P_{\beta_j}\| < \infty$, i.e., the Kalman filtering system is peak-covariance stable.

The proof of Theorem 1 is given in Appendix B. For condition (i) of Theorem 1, a quite heavy computational overhead may be incurred in searching for a satisfactory K. Condition (iv), as an LMI interpretation of condition (i), makes it possible to check the sufficient condition for PCS through an LMI feasibility criterion.

Remark 2: Theorem 1 establishes a direct connection between $\lambda_{H(K)}$, p, q, the most essential aspects of the system

$$\Psi \triangleq \begin{bmatrix} X & \sqrt{p}(A'Y + C'F_1) & \cdots & \sqrt{p(1-p)^{I_o-2}}\left((A^{I_o-1})'Y + (C^{(i)})'F_{I_o-1}\right) \\ \sqrt{p}(YA + F_1^*C) & Y & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{p(1-p)^{I_o-2}}\left(YA^{I_o-1} + F_{I_o-1}^*C^{(i)}\right) & 0 & \cdots & Y \end{bmatrix} \tag{9}$$

dynamic and channel characteristics on the one hand, and PCS on the other hand. These results cover the ones in [28], [29], and [31], as is evident using the subadditivity property of matrix norm, and the fact that the spectral radius is the infimum of all possible matrix norms. To see this, one should notice that

$$\begin{aligned}
&|\lambda_{H(k)}| \\
&\leq \; qp\left\|\left[(A\otimes A)^{-1}-(1-q)I\right]^{-1}\right\| \\
&\quad \sum_{i=1}^{\mathsf{l_o}-1}(1-p)^{i-1}\left\|\overline{(A^i+K^{(i)}C^{(i)})}\otimes(A^i+K^{(i)}C^{(i)})\right\| \\
&\leq \; qp\sum_{i=1}^{\infty}(1-q)^{i-1}\|A^i\otimes A^i\| \\
&\quad \sum_{i=1}^{\mathsf{l_o}-1}(1-p)^{i-1}\left\|\overline{(A^i+K^{(i)}C^{(i)})}\otimes(A^i+K^{(i)}C^{(i)})\right\| \\
&= \; q\sum_{i=1}^{\infty}(1-q)^{i-1}\|A^i\|^2 p\sum_{i=1}^{\mathsf{l_o}-1}(1-p)^{i-1}\|A^i+K^{(i)}C^{(i)}\|^2
\end{aligned}$$

in which the first inequality follows from $|\lambda_{H(K)}|\leq\|H(K)\|$ and the submultiplicative property of matrix norms, and the last equality holds because, for a matrix X, $\|\overline{X}^i\otimes X^i\|=\sqrt{\lambda_{(\overline{X}^i(X')^i)\otimes(X^i(X^*)^i)}}=\lambda_{X^i(X^*)^i}=\|X^i\|^2$. Comparison with the related results in the literature is also demonstrated by Example I in Section V.

In addition, in [28], [29], and [31], the criteria for PCS are difficult to check since some constants related to the operator g are hard to explicitly compute. A thorough numerical search may be computationally demanding. In contrast, the stability check of Theorem 1 uses an LMI feasibility problem, which can often be efficiently solved.

In the following proposition, we present another condition for PCS, which is, despite being conservative, easy to check. The new condition is obtained by making all $K^{(i)}$s in Theorem 1 take the value zero.

Proposition 1: If the following condition is satisfied

$$pq|\lambda_A|^2\sum_{i=1}^{\mathsf{l_o}-1}|\lambda_A|^{2i}(1-p)^{i-1}<1-|\lambda_A|^2(1-q), \qquad (10)$$

then the Kalman filtering system is peak-covariance stable.

Proof: The proof requires the following lemma.

Lemma 1 ([39, Th. 1.1.6]): Let $\mathfrak{p}(\cdot)$ be a given polynomial. If λ is an eigenvalue of a matrix A, then, $\mathfrak{p}(\lambda)$ is an eigenvalue of the matrix $\mathfrak{p}(A)$.

Define a sequence of polynomials of the matrix $A\otimes A$ as $\{\mathfrak{p}_n(A\otimes A)\}_{n\in\mathbb{N}}$, where

$$\begin{aligned}\mathfrak{p}_n(A\otimes A) = &\sum_{i=1}^{n}(A\otimes A)^i(1-q)^{i-1}q \\ &\times\sum_{j=1}^{\mathsf{l_o}-1}(A\otimes A)^j(1-p)^{j-1}p.\end{aligned}$$

In light of Lemma 1, the spectrum of $\mathfrak{p}_n(A\otimes A)$ is given by $\sigma(\mathfrak{p}_n(A\otimes A))=\{\mathfrak{p}_n(\lambda_i\lambda_j):\ \lambda_i,\lambda_j\in\sigma(A)\}$. Since A is a real matrix, its complex eigenvalues, if any, always occur in conjugate pairs. Therefore, $|\lambda_A|^2$ must be an eigenvalue of $A\otimes A$, and the spectral radius of $\mathfrak{p}_n(A\otimes A)$ can be computed as $|\lambda_{\mathfrak{p}_n(A\otimes A)}|=\sum_{i=1}^{n}|\lambda_A|^{2i}(1-q)^{i-1}q\sum_{j=1}^{\mathsf{l_o}-1}|\lambda_A|^{2j}(1-p)^{j-1}p$. It is evident that the sequence $\{|\lambda_{\mathfrak{p}_n(A\otimes A)}|\}_{n\in\mathbb{N}}$ is monotonically increasing. When $|\lambda_A|^2(1-q)<1$, we have

$$\lim_{n\to\infty}\mathfrak{p}_n(A\otimes A)=H(0) \qquad (11)$$

and

$$\lim_{n\to\infty}|\lambda_{\mathfrak{p}_n(A\otimes A)}|=\frac{q|\lambda_A|^2}{1-|\lambda_A|^2(1-q)}\sum_{j=1}^{\mathsf{l_o}-1}|\lambda_A|^{2j}(1-p)^{j-1}p. \qquad (12)$$

As $|\lambda_X|$ is continuous with respect to X, (11) and (12) altogether lead to $|\lambda_{H(0)}|=\frac{q|\lambda_A|^2}{1-|\lambda_A|^2(1-q)}\sum_{j=1}^{\mathsf{l_o}-1}|\lambda_A|^{2j}(1-p)^{j-1}p$. Letting $K^{(i)}=0\ \forall 1\leq i\leq \mathsf{l_o}-1$, the condition provided in Theorem 1 becomes: (i) $|\lambda_A|^2(1-q)<1$, and (ii) $|\lambda_{H(0)}|<1$. Since the left side of (10) is positive, it imposes the positivity of $1-|\lambda_A|^2(1-q)$, whereby the conclusion follows. ■

Although computationally friendly, Proposition 1 only provides a comparably rough criterion. It can be expected that, given the ability of searching for $K^{(i)}$s on the positive semidefinite cone, Theorem 1 is less conservative than Proposition 1; this is demonstrated by Example I in Section V.

Remark 3: The left side of (10) is strictly positive when $\mathsf{l_o}\geq 2$, while it vanishes when $\mathsf{l_o}=1$. In the latter case, plus the necessity as shown in [30], $|\lambda_A|^2(1-q)<1$ thereby becomes a necessary and sufficient condition for PCS. This observation is consistent with the conclusion of [31, Corollary 2].

Remark 4: Proposition 1 covers the result of [30, Th. 3.1]. To see this, notice that

$$\begin{aligned}
&\frac{q|\lambda_A|^2}{1-|\lambda_A|^2(1-q)}\sum_{j=1}^{\mathsf{l_o}-1}|\lambda_A|^{2j}(1-p)^{j-1}p \\
&=\sum_{i=1}^{\infty}|\lambda_A|^{2i}(1-q)^{i-1}q\sum_{j=1}^{\mathsf{l_o}-1}|\Lambda_A|^{2j}(1-p)^{j-1}p \\
&\leq\sum_{i=1}^{\infty}\|A^i\|^2(1-q)^{i-1}q\sum_{j=1}^{\mathsf{l_o}-1}\|A^j\|^2(1-p)^{j-1}p,
\end{aligned}$$

which implies the less conservativity of [30, Th. 3.1].

IV. Mean-Square Stability

In this section, we will discuss MSS of Kalman filtering with Markovian packet losses.

A. From PCS to MSS

Note that the PCS characterizes the filtering system at stopping times defined by (5), whereas MSS characterizes the property of stability at all sampling times. In the literature, the relationship between the two stability notations is still an open problem. In this section, we aim to establish a connection between PCS and MSS. First, we need the following definition for the defective eigenvalues of a matrix.

Definition 4: For $\lambda \in \sigma(A)$ where A is a matrix, if the algebraic multiplicity and the geometric multiplicity of λ are equal, then λ is called a semisimple eigenvalue of A. If λ is not semisimple, λ is called a defective eigenvalue of A.

We are now able to present the following theorem indicating that as long as A has no defective eigenvalues on the unit circle, i.e., the corresponding Jordan block is 1×1, PCS always implies MSS. In fact, we are going to prove this connection for *general* random packet drop processes $\{\gamma_k\}_{k\in\mathbb{N}}$, instead of limiting to the Gilbert–Elliott model.

