LQG and Medium Access Control ⋆

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Abstract:
The communication channel is a shared resource in networked control systems, and channel access at every instant cannot be guaranteed. In this paper, we propose a novel architecture for control over wireless networks with integrated medium access control (MAC). We evaluate the impact of constrained channel access on the cost of controlling a single plant over a network and establish that the separation principle holds under certain conditions on the MAC. We arrive at a classification of random access methods for networked control systems and identify a structure for each method. Then, by evaluating the increase in cost compared to a conventional setup, we identify an adaptive random access method which uses a threshold-based decision criteria on the current data to determine channel access. Finally, we give stability criteria for control applications using these medium access methods.

Keywords: Networked Control, Dual Predictor Structure, Medium Access Control, Linear Quadratic Gaussian Control, Random Access.

1. INTRODUCTION

In a traditional control system, the controller receives measurements of the plant state from sensors through dedicated channel resources. Large bandwidths, guaranteed channel access and negligible packet losses or delays characterize these systems, making them transparent to the control designer. With the advent of wireless networked control systems, the communication channel is far from ideal, and imposes constraints on the application layer. In addition to limited bandwidth, packet loss and delay, these networks do not guarantee channel access. To arbitrate channel access amongst multiple sources of data, networks use a multiple access mechanism, which is dealt with by the MAC Layer.

Determining a multiple access strategy for control over networks is an open problem, with little work published so far, an exception being Liu et al. (2004). There are many challenges to be overcome while integrating feedback control and multiple access. Most multiple access methods in communication literature try to divide the shared resource in a fair manner, with no notion of priority. Delay-aware methods are required to ensure that the channel is allotted to the packet whose delivery is critical to meet the system's performance constraints. But, how do we determine which packet is most critical to the performance of its control system?

To answer this question, we must evaluate the impact of a decision to transmit this packet or not. We are thus evaluating the impact of medium access methods on the cost of controlling a linear plant over a network. We consider random access methods in particular, as they are used by open architectures in the ISM band, like IEEE 802.15.4. When channel access is not guaranteed at every sampling instant, the plant must settle for limited channel access and a higher cost of controlling the process. We try to quantify this trade-off between channel access and control cost for random access methods.

We consider a single plant controlled over a network in a dual predictor architecture, which has a predictor at both ends of the channel. This architecture has been proposed earlier by others such as Xu et al. (2004). We design the MAC to be a decision process, which decides to transmit the current packet or not. This decision could be based on a coin flip, or on past channel use, or even on the data in the current packet. But, in all cases, a decision taken at the current step affects the plant state and possibly decisions to be taken in the future. What decision law would then be appropriate? Also, what control law should be used? Does the separation principle still hold, when the decision to transmit or not is taken based on the data in the current packet?

In this paper, we try to answer the above questions. We find that for a quadratic cost, under certain constraints, the separation principle does hold. We classify random access methods based on their decision criteria and derive the increase in cost for each MAC. We identify an adaptive MAC that attempts to reduce the increase in cost, with an adaptive decision law.

Random access methods from literature [Rom and Sidi (1990)] can be classified on the basis outlined in this paper. Carrier Sense Multiple Access (CSMA) methods with fixed or exponential backoff and p-persistent methods are examples of random access methods that belong to categories in our classification. But, the Adaptive MAC does not seem to exist in the literature today, as it is adapted to the application layer. However, as we show, it may hold significant advantages over other methods.