Theorem 2: Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a random process over an underlying probability space $(\mathscr{S}, \mathcal{S}, \mu)$ with each γ_k taking its value in $\{0, 1\}$. Suppose $\{\beta_j\}_{j\in\mathbb{N}}$ take finite values $\mu-$almost surely, and that A has no defective eigenvalues on the unit circle. Then, the PCS of the Kalman filter always implies MSS, i.e., $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| < \infty$ whenever $\sup_{j\in\mathbb{N}} \mathbb{E}\|P_{\beta_j}\| < \infty$.

Note that $\{\beta_j\}_{j\in\mathbb{N}}$ can be defined over any random packet loss processes, therefore, the PCS with packet losses that the filtering system is undergoing remains in accord with Definition 2.

Theorem 2 bridges the two stability notions of Kalman filtering with random packet losses in the literature. Particularly, this connection covers most of the existing models for packet losses, e.g., i.i.d. model [16], bounded Markovian [40], Gilbert–Elliott [28], and finite-state channel [25], [26]. Although $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\|$ and $\sup_{j\in\mathbb{N}} \mathbb{E}\|P_{\beta_j}\|$ are not equal in general, this connection is built upon a critical understanding that, no matter to which interarrival interval between two successive β_js the time k belongs, $\|P_k\|$ *is uniformly bounded from above by an affine function of the norm of the peak covariances at the starting and ending points thereof.* This point holds regardless of the model of packet loss process. The proof of Theorem 2 was given in Appendix C.

We also remark that there is difficulty in relaxing the assumption that A has no defective eigenvalues on the unit circle in Theorem 2. This is due to the fact that As defective eigenvalues on the unit circle will influence both the PCS and MSS in a nontrivial manner (see Fig. 3).

Remark 5: In [29], for a scalar model with i.i.d. packet losses, it has been shown that the PCS is equivalent to MSS, whereas for a vector system even with i.i.d. packet losses, the relationship between the two is unclear. In [31], the equivalence between the two stability notions was established for systems that are one-step observable, again for the i.i.d. case. Theorem 2 now fills the gap for a large class of vector systems under general random packet drops.

B. The Critical $p - q$ Curve

In this section, we first show that for a fixed q in the Gilbert–Elliott channel, there exists a critical failure rate p_c, such that if and only if the failure rate is below p_c, the Kalman filtering is mean-square stable. This conclusion is relatively independent of previous results, and the proof relies on a coupling argument and can be found in Appendix D.

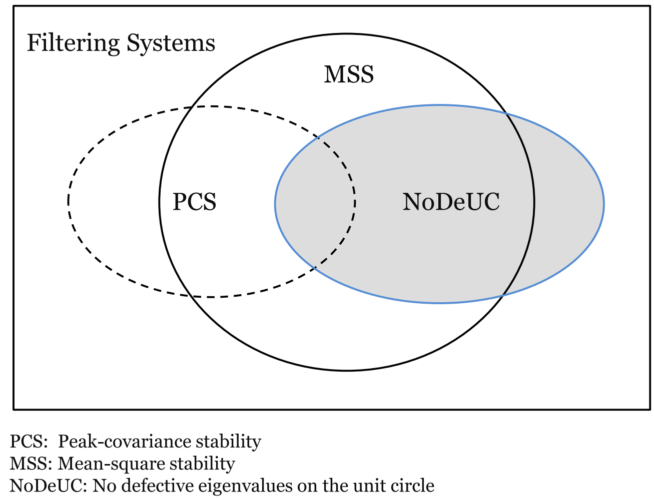

Fig. 3. Relationships between the PCS and MSS over the space of the filtering systems under consideration. Theorem 2 indicates that the intersection of the two sets, PCS and NoDEUC, is contained in the set MSS.

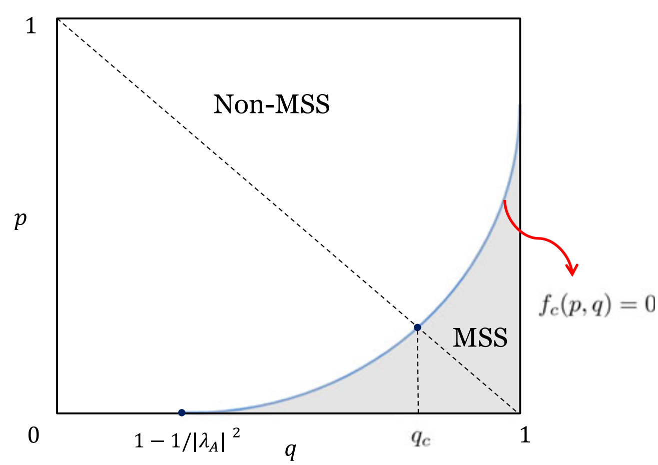

Fig. 4. $p - q$ plane is divided into MSS and non-MSS regions by the critical curve $f_c(p, q) = 0$. When $p + q = 1$, the Markovian packet loss process is reduced to an i.i.d. process. As a result, the intersection point of the curves $f_c(p, q) = 0$ and $p + q = 1$ gives the critical packet drop probability established in [16].

Proposition 2: Let the recovery rate q satisfy $|\lambda_A|^2(1 - q) < 1$. Then, there exists a critical value $p_c \in (0, 1]$ for the failure rate in the sense that

i) $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| < \infty$ for all $\Sigma_0 \geq 0$ and $0 < p < p_c$; and

ii) For each $p \in (p_c, 1)$, there exists $\Sigma_0 \geq 0$ such that $\sup_{k\in\mathbb{N}} \mathbb{E}\|P_k\| = \infty$.

It has been shown in [32] that a necessary condition for MSS of the filtering system is $|\lambda_A|^2(1 - q) < 1$, which is only related to the recovery rate q. For Gilbert–Elliott channels, a critical value phenomenon with respect to q is also expectable. Theorem 3 proves the existence of the critical $p - q$ curve and Fig. 4 illustrates this critical curve in the $p - q$ plane. The proof, analogous to that of Proposition 2, is given in Appendix E.

Theorem 3: There exists a critical curve defined by $f_c(p,q)=0$, which reads two nondecreasing functions $p=p_c(q)$ and $q=q_c(p)$ with $q_c(\cdot)=p_c^{-1}(\cdot)$, dividing $(0,1)^2$ into two disjoint regions such that

i) If $(p,q)\in\{f_c(p,q)>0\}$, then, $\sup_{k\in\mathbb{N}}\mathbb{E}\|P_k\|<\infty$ for all $\Sigma_0\geq 0$;

ii) If $(p,q)\in\{f_c(p,q)<0\}$, then, there exists $\Sigma_0\geq 0$ under which $\sup_{k\in\mathbb{N}}\mathbb{E}\|P_k\|=\infty$.

Remark 6: If the packet loss process is an i.i.d. process, where $p+q=1$ in the transition probability matrix defined in (2), Proposition 2 and Theorem 3 recover the result of [16, Th. 2]. It is worth pointing out that whether MSS holds or not exactly on the curve $f_c(p,q)=0$ is beyond the reach of the current analysis (even for the i.i.d. case with $p+q=1$): such an understanding relies on the compactness of the stability or nonstability regions.

C. MSS Conditions

We can now make use of the PCS conditions we obtained in the last section, and the connection between PCS and MSS indicated in Theorem 2, to establish MSS conditions for the considered Kalman filter. *It turns out that the assumption requiring no defective eigenvalues on the unit circle can be relaxed by an approximation method.* We present the following result.

Theorem 4: Let the recovery rate q satisfy $|\lambda_A|^2(1-q)<1$. Then, there holds $\underline{p}\leq p_c$, where

$$\underline{p}\triangleq\sup\left\{p:\exists(K,P)\text{ s.t. }\mathcal{L}_K(P)<P, P>0\right\}\qquad(13)$$

i.e., for all $\Sigma_0\geq 0$ and $0<p<\underline{p}$, the Kalman filtering system is mean-square stable.

The proof of Theorem 4 is given in Appendix F.

Remark 7: For second-order systems and certain classes of higher order systems, such as nondegenerate systems, necessary and sufficient conditions for MSS have been derived in [32] and [34]. However, these results rely on a particular system structure and fail to apply to general LTI systems. Theorem 4 gives a stability criterion for general LTI systems.

V. Numerical Examples

In this section, we present two numerical examples to demonstrate the theoretical results we established in Sections III and IV.

A. Example I: A Second-order System

To compare with the works in [28]–[30], we will examine the same vector example considered therein. The parameters are specified as follows

$$A=\begin{bmatrix}1.3 & 0.3\\ 0 & 1.2\end{bmatrix},\quad C=[1,1]$$

$Q=I_{2\times2}$ and $R=1$. As illustrated in [29], it is easily checked that $\mathsf{l_o}=2$ and the spectrum of A is $\sigma(A)=\{1.2,1.3\}$, and that $\lambda_A=1.3$.