The analysis in this paper can be assigned other interpretations, such as packet losses over an independent and identically distributed (IID) erasure channel and an erasure channel with

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Table 1. Signal Flow in the Dual Predictor Architecture

k	0	1	...	$n-1$	n	...	$N-1$	N
Plant State	x_0	x_1		x_{n-1}	x_n		x_{N-1}	x_N
Prediction at Sensor	$\hat{x}_1^s$	$\hat{x}_2^s$		$\hat{x}_n^s$	$\hat{x}_{n+1}^s$		$\hat{x}_N^s$	
Decision on Transmission	γ_0	γ_1		$\gamma_{n-1} = 1$	γ_n		γ_{N-1}	
Prediction at Controller	$\hat{x}_0^c = \hat{x}_0^s$	$\hat{x}_1^c$		$\hat{x}_{n-1}^c$	$\hat{x}_n^c = \hat{x}_n^s$		$\hat{x}_{N-1}^c$	
Control Signal	u_0	u_1		u_{n-1}	u_n		u_{N-1}	
Last Received Index	$\tau_0 = -1$	τ_1		τ_{n-1}	$\tau_n = n-1$		τ_{N-1}	

memory. These problems have been dealt with in the past by Sinopoli et al. (2005) and Gupta et al. (2005). We also identify conditions for stability, given packet losses or random channel access, and find that we obtain results for the former consistent with previous studies.

The rest of the paper is organized as follows. In Section 2, we describe the system architecture and notation used in the paper. We derive the Linear Quadratic Gaussian (LQG) cost function with medium access control, and evaluate the increase in cost due to limited channel access in Section 3. We present some simulation results in Section 4 and conclude in Section 5.

2. DUAL PREDICTOR WITH MAC

In this section, we describe each block of the dual predictor along with the MAC and introduce the notation we use to define our problem.

We consider a single plant and controller, which communicate over a network, as shown in Fig. 1. The plant has a state $x \in \mathbb{R}^n$, which evolves as

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is the state transition matrix and $B \in \mathbb{R}^{n\times p}$. $u \in \mathbb{R}^p$ is the control signal. The process noise w is zero mean white Gaussian with covariance $R_w \in \mathbb{R}^{n\times n}$.

The smart sensor attached to the plant consists of a sensor and a predictor. The sensor measures $y \in \mathbb{R}^m$, which can be related to the plant state by

$$y_k = Cx_k + v_k \tag{2}$$

where $C \in \mathbb{R}^{m\times n}$. The measurement noise v, uncorrelated to the process noise w, is zero mean white Gaussian with covariance $R_v \in \mathbb{R}^{m\times m}$. The predictor at the sensor (KF^s) tracks the best possible linear estimate of the state $\hat{x}^s$, with access to all measurements until time $k-1$. The state estimate is updated as

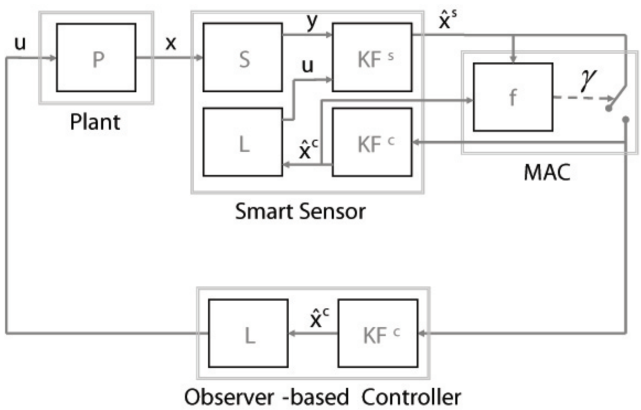

Fig. 1. The system architecture (dual predictor) with MAC

$$\begin{aligned} \hat{x}_{k+1}^s &= A\hat{x}_k^s + Bu_k + K_k(y_k - C\hat{x}_k^s) \\ P_{k+1}^s &= AP_k^s A^T - K_k R_{e_k} K_k^T + R_w \\ R_{e_k} &= CP_k^s C^T + R_v \\ K_k &= AP_k^s C^T R_{e_k}^{-1} \end{aligned} \tag{3}$$

The control signal u_k is generated by running a copy of the observer based controller at the smart sensor.