Note that $|\lambda_A|^2(1-q)<1$ is a necessary condition for MSS. We take $q=0.65$ as was done in [29]. As for the failure rate p, [29] concludes that $p<0.04$ guarantees PCS; while Proposition 1 requires $p<\frac{1-|\lambda_A|^2(1-q)}{|\lambda_A|^4 q}$, which generates a less conservative condition $p<0.22$. Similarly, the example in [30] allows $p=0.1191$ at most. We also note that it is rather convenient to check the condition in Proposition 1 even with manual calculation; in contrast, to check the conditions in [29] and [30] involves a considerable amount of numerical calculation. Additionally, we use the criterion established in Theorem 1 to check for the PCS. We obtain that when $p=1$, the LMI in (ii) of Theorem 1 is still feasible.[2] Fig. 5 illustrates sample paths of $\|P_k\|$ with $(p,q)=(0.99,0.65)$, which shows that even a high value of p may not affect the PCS in this example.

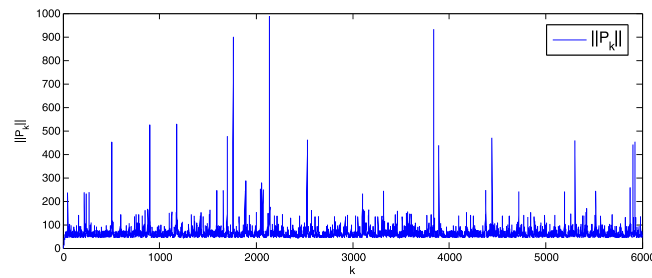

Fig. 5. Sample path of $\|P_k\|$ with $p=0.99$, $q=0.65$ in Example I.

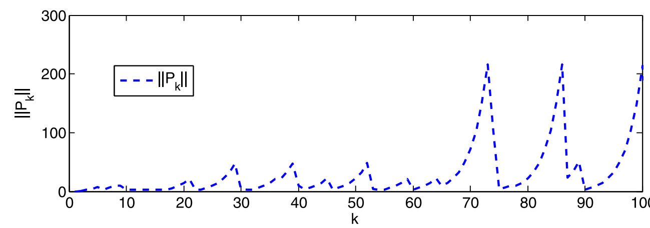

Fig. 6. Sample path of $\|P_k\|$ with $p=0.45$, $q=0.5$ in Example II.

B. Example II: A Third-order System

To compare the work in Section IV with the result of[32] and [34], we will use the following example, where the parameters are given by

$$A=\begin{bmatrix}1.2 & 0 & 0\\ 0 & 1.2 & 0\\ 0 & 0 & -1.2\end{bmatrix},\quad C=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}\qquad(14)$$

$Q=I_{3\times3}$ and $R=I_{2\times2}$.

1) Mean-Square Stability: The system described in (14) is observable and degenerate [22]. Therefore, the MSS conditions presented in [32] and [34] for nondegenerate systems are not applicable in this example. In contrast, our Theorem 4 provides a universal criterion for MSS. Fixing $q=0.5$, we can conclude from Theorem 4 that if $p\leq 0.465$ the Kalman filter is mean-square stable. Fig. 6 illustrates a stable sample path of $\|P_k\|$ with $(p,q)=(0.45,0.5)$. On the other hand, when $(p,q)=(0.99,0.5)$ the expected prediction error covariance matrices diverge (as illustrated in Fig. 7). One can check that

[2]To satisfy the assumption (A2), we need to configure $p=1-\epsilon$ for an arbitrary small positive ϵ.

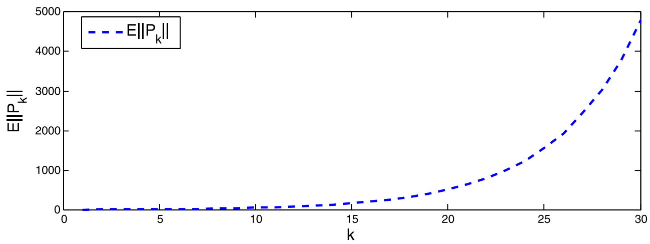

Fig. 7. Divergence of $\mathbb{E}\|P_k\|$ with $p = 0.99$, $q = 0.5$ in Example II.

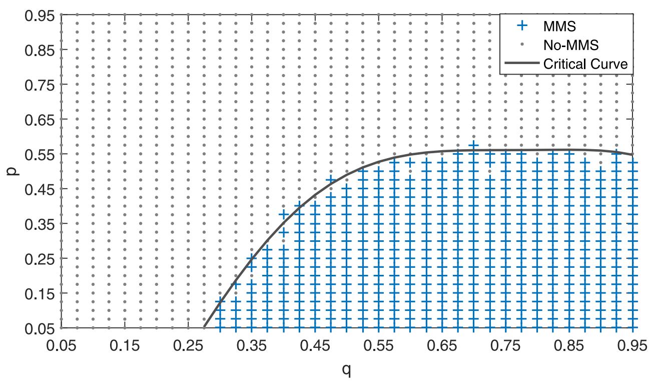

Fig. 8. Blue-colored crosses represent (p, q) points where the Kalman filter is mean-square stable; the gray dots represent (p, q) points at which the Kalman filter is mean-square unstable. Clearly a critical curve emerges.

when $(p, q) = (0.99, 0.5)$ the criterion in Theorem 4 is violated. This verifies the contrapositive of Theorem 4.

2) Critical Curve: We now illustrate the result of Theorem 3 by Monte Carlo simulations. We quantize the $p-q$ plane with a step size 0.025 along each axis. For each discretized (p, q) value, we run Monte Carlo simulations for $N = 100\,000$ repeated rounds, and each sample goes through 70 time steps. Noticing that without packet loss, the norm of the steady estimation error covariance is around 3.7, we use

$$P_{\text{app}} = \frac{1}{N}\sum_{i=1}^{N} \|P_k(\omega_i)\|$$

with $k = 70$ as the empirical approximation of $\mathbb{E}[P_k]$ and set a criterion $P_{\text{app}} \leq 100$ for MSS. Clearly the $p-q$ plane consists of a stable and another unstable regions, separated by an emerging critical curve (see Fig. 8).

VI. Conclusion

We have investigated the stability of Kalman filtering over Gilbert–Elliott channels. Random packet drops follow a time-homogeneous two-state Markov chain where the two states indicate successful or failed packet transmissions. We established a relaxed condition guaranteeing PCS described by an inequality in terms of the spectral radius of the system matrix and transition probabilities of the Markov chain, and then showed that the condition can be reduced to an LMI feasibility problem. It was proved that PCS implies MSS if the system matrix has no defective eigenvalues on the unit circle. This connection holds for general random packet drop processes. We also proved that there exists a critical region in the $p-q$ plane such that if and only if the pair of recovery and failure rates falls into that region the expected prediction error covariance matrices are uniformly bounded. By fixing the recovery rate, a lower bound for the critical failure rate was obtained making use of the relationship between two stability criteria for general LTI systems. Numerical examples demonstrated significant improvement on the effectiveness of our approach compared with the existing literature.

Appendix A
Auxiliary Lemmas

In this section, we collect some lemmas that are used in the proofs of our main results.

Lemma 2 (Lemma A.1 in [33]): For any matrices $X \geq Y \geq 0$, the following inequalities hold

$$h(X) \geq h(Y) \tag{15}$$

$$g(X) \geq g(Y) \tag{16}$$

$$h(X) \geq g(X) \tag{17}$$

where the operators h and g are defined in (3) and (4), respectively.

Lemma 3: Consider the operator

$$\phi_i(K^{(i)}, P) \triangleq (A^i + K^{(i)}C^{(i)})X(\cdot)^* + [A^{(i)}\ K^{(i)}]\begin{bmatrix} Q^{(i)} & Q^{(i)}(D^{(i)})' \\ * & D^{(i)}(Q^{(i)})(D^{(i)})' + R^{(i)} \end{bmatrix}[A^{(i)}\ K^{(i)}]^*$$

for all $i \in \mathbb{N}$, where $C^{(i)} = [C',\ A'C', \ldots, (A')^{i-1}C']'$, $A^{(i)} = [A^{i-1}, \ldots, A,\ I]$, $D^{(i)} = 0$ for $i = 1$ otherwise

$$D^{(i)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-2} & CA^{i-3} & \cdots & 0 \end{bmatrix},$$

$Q^{(i)} = \operatorname{diag}(\underbrace{Q, \ldots, Q}_{i})$, $R^{(i)} = \operatorname{diag}(\underbrace{R, \ldots, R}_{i})$, and $K^{(i)}$ are of compatible dimensions. For any $X \geq 0$ and $K^{(i)}$, it always holds that $g^i(X) = \min_{K^{(i)}} \phi_i(K^{(i)}, X) \leq \phi_i(K^{(i)}, X)$.

Proof: The result is readily established when setting $B = I$ in [40, Lemmas 2 and 3]. For $i = 1$, The result is well known as in [16, Lemma 1].

Lemma 4 ([41]): For any $A \in \mathbb{C}^{n \times n}$, $\epsilon > 0$ and $k \in \mathbb{N}$, it holds that $\|A^k\| \leq \sqrt{n}(1 + 2/\epsilon)^{n-1}\Big(|\lambda_A| + \epsilon\|A\|\Big)^k$.

Lemma 5 (Lemma 2 in [42]): For any $G \in \mathbb{C}^{n \times n}$, there exist $G_i \in \mathbb{S}_+^n$, $i = 1, 2, 3, 4$ such that $G = (G_1 - G_2) + (G_3 - G_4)\mathbf{i}$, where $\mathbf{i} = \sqrt{-1}$.

Appendix B
Proof of Theorem 1

The proof is lengthy and is divided into two parts. In the first part, we will show that condition i) is a sufficient condition for

PCS. In the second part, we will show the equivalence between conditions i)–iv). The proof relies on the following two lemmas.

Lemma 6 ([29, Lemma 5]): Assume that (C, A) is observable and $(A, Q^{1/2})$ is controllable. Define

$$\mathbb{S}_0^n \triangleq \{P : 0 \le P \le AP_0A' + Q, \text{ for some } P_0 \ge 0\}.$$

Then, there exists a constant $\mathsf{L} > 0$ such that

i) for any $X \in \mathbb{S}_0^n$, $g^k(X) \le \mathsf{L}I$ for all $k \ge \mathsf{l_o}$; and

ii) for any $X \in \mathbb{S}_+^n$, $g^{k+1}(X) \le \mathsf{L}I$ for all $k \ge \mathsf{l_o}$

where the operator g is defined in (4).

Lemma 7: For $q \in (0, 1)$ and $A \in \mathbb{R}^{n\times n}$, the series of matrices $\sum_{i=1}^{\infty}(A\otimes A)^i(1-q)^{i-1}q$ and $\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}(A\otimes A)^j(1-q)^{i-1}q$ converge if and only if $|\lambda_A|^2(1-q) < 1$.

Proof: First observe that $\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}(A\otimes A)^j(1-q)^{i-1}q = \sum_{i=0}^{\infty}(A\otimes A)^i(1-q)^i$. The geometric series generated by $(A\otimes A)(1-q)$ converges if and only if $|\lambda_{A\otimes A}|(1-q) < 1$. Therefore, the conclusion follows from the fact that $|\lambda_{A\otimes A}| = \max\{|\lambda_i\lambda_j| : \lambda_i, \lambda_j \in \sigma(A)\} = |\lambda_A|^2$. ■

Now, fix $j \ge 1$. First note that, for any $k \in [\tau_{j+1}, \beta_{j+1} - 1]$, $\gamma_k = 0$. Hence, we have

$$\begin{aligned} P_{\beta_{j+1}} &= \sum_{i=1}^{\infty} 1_{\{\beta_{j+1}^*=i\}} h^i(P_{\tau_{j+1}}) \\ &\triangleq \sum_{i=1}^{\infty} 1_{\{\beta_{j+1}^*=i\}} A^i P_{\tau_{j+1}}(A^i)' + \sum_{i=1}^{\infty} 1_{\{\beta_{j+1}^*=i\}} V_i \end{aligned} \quad (18)$$

where $V_i \triangleq \sum_{l=0}^{i-1} A^l Q(A^l)'$. Now, let us consider the interval $[\beta_j, \tau_{j+1} - 1]$ over which τ_{j+1}^* packets are successfully received. We will analyze the relationship between $P_{\tau_{j+1}}$ and P_{β_j} in two separate cases, which are $\tau_{j+1}^* \le \mathsf{l_o} - 1$ and $\tau_{j+1}^* \ge \mathsf{l_o}$. Computation yields the following result

$$\begin{aligned} P_{\tau_{j+1}} &= \sum_{i=1}^{\mathsf{l_o}-1} 1_{\{\tau_{j+1}^*=i\}} g^i(P_{\beta_j}) + \sum_{l=\mathsf{l_o}}^{\infty} 1_{\{\tau_{j+1}^*=l\}} g^l(P_{\beta_j}) \\ &\le \sum_{i=1}^{\mathsf{l_o}-1} 1_{\{\tau_{j+1}^*=i\}} \phi_i(K^{(i)}, P_{\beta_j}) + \mathsf{L}I \sum_{l=\mathsf{l_o}}^{\infty} 1_{\{\tau_{j+1}^*=l\}} \\ &= \sum_{i=1}^{\mathsf{l_o}-1} 1_{\{\tau_{j+1}^*=i\}} (A^i + K^{(i)}C^{(i)}) P_{\beta_j} (A^i + K^{(i)}C^{(i)})^* \\ &\quad + \mathsf{L}I \sum_{j=\mathsf{l_o}}^{\infty} 1_{\{\tau_{j+1}^*=l\}} + \sum_{i=1}^{\mathsf{l_o}-1} 1_{\{\tau_{j+1}^*=i\}} [A^i \ K^{(i)}] J_i [A^i \ K^{(i)}]^* \\ &\triangleq (A^i + K^{(i)}C^{(i)}) P_{\beta_j} (A^i + K^{(i)}C^{(i)})^* + U \end{aligned} \quad (19)$$

where $J_i \triangleq \begin{bmatrix} Q^{(i)} & Q^{(i)}(D^{(i)})' \\ D^{(i)}(Q^{(i)}) & D^{(i)}(Q^{(i)})(D^{(i)})' + R^{(i)} \end{bmatrix}$ and

$$U \triangleq \mathsf{L}I \sum_{j=\mathsf{l_o}}^{\infty} 1_{\{\tau_{j+1}^*=l\}} + \sum_{i=1}^{\mathsf{l_o}-1} 1_{\{\tau_{j+1}^*=i\}} [A^i \ K^{(i)}] J_i [A^i \ K^{(i)}]^*$$

is bounded. The first inequality is from Lemmas 3 and 6. By substituting (19) into (18), it yields

$$\begin{aligned} P_{\beta_{j+1}} &\le W + \sum_{i=1}^{\infty} 1_{\{\beta_{j+1}^*=i\}} A^i \\ &\quad \times \left[\sum_{l=1}^{\mathsf{l_o}-1} 1_{\{\tau_{j+1}^*=l\}} (\cdot) P_{\beta_j} (A^l + K^{(l)}C^{(l)})^* \right] (A^i)' \end{aligned} \quad (20)$$

where $W \triangleq \sum_{i=1}^{\infty} 1_{\{\beta_{j+1}^*=i\}} A^i U (A^i)' + \sum_{i=1}^{\infty} 1_{\{\beta_{j+1}^*=i\}} V_i$. To facilitate discussion, we force $P_{\beta_{j+1}}$ in (20) to take the maximum. For other cases in (20), the subsequent conclusion still holds as it renders an upper envelop of $\{P_{\beta_j}\}_{j\in\mathbb{N}}$ by imposing (20) to take equality.

We introduce the vectorization operator. Let $X = [x_1 \ x_2 \ \cdots \ x_n] \in \mathbb{C}^{m\times n}$ where $x_i \in \mathbb{C}^m$. Then, we define $\text{vec}(X) \triangleq [x_1', x_2', \ldots, x_n']' \in \mathbb{C}^{mn}$. Notice that $\text{vec}(AXB) = (B' \otimes A)\text{vec}(X)$. For Kronecker product, we have $(A_1A_2) \otimes (B_1B_2) = (A_1 \otimes B_1)(A_2 \otimes B_2)$. Take expectations and vectorization operators over both sides of (20). From [29, Lemma 2], we obtain

$$\begin{aligned} &\mathbb{E}[\text{vec}(P_{\beta_{j+1}})] = \mathbb{E}[\text{vec}(W)] \\ &\quad + \sum_{i=1}^{\infty} (A\otimes A)^i (1-q)^{i-1} q \\ &\quad \cdot \sum_{l=1}^{\mathsf{l_o}-1} \overline{(A^l + K^{(l)}C^{(l)})} \otimes (\cdot) p(1-p)^{l-1}\, \mathbb{E}[\text{vec}(P_{\beta_j})]. \end{aligned} \quad (21)$$

In the above-mentioned equation $\mathbb{E}[\text{vec}(W)]$ can be written as

$$\begin{aligned} \mathbb{E}[\text{vec}(W)] &= \sum_{i=1}^{\infty} (A\otimes A)^i (1-q)^{i-1} q\, \text{vec}(U) \\ &\quad + \sum_{i=1}^{\infty}\sum_{l=0}^{i-1} (A\otimes A)^l (1-q)^{i-1} q\, \text{vec}(Q). \end{aligned} \quad (22)$$

In Lemma 7, we show that both of the two terms in (22) converge if $|\lambda_A|^2(1-q) < 1$.

For $j = 1$, following the similar argument as abovementioned, we have

$$\begin{aligned} \mathbb{E}\|P_{\beta_1}\| \le\ & \mathbb{E}\|P_{\tau_1}\| \sum_{i=1}^{\infty} \|A^i\|^2 (1-q)^{i-1} q \\ & + \left\| \sum_{i=1}^{\infty} (1-q)^{i-1} q V_i \right\| \end{aligned}$$

where V_is are defined in (18). Moreover, by Lemma 6 and (17), it holds that

$$\begin{aligned} \mathbb{E}\|P_{\tau_1}\| \le\ & \|\mathsf{L}I\| + \sum_{i=1}^{\mathsf{l_o}} \|g^i(\Sigma_0)\| (1-p)^{i-1} p \\ \le\ & \|\mathsf{L}I\| + \|\Sigma_0\| \sum_{i=1}^{\mathsf{l_o}} \|A^i\|^2 (1-p)^{i-1} p + \|V_{\mathsf{l_o}}\| \end{aligned}$$

showing that $\mathbb{E}\|P_{\tau_1}\|$ is bounded. To sum up, $\mathbb{E}\|P_{\beta_1}\|$ is bounded if $|\lambda_A|^2(1-q)<1$. By applying the Cauchy–Schwarz inequality to the inner product of random variables, the boundness of $\mathbb{E}\|P_{\beta_1}\|$ implies the boundness of each element of $\mathbb{E}[P_{\beta_1}]$. So is $\mathbb{E}[\mathrm{vec}(P_{\beta_1})]$ if $|\lambda_A|^2(1-q)<1$.