The sensor transmits the prediction $\hat{x}^s$ over the network. Channel Access is controlled by a MAC block, whose functionality is represented as

$$\gamma_k = f(\tau_k, \hat{x}_{k+1|k}^s - \hat{x}_{k+1|\tau_k}^c) \tag{4}$$

where γ_k is a binary random variable that assumes a value of 1 when the packet is granted channel access. Thus, we limit our analysis to the class of random access methods in this paper. The parameter, $\hat{x}^c$, is the predicted estimate at the controller and τ_k is the index of the last packet to have been successfully transmitted over the channel. The last received index τ_k can be defined as

$$\begin{aligned} \tau_k &= \max\{i : \gamma_i = 1 \text{ and } -1 \le i \le k-1\} \\ \tau_{k+1} &= \bar{\gamma}_k \tau_k + \gamma_k k, \tau_0 = -1 \end{aligned} \tag{5}$$

The observer-based controller consists of an observer (KF^c) and a time-varying control gain L_k. The observer determines an estimate $\hat{x}^c$ of the plant state, which is pegged to the estimate at the sensor when the packet has access to the channel. When the packet is denied channel access, this estimate is updated with knowledge of the state model. Thus, we have

$$\hat{x}_{k+1}^c = \bar{\gamma}_k(A\hat{x}_k^c + Bu_k) + \gamma_k \hat{x}_{k+1}^s \tag{6}$$

where $\bar{\gamma}_k$ is the binary complement of γ_k. A control signal $u_k = -L_k\hat{x}_k^c$ is generated, where L_k is derived in the next section.

This paper primarily investigates the MAC Layer, and hence we assume that the physical channel is perfect, whenever accessed. We also do not consider random access for the control signal.

Table 1 depicts the signal flow in this architecture. From the table, we define the information sets $\mathbb{I}_k^s$ and $\mathbb{I}_k^c$, which describe the information available to the predictors at the sensor and the controller respectively.

$$\begin{aligned} \mathbb{I}_k^s &= \{\mathbb{Y}_k, \Gamma_{k-1}\}; \quad \mathbb{I}_k^c = \{\mathbb{Y}_{\tau_k}, \Gamma_{k-1}\} \\ \mathbb{Y}_i &= [y_0 ... y_i]^T; \quad \Gamma_i = [\gamma_0 ... \gamma_i]^T \end{aligned} \tag{7}$$

Now, our system definition is complete but for the function f in the MAC block (4) and the time varying control gain L_k. In the following sections, we evaluate the impact of the binary decision variable γ on the cost of controlling the plant P, for the generic function f defined in (4). While doing so, we derive the time-varying optimal control gain L_k. Furthermore, we classify the MAC based on the arguments of the function f and identify suitable functions for each class, thus designing the MAC for the dual predictor structure described above.

3. LQG AND MEDIUM ACCESS CONTROL

In this section, we introduce a classification of medium access control methods and derive the Linear Quadratic Gaussian cost for each. We identify three classes of random access methods, relevant to this analysis.

- *Static MAC:* Static MAC protocols are random access methods with a fixed channel access probability. The access probability is independent of the current data or the past history of transmissions. The functional block in this method is a binary random number generator, such as a coin flip.
- *Dynamic MAC:* Dynamic MAC protocols are random access methods with a channel access probability that evolves over time. The access probability is still independent of the current data, but now depends on the past history of transmissions. The functional block here is $\gamma_k = f(\tau_k, 0)$.
- *Adaptive MAC:* Adaptive MAC protocols are random access methods with a channel access probability that depends on the current data packet, and possibly, evolve over time as well. The functional block here is given in (4).

We use the LQG cost criterion for the following analysis. The expected loss criterion for a control loop with process state x and control signal u is

$$\mathrm{E}\left[x_N^T Q_0 x_N + \sum_{s=0}^{N-1} (x_s^T Q_1 x_s + u_s^T Q_2 u_s)\right] \tag{8}$$

where Q_0, Q_1 and Q_2 denote positive semi-definite matrices of appropriate orders that penalize the path taken by the state from x_0 to x_N and the required control signal. The optimal LQG cost achieved with channel access at every sampling instant is denoted J_0 and defined as

$$J_0 = \hat{x}_0^T S_0 \hat{x}_0 + \mathrm{tr}\{S_0 P_0\} + \sum_{s=0}^{\text{N-1}} \mathrm{tr}\{Q_1 P_s^s\} + \sum_{s=0}^{\text{N-1}} \mathrm{tr}\{S_{s+1} K_s R_{e_s} K_s^T\} + \mathrm{tr}\{Q_0 P_N^s\}$$

where S_k and P_k are solutions to the control and filter Riccati equations, respectively. J is the LQG cost obtained with constrained channel access. For all three classes of MAC protocols, we state the following theorem.

Theorem 1. Consider the system defined by (1)–(6). Suppose that

i. γ_k is independent of $\hat{x}^c_{k+1|\tau_k}$, as in (4).
ii. $f(\tau, e)$ is symmetric in e for all τ.

Then, the LQG cost is equal to $J = J_0 + J_\varepsilon$, where J_0 is given above and J_ε is given by

$$\begin{aligned} J_\varepsilon = & -\sum_{k=0}^{N-1} \mathrm{tr}\{S_{k+1} K_k R_{e_k} K_k^T\} + \sum_{k=0}^{N-1} \mathrm{E}[\mathrm{tr}\{S_{k+1} \\ & \times \mathrm{E}[\gamma_k^2 \sum_{s=1}^{k-\tau_k} \sum_{r=1}^{k-\tau_k} A^{s-1} K_{k+1-s} e_{k+1-s} e_{k+1-r}^T K_{k+1-r}^T A^{r-1^T}]\}] \\ & + \mathrm{E}[\mathrm{tr}\{Q_0 \mathrm{E}[\bar{\gamma}_{N-1}^2 \sum_{s=1}^{N-1-\tau_{N-1}} \sum_{r=1}^{N-1-\tau_{N-1}} A^{s-1} K_{N-s} e_{N-s} \\ & \times e_{N-r}^T K_{N-r}^T A^{r-1^T}]\}] + \sum_{k=1}^{N-1} \mathrm{E}[\mathrm{tr}\{Q_1 \\ & \times \mathrm{E}[\bar{\gamma}_{k-1}^2 \sum_{s=1}^{k-1-\tau_{k-1}} \sum_{r=1}^{k-1-\tau_{k-1}} A^{s-1} K_{k-s} e_{k-s} e_{k-r}^T K_{k-r}^T A^{r-1^T}]\}] \end{aligned} \tag{9}$$

Proof. We follow the approach outlined in Åström (2006), in the proof of the separation principle, and express the cost at any time k in terms of the cost at time $k+1$. Using the property

$$\min_u \mathrm{E}[f(x,u)] = \mathrm{E}_\theta \left[\min_u \mathrm{E}[f(x,u)|\theta] \right]$$

we can write $\mathrm{E}[V_k]$, the minimum cost at time k, where

$$\mathrm{E}[V_k] = \min_{u_k,..,u_{N-1}} \mathrm{E}[x_N^T Q_0 x_N + \sum_{s=k}^{N-1} (x_s^T Q_1 x_s + u_s^T Q_2 u_s)]$$

$$V_k = \min_{u_k} \mathrm{E}[x_k^T Q_1 x_k + u_k^T Q_2 u_k + V_{k+1} \mid \mathbb{I}_k^c]$$

is the Bellman Equation. The net cost is then given by $J = \mathrm{E}[V_0]$, where V_0 is the minimal cost at time 0.

The dimension of the set $\mathbb{I}_{k+1}^c$ increases with time. But, note that any new measurements y_i can be written as $y_i = \hat{y}_i^c + e_i^c$ for $\tau_k \le i \le \tau_{k+1}$. Using predictor (6), the innovation in the measurements (e_i^c) is independent of the predicted estimate at the controller $(\hat{x}_i^c)$ by design, and we note that $\hat{x}_i^c$ contains all the information in the set $\mathbb{Y}_{\tau_i}$. Also, from τ_i, we can deduce γ_{i-1}. Then, $\{\hat{x}_k^c, \tau_k\}$ are a sufficient statistic for the set $\mathbb{I}_k^c$, and we can write

$$\mathrm{E}[V_k] = \min_{u_k} \mathrm{E}[x_k^T Q_1 x_k + u_k^T Q_2 u_k + V_{k+1} \mid \hat{x}_k^c, \tau_k]$$

Note that, for e_i^c to be independent of $\hat{x}_i^c$, γ_{i-1} must be independent of $\hat{x}_i^c$ (Condition (*i*) of Theorem 1).