We have shown that $\mathbb{E}[\mathrm{vec}(P_{\beta_j})]$ for $j\in\mathbb{N}$ evolves following (21), and that $\mathbb{E}[\mathrm{vec}(W)]$ in (21) and $\mathbb{E}[\mathrm{vec}(P_{\beta_1})]$ are bounded if $|\lambda_A|^2(1-q)<1$. We conclude that if $|\lambda_A|^2(1-q)<1$ and there exists an $K\triangleq[K^{(1)},\ldots,K^{(\mathsf{l_o}-1)}]$ such that $|\lambda_{H(K)}|$ defined in (7) is less than 1, then the spectral radius of

$$\sum_{i=1}^{\infty}(A\otimes A)^i(1-q)^{i-1}q\sum_{l=1}^{\mathsf{l_o}-1}\overline{(A^l+K^{(l)}C^{(l)})}\otimes(\cdot)(1-p)^{l-1}p$$

is less than 1, all the above-mentioned observations lead to $\sup_{j\geq1}\mathbb{E}[\mathrm{vec}_i(P_{\beta_j})]<\infty\ \forall 1\leq i\leq n^2$, where $\mathrm{vec}_i(X)$ represents the ith element of $\mathrm{vec}(X)$. In addition, there holds

$$\mathbb{E}\|P_{\beta_j}\|\leq\mathbb{E}\left[\mathrm{tr}(P_{\beta_j})\right]=[e_1',\ldots,e_n']\,\mathbb{E}[\mathrm{vec}(P_{\beta_j})]$$

where e_i denotes the vector with 1 in the ith coordinate and 0s elsewhere, so the desired result follows.

ii) $\Rightarrow$ i). It suffices to show $|\lambda_{H^*(K)}|<1$. The hypothesis means that for any $X\in\mathbb{S}_+^n$

$$\lim_{k\to\infty}(H^*(K))^k\,\mathrm{vec}(X)=0. \tag{23}$$

In light of Lemma 5, for any $G\in\mathbb{C}^{n\times n}$ there exist $G_1,G_2,G_3,G_4\in\mathbb{S}_+^n$ such that $G=(G_1-G_2)+(G_3-G_4)\mathbf{i}$. It can been seen from (23) that

$$\begin{aligned}&\lim_{k\to\infty}(H^*(K))^k\,\mathrm{vec}(G)\\ =&\lim_{k\to\infty}(H^*(K))^k\,(\mathrm{vec}(G_1)-\mathrm{vec}(G_2))\\ &+\lim_{k\to\infty}(H^*(K))^k\,(\mathrm{vec}(G_3)-\mathrm{vec}(G_4))\,\mathbf{i}=0\end{aligned}$$

which implies i).

i) $\Rightarrow$ iii). Since $|\lambda_{H^*(K)}|<1$ by the hypothesis in (i), $(I-H^*(K))^{-1}$ exist and it equals to $\sum_{i=0}^{\infty}(H^*(K))^i$. Due to the nonsingularity of $(I-H^*(K))^{-1}$ and the one-to-one correspondence of vectorization operator, for any positive definite matrix $V\in\mathbb{C}^{n\times n}$, there exists a unique matrix $P\in\mathbb{C}^{n\times n}$ such that

$$\mathrm{vec}(V)=(I-H^*(K))\,\mathrm{vec}(P). \tag{24}$$

The property of Kronecker product gives $\mathrm{vec}(V)=\mathrm{vec}\,(P-\mathcal{L}_K(P))$. Since, vectorization is one-to-one correspondence, we then have $V=P-\mathcal{L}_K(P)>0$. It still remains to show $P>0$. It follows from (25) that

$$\begin{aligned}\mathrm{vec}(P)=&\ (I-H^*(K))^{-1}\,\mathrm{vec}(V)\\ =&\ \sum_{i=0}^{\infty}(H^*(K))^i\,\mathrm{vec}(V)\\ =&\ \mathrm{vec}\left(\sum_{i=0}^{\infty}\mathcal{L}_K^i(V)\right)\end{aligned} \tag{25}$$

which yields $P=\sum_{i=0}^{\infty}\mathcal{L}_K^i(V)>0$.

iii) $\Rightarrow$ ii). If there exist $K=[K^{(1)},\ldots,K^{(\mathsf{l_o}-1)}]$ with each matrix $K^{(i)}$ having compatible dimensions and $P>0$ such that $\mathcal{L}_K(P)<P$, then there must exist a $\mu\in(0,1)$ satisfying $\mathcal{L}_K(P)<\mu P$. Choose $c>0$ such that $X\leq cP$. Then, due to the linearity and nondecreasing properties of $\mathcal{L}_K(X)$ with respect to X on the positive semidefinite cone, for $k\in\mathbb{N}$

$$\mathcal{L}_K^k(X)\leq\mathcal{L}_K^k(cP)=c\mathcal{L}_K^k(P)<c\mathcal{L}_K^{k-1}(\mu P)<\cdots<c\mu^kP$$

which leads to $\lim_{k\to\infty}\mathcal{L}_K^k(X)=0$.

iii) $\Rightarrow$ iv). The proof is similar to the proof of b) $\Rightarrow$ c) in [16, Th. 5] and is omitted here.

iv) $\Rightarrow$ iii). Note that, by the Schur complement lemma and $X,Y>0$, (8) holds if and only if

$$Y\geq(1-q)A'YA+qA'XA. \tag{26}$$

Similarly, (9) holds if and only if

$$p\sum_{i=1}^{\mathsf{l_o}-1}(1-p)^{i-1}(A^i+K^{(i)}C^{(i)})^*Y(A^i+K^{(i)}C^{(i)})<X \tag{27}$$

where $K^{(i)}=Y^{-1}F_i^*,\ i=1,\ldots,\mathsf{l_o}-1$. Applying the inequality of (26) for k times, it results in

$$Y\geq(1-q)^k(A')^kYA^k+q\sum_{j=1}^{k}(1-q)^{j-1}(A')^jXA^j.$$

As Y is bounded, taking limitation on the right sides, it yields

$$Y\geq q\sum_{j=1}^{\infty}(1-q)^{j-1}(A')^jXA^j. \tag{28}$$

Combining (27) and (28), we obtain $\mathcal{L}_K(X)<X$.

Appendix C
Proof of Theorem 2

To prove this theorem, we need some preliminary lemmas.

Lemma 8: Suppose that there exist constants d_1, $d_0\geq0$ such that, for any $j\in\mathbb{N}$ and $k\in[\beta_j,\beta_{j+1}]$, $\|P_k\|\leq\max_{i=j,j+1}\{d_1\|P_{\beta_i}\|+d_0\}$ holds $\mu-$almost surely. If $\sup_{j\in\mathbb{N}}\mathbb{E}\|P_{\beta_j}\|<\infty$, then $\sup_{k\in\mathbb{N}}\mathbb{E}\|P_k\|<\infty$ holds.

Proof: Since $\sup_{j\in\mathbb{N}}\mathbb{E}\|P_{\beta_j}\|<\infty$, there exists a uniform bound α for $\{\mathbb{E}\|P_{\beta_j}\|\}_{j\in\mathbb{N}}$, i.e., $\mathbb{E}\|P_{\beta_j}\|\leq\alpha$ for all $j\in\mathbb{N}$. By the definition of β_j in (5), k should be no larger than β_k for all $k\in\mathbb{N}$. Then, one obtains

$$\begin{aligned}\mathbb{E}\|P_k\|=&\ \mathbb{E}\left[\sum_{j=0}^{k-1}\mathbb{E}\Big[\|P_k\|\mid\beta_j\leq k\leq\beta_{j+1}\Big]1_{\{\beta_j\leq k\leq\beta_{j+1}\}}\right]\\ \leq&\ \sum_{j=0}^{k-1}\mathbb{E}\left[\max_{i=j,j+1}\{d_1\|P_{\beta_i}\|+d_0\}\mid\beta_j\leq k\leq\beta_{j+1}\right]\end{aligned}$$

$$
\begin{aligned}
&\cdot\mu(\beta_j \le k \le \beta_{j+1})\\
\le\ & \sum_{j=0}^{k-1} \left(d_1 \mathbb{E}\|P_{\beta_j}\| + d_1 \mathbb{E}\|P_{\beta_{j+1}}\| + d_0\right)\\
&\cdot\mu(\beta_j \le k \le \beta_{j+1})\\
\le\ & \left(2d_1 \sup_{j\le k} \mathbb{E}\|P_{\beta_j}\| + d_0\right) \sum_{j=0}^{k-1} \mu(\beta_j \le k \le \beta_{j+1})\\
\le\ & 2d_1\alpha + d_0
\end{aligned}
$$

which completes the proof. ■

Before proceeding to the proof of the theorem, let us provide some properties related to the discrete-time algebraic Riccati equation (DARE). The proof, provided in [43], is omitted.