We will show, using induction, that the solution to the Bellman Equation is a quadratic function $\mathrm{E}[V_k] = \hat{x}_k^{c^T} S_k \hat{x}_k^c + s_k$ for $0 \le k \le N$, where S is a positive semi-definite matrix. The initial condition at time N follows from this relationship

$$\mathrm{E}[x_i^T Q x_i] = m_i^T Q m_i + \mathrm{tr}\{Q\ \mathrm{E}[(x_i - m_i)(x_i - m_i)^T]\}$$

where m_i is the mean of the state x_i, and Q is a positive semi-definite matrix. Assuming that our conjecture is true at time $k+1$, we have at time k -

$$\mathrm{E}[V_k] = \min_{u_k} \mathrm{E}[x_k^T Q_1 x_k + u_k^T Q_2 u_k + \hat{x}_{k+1}^{c^T} S_{k+1} \hat{x}_{k+1}^c + s_{k+1} | \hat{x}_k^c, \tau_k]$$

We can express x_k in terms of $\hat{x}_k^c$ and τ_k as follows -

$$\begin{aligned} x_k &= \hat{x}_k^c + \tilde{x}_k^c \\ \tilde{x}_k^c &= \tilde{x}_k^s + \bar{\gamma}_{k-1} \sum_{s=1}^{k-1-\tau_{k-1}} A^{s-1} K_{k-s} e_{k-s} \end{aligned} \tag{10}$$

$$P_k^c = P_k^s + \mathrm{E}[\bar{\gamma}_{k-1}^2 \sum_{s=1}^{k-1-\tau_{k-1}} \sum_{r=1}^{k-1-\tau_{k-1}} A^{s-1} K_{k-s} e_{k-s} e_{k+1-r}^T K_{k-r}^T A^{r-1^T}]$$

where $\tilde{x}$ denotes the estimation errors. Here, it cannot be established that $\tilde{x}_k^c$ has zero mean, until we evaluate the expected value of the estimation error between the predictor at the controller and the sensor ($\mathrm{E}[\bar{\gamma}_{k-1} \sum_{s=1}^{k-1-\tau_{k-1}} A^{s-1} K_{k-s} e_{k-s}]$). Similarly, $\hat{x}_{k+1}^c$ can be written as

$$\begin{aligned} \hat{x}_{k+1}^c &= A\hat{x}_k^c + Bu_k + \gamma_k \sum_{s=1}^{k-\tau_k} A^{s-1} K_{k+1-s} e_{k+1-s} \\ &= \hat{x}_{k+1}^s - \bar{\gamma}_k \sum_{s=1}^{k-\tau_k} A^{s-1} K_{k+1-s} e_{k+1-s} \end{aligned} \tag{11}$$

The expressions within expectations in (10) and (11) can be simplified depending on the relationship between γ_k, $\{e_k\}$ and $\mathbb{I}_k^c$, where by $\{e_k\}$, we refer to the set of innovations which have not been communicated to the observer-based controller. For a generic medium access controller, γ_k could depend on $\{e_k\}$ and $\mathbb{I}_k^c$. Hence, it is not possible to simplify the expressions in (11). This also means that it is not trivial to define the mean and variance of the conditional distribution of $(x_k|\hat{x}_k^c, \tau_k)$ or $(\hat{x}_{k+1}^c|\hat{x}_k^c, \tau_k)$. We simplify the problem by assuming that the decision criterion in our scheduling strategy is symmetric in $\{e_k\}$ (Condition (*ii*) of Theorem 1). This ensures that $\mathrm{E}[\bar{\gamma}_{k-1}\sum_{s=1}^{k-1-\tau_{k-1}} A^{s-1}K_{k-s}e_{k-s}] = 0$ and consequently, in (10) above, $\tilde{x}_k^c$ is zero mean. Then, we have