Lemma 9: Consider the following DARE

$$P = APA' + Q - APC'(CPC' + R)^{-1}CPA'. \tag{29}$$

If $(A, Q^{1/2})$ is controllable and (C, A) is observable, then, it has a unique positive definite solution $\tilde{P}$ and $A + \tilde{K}C$ is stable, where $\tilde{K} = -A\tilde{P}C'(C\tilde{P}C' + R)^{-1}$.

Fix $j \ge 0$. First of all, we shall show that, for $k \in [\beta_j + 1, \tau_{j+1}]$, $\|P_k\|$ is uniformly bounded by an affine function of $\|P_{\beta_j}\|$. By Lemmas 3 and 9, we have $g(P_{k-1}) \le \phi_1(\tilde{K}, P_{k-1})$ and that $A + \tilde{K}C$ is stable. In light of (16) in Lemma 2, we further obtain $g^i(P_{k-1}) \le \phi_1^i(\tilde{K}, P_{k-1})$ for all $i \in \mathbb{N}$. Therefore, an upper bound of $\|P_k\|$ is given as follows:

$$
\begin{aligned}
\|P_k\| =\ & \|g^{k-\beta_j}(P_{\beta_j})\| \le \|\phi_1^{k-\beta_j}(\tilde{K}, P_{\beta_j})\|\\
\le\ & \left\|(A + \tilde{K}C)^{k-\beta_j} P_{\beta_j} (A' + C'\tilde{K}^*)^{k-\beta_j}\right\|\\
&+ \left\| \sum_{i=0}^{k-\beta_j-1} (A + \tilde{K}C)^i (Q + \tilde{K}R\tilde{K}^*)(A' + C'\tilde{K}^*)^i \right\|\\
\le\ & \|(A + \tilde{K}C)^{k-\beta_j}\|^2 \|P_{\beta_j}\|\\
&+ \sum_{i=0}^{k-\beta_j-1} \|(A + \tilde{K}C)^i\|^2 \|Q + \tilde{K}R\tilde{K}^*\|\\
\le\ & m_0\, \alpha_0^{2k-2\beta_j} \|P_{\beta_j}\| + m_0 \|Q + \tilde{K}R\tilde{K}^*\| \sum_{i=0}^{k-\beta_j-1} \alpha_0^{2i}
\end{aligned}
$$

where $\alpha_0 = |\lambda_{A+\tilde{K}C}| + \epsilon_0 \|A + \tilde{K}C\|$ and $m_0 = n(1 + 2/\epsilon_0)^{2n-2}$ with a positive number ϵ_0 satisfying $|\lambda_{A+\tilde{K}C}| + \epsilon_0\|A + \tilde{K}C\| < 1$ (such an ϵ_0 must exist because $|\lambda_{A+\tilde{K}C}| < 1$), and the last inequality holds due to Lemma 4. Observe that $\sum_{i=0}^{k-\beta_j-1} \alpha_0^{2i} \le \frac{1}{1-\alpha_0^2}$. As $\alpha_0 < 1$, $\alpha_0^{2k-2\beta_j} < 1$ for any $k \in [\beta_j + 1, \tau_{j+1}]$. Therefore

$$\|P_k\| \le m_0 \|P_{\beta_j}\| + n_0 \tag{30}$$

where $n_0 \triangleq \frac{m_0}{1-\alpha_0^2} \|Q + \tilde{K}R\tilde{K}'\|$.

Next, we shall show that, for $k \in [\tau_{j+1} + 1, \beta_{j+1}]$, $\|P_k\|$ is bounded by an affine function of $\|P_{\beta_{j+1}}\|$. To do this, let us look at the relationship between $P_{\beta_{j+1}}$ and P_k. Since $\gamma_k = 0$ for all $k \in [\tau_{j+1}, \beta_{j+1} - 1]$, the relation is given by

$$P_{\beta_{j+1}} = A^{\beta_{j+1}-j} P_k (A')^{\beta_{j+1}-k} + \sum_{i=0}^{\beta_{j+1}-k-1} A^i Q (A')^i$$

from which we obtain $P_{\beta_{j+1}} \ge A^{\beta_{j+1}-k} P_k (A')^{\beta_{j+1}-k}$. Then, it yields

$$
\begin{aligned}
\|P_{\beta_{j+1}}\| \ge\ & \|A^{\beta_{j+1}-k} P_k (A')^{\beta_{j+1}-k}\|\\
\ge\ & \frac{1}{n} \mathrm{Tr}(A^{\beta_{j+1}-k} P_k (A')^{\beta_{j+1}-k})\\
=\ & \frac{1}{n} \mathrm{Tr}(P_k^{1/2} (A')^{\beta_{j+1}-k} A^{\beta_{j+1}-k} P_k^{1/2})\\
\ge\ & \frac{1}{n} \|A^{k-\beta_{j+1}}\|^{-2} \mathrm{Tr}(P_j)\\
\ge\ & \frac{1}{n} \|A^{k-\beta_{j+1}}\|^{-2} \|P_k\|
\end{aligned}
$$

where the second and the last inequality allows from the fact that $\|X\| = \lambda_X \ge \frac{1}{n}\mathrm{Tr}(X)$ and $\mathrm{Tr}(X) \ge \|X\|$ for any $X \in \mathbb{S}_+^n$; the third one holds since

$$
\begin{aligned}
(A')^{\beta_{j+1}-k} A^{\beta_{j+1}-k} \ge\ & \min \sigma\left(A^{\beta_{j+1}-k}(\cdot)'\right) I\\
=\ & \frac{1}{\lambda_{A^{k-\beta_{j+1}}(\cdot)'}} I\\
=\ & \|A^{k-\beta_{j+1}}\|^{-2} I.
\end{aligned}
$$

If A has no eigenvalues on the unit circle, by Lemma 4, there holds $\|A^{k-\beta_{j+1}}\| \le n_1\, \alpha_1^{\beta_{j+1}-k}$ where $n_1 \triangleq \sqrt{n}(1 + 2/\epsilon_1)^{n-1}$ and $\alpha_1 \triangleq |\lambda_{A^{-1}}| + \epsilon_1 \|A^{-1}\|$ with a positive number ϵ_1 so that $\alpha_1 < 1$ (such an ϵ_1 must exist since $|\lambda_{A^{-1}}| < 1$ by assumption (A1)). As $\alpha_1 < 1$, $\|A^{k-\beta_{j+1}}\| \le n_1\, \alpha_1^{\beta_{j+1}-k} < n_1$ for all $k \in [\beta_j + 1, \tau_{j+1}]$. If A has semisimple eigenvalues on the unit circles, we denote the Jordan form of A as $J = \mathrm{diag}(J_{11}, J_{22})$, where J_{11} has no eigenvalues on the unit circle and J_{22} is diagonal with all semisimple eigenvalues on the unit circle, i.e., there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that $J = SAS^{-1}$. In this case

$$\|J^{k-\beta_{j+1}}\| = \max\left\{\|J_{11}^{k-\beta_{j+1}}\|, \|J_{22}^{k-\beta_{j+1}}\|\right\} \le n_1$$

where, similarly, $n_1 \triangleq \sqrt{n}(1 + 2/\epsilon_1)^{n-1}$ with a positive number ϵ_1 so that $|\lambda_{J_{11}^{-1}}| + \epsilon_1 \|J_{11}^{-1}\| = 1$. Since $\|S^{-1} \cdot S\|$ can be considered a matrix norm, we have

$$\|A^{k-\beta_{j+1}}\| = \|S^{-1} J^{k-\beta_{j+1}} S\| \le c \|J^{k-\beta_{j+1}}\| \le c n_1$$

where $c = \sup_{X\in\mathbb{C}^{n\times n}} \frac{\|S^{-1}XS\|}{\|X\|} < \infty$ due to the equivalence of matrix norms on a finite dimensional vector space. Then, we have the following upper bound for $\|P_k\|$ for all $k \in [\tau_{j+1} + 1, \beta_{j+1}]$

$$\|P_k\| \le n \|A^{k-\beta_{j+1}}\|^2 \|P_{\beta_{j+1}}\| \le m_1 \|P_{\beta_{j+1}}\| \tag{31}$$

where $m_1 \triangleq nc^2 n_1^2$.

According to (30) and (31), it can be seen that, when $k \in [\beta_j, \beta_{j+1}]$, $\|P_k\| \le \max\{m_0, m_1\} \max\left\{\|P_{\beta_j}\|, \|P_{\beta_{j+1}}\|\right\} + n_0$. Since, j is arbitrarily chosen, by invoking Lemma 8, the desired conclusion follows.

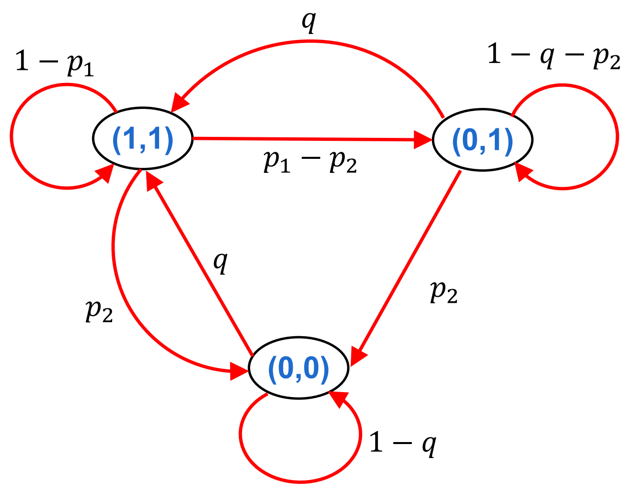

Fig. 9. Transitions of the Markov chain $\{(z_k, \tilde{z}_k)\}_{k\in\mathbb{N}}$ when $p_2 + q \le 1$.

Appendix D
Proof of Proposition 2

If $p = 0$, we have the standard Kalman filter, which evidently converges to a bounded estimation error covariance. On the other hand, if $p = 1$, then the Kalman filter reduces to an open-loop predictor after time step $k = 2$, which suggests that there exists a transition point for p beyond which the expected prediction error covariance matrices are not uniformly bounded. It remains to show that with a given q this transition point is unique. Fix $0 \le p_1 < 1$ such that $\sup_{k\in\mathbb{N}} \mathbb{E}_{p_1}\|P_k\| < \infty\ \forall \Sigma_0 \ge 0$. It suffices to show that, for any $p_2 < p_1$, $\sup_{k\in\mathbb{N}} \mathbb{E}_{p_2}\|P_k\| < \infty$ for all $\Sigma_0 \ge 0$. To differentiate two Markov chains with different failure rates in (2), we use the notation $\{\gamma_k(p_i)\}_{k\in\mathbb{N}}$ instead to represent the packet loss process so as to indicate the configuration $p = p_i$ in (2). We will prove the aforementioned statement using a coupling argument. We define a sequence of random vectors $\{(z_k, \tilde{z}_k)\}_{k\in\mathbb{N}}$ over a probability space $(\mathscr{G}, \mathcal{G}, \pi)$ with $\mathscr{G} = \{(0,0), (0,1), (1,1)\}^{\mathbb{N}}$ and $\mathcal{G}_k$ representing the filtration generated by $(z_1, \tilde{z}_1), \ldots, (z_k, \tilde{z}_k)$. We also define

$$\varphi_1(\{z_k, \tilde{z}_k\}_{k=1:t}) = \psi_{z_t} \circ \cdots \circ \psi_{z_1}(\Sigma_0)$$

and

$$\varphi_2(\{z_k, \tilde{z}_k\}_{k=1:t}) = \psi_{\tilde{z}_t} \circ \cdots \circ \psi_{\tilde{z}_1}(\Sigma_0)$$

where $\psi_z = zg + (1-z)h$ with h, g defined in (3), (4), and $z = \{0, 1\}$. Due to Lemma 2 and $z_k \le \tilde{z}_k$ in $\mathscr{G}$, we have $\varphi_1(\{z_k, \tilde{z}_k\}_{k=1:t}) \ge \varphi_2(\{z_k, \tilde{z}_k\}_{k=1:t})$.

When $p_2 + q \le 1$, we let the evolution of $\{(z_k, \tilde{z}_k)\}_{k\in\mathbb{N}}$ follow the Markov chain illustrated in Fig. 9, whereby it can seen that $\pi(z_{k+1} = j | z_k = i)$s for $i, j = \{0, 1\}$ are constants independent of $\tilde{z}_k$s, and conversely that $\pi(\tilde{z}_k = j | \tilde{z}_k = i)$s for $i, j = \{0, 1\}$ are constants independent of z_ks. Therefore, both the marginal distributions of $\{z_k\}_{k\in\mathbb{N}}$ and $\{\tilde{z}_k\}_{k\in\mathbb{N}}$ are Markovian, and moreover

$$\pi(z_{k+1} = j | z_k = i) = \mathbb{P}_{p_1}(\gamma_{k+1}(p_1) = j | \gamma_k(p_1) = i)$$

and

$$\pi(\tilde{z}_{k+1} = j | \tilde{z}_k = i) = \mathbb{P}_{p_2}(\gamma_{k+1}(p_2) = j | \gamma_k(p_2) = i)$$

for all $i, j = \{0, 1\}$ and $k \in \mathbb{N}$. It can be seen that the Markov chain in Fig. 9 is ergodic and has a unique stationary distribution

$$\begin{aligned}\pi_\infty((0,0)) &= \frac{p_2}{p_2+q}, \quad \pi_\infty((0,1)) = \frac{p_1}{p_1+q} - \frac{p_2}{p_2+q} \\ \pi_\infty((1,1)) &= \frac{q}{p_1+q}.\end{aligned} \tag{32}$$

We assume that the Markov chain starts at the stationary distribution. Then, the distribution of $(z_k, \tilde{z}_k)$ for $k \ge 2$ is the same as $(z_1, \tilde{z}_1)$, which gives

$$\begin{aligned}\mathbb{E}^\infty_{p_1}\|P_k\| &= \int_\Omega \left\|\psi_{\gamma_k(p_1)} \circ \cdots \circ \psi_{\gamma_1(p_1)}(\Sigma_0)\right\| \, \mathrm{d}\mathbb{P}_{\mathrm{p}_1} \\ &= \int_{\mathscr{G}} \left\|\varphi_1(\{z_j, \tilde{z}_j\}_{j=1:k})\right\| \mathrm{d}\pi \\ &\ge \int_{\mathscr{G}} \left\|\varphi_2(\{z_j, \tilde{z}_j\}_{j=1:k})\right\| \mathrm{d}\pi \\ &= \int_\Omega \left\|\psi_{\gamma_k(p_2)} \circ \cdots \circ \psi_{\gamma_1(p_2)}(\Sigma_0)\right\| \, \mathrm{d}\mathbb{P}_{\mathrm{p}_2} \\ &= \mathbb{E}^\infty_{p_2}\|P_k\|\end{aligned}$$

where $\mathbb{E}^\infty$ means that the expectations is taken conditioned on the stationarily distributed γ_1.

When $p_2 + q > 1$, we abuse the definition of probability measure and allow the existence of negative probabilities in the Markov chain described in Fig. 9, generating $\{(z_k, \tilde{z}_k)\}_{k\in\mathbb{N}}$. It can be easily shown by direct computation that the eigenvalues of transition probability matrix, denoted by $\mathbf{M} \in \mathbb{R}^{3\times 3}$, of this Markov chain are $1 - q - p_1$, $1 - q - p_2$, and 1, respectively. As a result, $\mathbf{M}^k$ converges to a limit as k tends to infinity indicating that the generalized Markov chain has a unique stationary distribution, which is the same as the one given in (32). Therefore, although the Markov chain is only formally defined without corresponding physical meaning with $p_2 + q > 1$, the most important basic property for this coupling still holds, that is

$$\pi(z_1 = i_1, \cdots, z_t = i_t) = \mathbb{P}_{p_1}(\gamma_1 = i_1, \cdots, \gamma_t = i_t)$$

and

$$\pi(\tilde{z}_1 = i_1, \cdots, \tilde{z}_t = i_t) = \mathbb{P}_{p_2}(\gamma_1 = i_1, \cdots, \gamma_t = i_t)$$

for all $t \in \mathbb{N}$ and $i_1, \cdots, i_t \in \{0, 1\}$. Thus, the inequality $\mathbb{E}^\infty_{p_1}\|P_k\| \ge \mathbb{E}^\infty_{p_2}\|P_k\|$ still proves true in this case.

Finally, in order to show $\sup_{k\in\mathbb{N}} \mathbb{E}_{p_2}\|P_k\| < \infty$ with packet losses initialized by $\gamma_1 = 1$, we only need to recall the following lemma.

Lemma 10 (Lemma 2 in [32]): The statement holds that $\sup_{k\in\mathbb{N}} \mathbb{E}^\infty\|P_k\| < \infty$ if and only if $\sup_{k\in\mathbb{N}} \mathbb{E}^1\|P_k\| < \infty$ and $\sup_{k\in\mathbb{N}} \mathbb{E}^0\|P_k\| < \infty$, where $\mathbb{E}^1$ and $\mathbb{E}^0$ denote the expectations conditioned on $\gamma_1 = 1$ and 0, respectively.

The proof is now complete.

Appendix E
Proof of Theorem 3

Let $p = p_c(q)$ be the critical value established in Proposition 2. For any given p, fix $1 - 1/|\lambda_A|^2 < q_1 < 1$ so that

$\sup_{k\in\mathbb{N}} \mathbb{E}_{q_1}\|P_k\| < \infty$ for all $\Sigma_0 \geq 0$. From a symmetrical coupling argument as the proof of Proposition 2, for any $q_1 \leq q_2 < 1$, $\sup_{k\in\mathbb{N}} \mathbb{E}_{q_2}\|P_k\| < \infty$ also holds for all $\Sigma_0 \geq 0$. As a result, $p_c(q)$ is a nondecreasing function of q. Consequently, $p_c(\cdot)$ yields an inverse function, denoted by $q_c(\cdot)$, which is also nondecreasing. The desired conclusion then follows immediately, e.g., we can simply choose $f_c(p,q) = p_c(q) - p$.

Appendix F
Proof of Theorem 4

We shall first show that

$$\lim_{\eta\to 1^+} p(\eta) = \underline{p} \tag{33}$$

holds where $\eta > 1$ is properly taken so that $\eta^2|\lambda_A|^2(1-q) < 1$, and

$$p(\eta) \triangleq \sup\left\{p : \exists (K,P) \text{ s.t. } \mathcal{L}(\eta A, K, P, p) < P, P > 0\right\}$$

with the notation $\mathcal{L}(A,K,P,p)$ used to alter $\mathcal{L}_K(P)$ in the proof so as to emphasize the relevance of $\mathcal{L}_K(P)$ to A and p. To this end, first note that $p(\eta)$ is a nonincreasing function of η. Thus, $\lim_{\eta\to 1^+} p(\eta)$ must exist. To show the equality in (33), we require the following lemma.

Lemma 11: Suppose $X, \tilde{X} > 0$ are the solutions to the following Lyapunov equations, respectively:

$$X = (1-q)AXA' + \tilde{Q}, \quad \eta^{-1}\tilde{X} = (1-q)A\tilde{X}A' + \tilde{Q}$$

where $\tilde{Q} > 0$, $0 < q < 1$ and $\eta > 1$ are properly taken so that $\eta(1-q)|\lambda_A|^2 < 1$. Then, for any $\epsilon > 0$ there always exists a $\delta > 0$ such that $\eta \leq 1+\delta$ implies $\tilde{X} \leq (1+\epsilon)X$.

Proof: First we shall find an upper bound for X and a lower bound for $\tilde{X}$. It is straightforward that

$$X \geq \tilde{Q} \geq \|\tilde{Q}^{-1}\|^{-1} I. \tag{34}$$

Let $d \triangleq \left[(1-q)|\lambda_A|^2\right]^{-1/4} > 1$ and restrict $1 < \eta \leq d$. By Lemma 4, we have

$$\begin{aligned}\tilde{X} &= \sum_{i=0}^{\infty} \eta^i (1-q)^i A^i \tilde{Q}(A^i)' \leq \|\tilde{Q}\| \sum_{i=0}^{\infty} \eta^i (1-q)^i \|A^i\|^2 I \\ &\leq n\|\tilde{Q}\|(1+2/\varepsilon)^{2n-2} \sum_{i=0}^{\infty} d^i (1-q)^i (|\lambda_A| + \varepsilon\|A\|)^{2i} I\end{aligned}$$

for any $\varepsilon > 0$. Taking $\varepsilon = \frac{(d-1)|\lambda_A|}{\|A\|}$, it yields that $\tilde{X} \leq \frac{cd}{d-1}\|\tilde{Q}\|$ with $c = n(1+2/\varepsilon)^{2n-2}$. Note that $\tilde{X} - X$ is bounded in the following way

$$\begin{aligned}\tilde{X} - X &= (1-q)A(\tilde{X}-X)A' \\ &\quad + (\eta-1)\left[(1-q)A\tilde{X}A' + \tilde{Q}\right] \\ &\leq (1-q)A(\tilde{X}-X)A' + (\eta-1)\tilde{X} \\ &\quad \vdots \\ &\leq (1-q)^k A^k(\tilde{X}-X)(\cdot)' \\ &\quad + (\eta-1)\sum_{i=0}^{k-1}(1-q)^i A^i \tilde{X}(A^i)'.\end{aligned}$$

Taking limitation on both sides, we obtain that $\tilde{X} - X \leq (\eta - 1)\sum_{i=0}^{\infty}(1-q)^i A^i \tilde{X}(A^i)'$, which by Lemma 4 and (34) gives

$$\begin{aligned}\tilde{X} - X &\leq (\eta-1)\|\tilde{Q}\|\frac{cd}{d-1}\sum_{i=0}^{\infty}(1-q)^i\|A^i\|^2 I \\ &\leq \frac{c^2d^3}{(d^2-1)(d-1)}(\eta-1)\|\tilde{Q}\| I \\ &\leq \frac{c^2d^3}{(d^2-1)(d-1)}(\eta-1)\|\tilde{Q}\|\|\tilde{Q}^{-1}\| X\end{aligned}$$

where the last inequality holds because of (34). Due to the positive definiteness of $\tilde{Q}$, the assertion follows by letting $1 < \eta \leq \min\left\{\frac{(d^2-1)(d-1)}{c^2d^3\|\tilde{Q}\|\|\tilde{Q}^{-1}\|}\epsilon + 1, d\right\}$.

By the definition of $\underline{p}$ one can verify that for any $0 < \varepsilon < \underline{p}$ there always exists at least a $p \in (\underline{p}-\varepsilon, \underline{p})$ so that there exist K and $P > 0$ satisfying $\mathcal{L}(A,K,P,p) < P$; otherwise it contradicts (13). We take an $\epsilon > 0$ so that $(1+\epsilon)\mathcal{L}(A,K,P,p) < P$ still holds. Then, from Lemma 11, there always exists an $\eta_0 > 1$ satisfying $\Phi_{\eta_0} \leq \sqrt{1+\epsilon}\Phi$ where Φ and Φ_{η_0} are the positive definite solutions to the following equations, respectively:

$$\begin{aligned}\Phi &= (1-q)A'\Phi A + A'PA, \\ \eta_0^{-2}\Phi_{\eta_0} &= (1-q)A'\Phi_{\eta_0}A + A'PA.\end{aligned}$$

In addition, there exists an $\eta_1 > 1$ such that $\eta_1^{2\mathsf{l_o}-2} \leq \sqrt{1+\epsilon}$. Letting $\tilde{\eta} = \min\{\eta_0, \eta_1\}$, we have

$$\begin{aligned}P &> (1+\epsilon)\mathcal{L}(A,K,P,p) \\ &\geq p\sum_{i=1}^{\mathsf{l_o}-1}(1-p)^{i-1}(\tilde{\eta}^iA^i + \tilde{\eta}^iK^{(i)}C^{(i)})^*(\Phi_{\tilde{\eta}})(\cdot)\end{aligned}$$

which implies that $\mathcal{L}(\tilde{\eta}A, K_{\tilde{\eta}}, P, p) < P$ with $K_{\tilde{\eta}} \triangleq [\tilde{\eta}K^{(1)}, \cdots, \tilde{\eta}^{\mathsf{l_o}-1}K^{(\mathsf{l_o}-1)}]$ and therefor that $p(\tilde{\eta}) > \underline{p} - \varepsilon$. As ε is any positive real number, $\lim_{\eta\to 1^+} p(\eta) = \underline{p}$ consequently holds. Since ηA has no defective eigenvalues on the unit circle, combining Theorems 1 and 2, we obtain that $p_c(\eta A, C, q) \geq p(\eta)$ holds.

To conclude, we also need to establish the following lemma.

Lemma 12: For a given recovery rate q satisfying $|\lambda_A|^2(1-q) < 1$, denote $q_c(A,C,q)$ as the critical value of q_c for a system (C,A). Then, we have

$$p_c(A,C,q) \geq \lim_{\eta\to 1^+} p_c(\eta A, C, q). \tag{35}$$

Proof: To emphasize the relevance of $h(X)$ and $g(X)$ to A, we will change the notations $h(X)$ and $g(X)$ into $h(A,X)$ and $g(A,X)$, respectively, in the proof. Note that $h(\eta A, X)$

and $g(\eta A, X)$ are both nondecreasing function of $\eta > 1$, where η is properly chosen so that $\eta^2|\lambda_A|^2(1-q) < 1$, since, for all $1 \leq \eta_1 \leq \eta_2 \leq 1+\delta$ and $X \geq 0$, one has

$$h(\eta_2 A, X) - h(\eta_1 A, X) = (\eta_2^2 - \eta_1^2)AXA' \geq 0$$

and

$$\begin{aligned}&(\eta_2 A, X) - g(\eta_1 A, X)\\ =\ & (\eta_2^2 - \eta_1^2)(AXA' - AXC'(CXC' + R)^{-1}CXA') \geq 0.\end{aligned}$$

According to the fact that $P_k = (1-\gamma_k)h(A, P_{k-1}) + \gamma_k g(A, P_{k-1})$ and that $h(A, X)$ and $g(A, X)$ are nondecreasing functions of X from Lemma 2, we can easily show by induction that P_k is also nondecreasing in η. Therefore, the limitation on the right side of (35) always exists, and then, the conclusion follows. ■

From Lemma 12 and what has been proved previously, it can be seen that $\lim_{\eta\to 1^+} p_c(\eta A, C, q)$ and $\lim_{\eta\to 1^+} p(\eta)$ exist and moreover that

$$p_c(A, C, q) \geq \lim_{\eta\to 1^+} p_c(\eta A, C, q) \geq \lim_{\eta\to 1^+} p(\eta) = \underline{p}$$

whereby the desired result follows.

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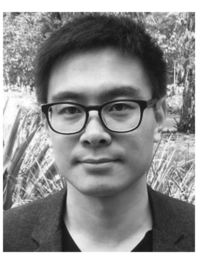

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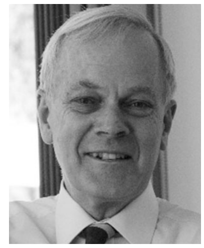

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