$$\begin{aligned}
\mathrm{E}[x_k|\hat{x}_k^c,\tau_k] = \hat{x}_k^c \quad \text{and} \quad \mathrm{E}[\tilde{x}_k^c\tilde{x}_k^{c^T}|\hat{x}_k^c,\tau_k] = P_k^c \\
\mathrm{E}[\hat{x}_{k+1}^c|\hat{x}_k^c,\tau_k] = A\hat{x}_k^c + Bu_k \qquad (12)\\
\mathrm{E}[(\hat{x}_{k+1}^c - A\hat{x}_k^c - Bu_k)\,(\hat{x}_{k+1}^c - A\hat{x}_k^c - Bu_k)^T|\hat{x}_k^c,\tau_k] \\
= \mathrm{E}[\gamma_k^2 \sum_{s=1}^{k-\tau_k}\sum_{r=1}^{k-\tau_k} A^{s-1}Ke_{k+1-s}e_{k+1-r}^T K^T A^{r-1^T}]
\end{aligned}$$

Now, we can simplify the cost at k, to get

$$\begin{aligned}
\mathrm{E}[V_k] &= \min_{u_k}\,[\hat{x}_k^{c^T} Q_1\hat{x}_k^c + \mathrm{tr}\{Q_1P_k^c\} + u_k^TQ_2u_k \\
&+ s_{k+1} + (A\hat{x}_k^c + Bu_k)^T S_{k+1}(A\hat{x}_k^c + Bu_k) \\
&+ \mathrm{tr}\{S_{k+1}\,\mathrm{E}[\gamma_k^2 \sum_{s=1}^{k-\tau_k}\sum_{r=1}^{k-\tau_k} A^{s-1}Ke_{k+1-s}e_{k+1-r}^T K^T A^{r-1^T}]\}] \\
&= \hat{x}_k^{c^T}[Q_1 + A^TS_{k+1}A - A^TS_{k+1}B(Q_2 + B^TS_{k+1}B)^{-1} \\
&\quad \times B^TS_{k+1}A]\hat{x}_k^c + \mathrm{tr}\{Q_1P_k^c\} + s_{k+1} \\
&+ \mathrm{tr}\{S_{k+1}\,\mathrm{E}[\gamma_k^2 \sum_{s=1}^{k-\tau_k}\sum_{r=1}^{k-\tau_k} A^{s-1}Ke_{k+1-s}e_{k+1-r}^T K^T A^{r-1^T}]\} \\
&= \hat{x}_k^{c^T}S_k\hat{x}_k^c + s_k \qquad \text{and} \qquad L_k = (Q_2 + B^TS_{k+1}B)^{-1}B^TS_{k+1}A
\end{aligned}$$

Thus, the induction is true at step k, and we have a solution to the Bellman Equation. The control signal u_k that minimizes the cost here is a linear function of the estimated state at the controller. Thus, we claim that the separation principle does hold for random medium access methods under two constraints. The first constraint requires the MAC decision to be independent of the estimate at the controller at any time k. Thus, a decision to transmit a packet must depend on the innovation in the data, or the amount of data in the packet which is not already available to the controller. The second constraint requires the criterion (function f) to be a symmetric function of the innovations set $\{e_k\}$. This constraint is sufficient, along with the first necessary constraint, for the proof outlined above. It simply requires that the absolute value of innovation, and not its sign, be taken into account.

Now that we have the cost at any time k, we can find the net cost of controlling this plant.

$$\begin{aligned}
J = E[V_0] &= \hat{x}_0^TS_0\hat{x}_0 + \mathrm{tr}\{S_0P_0\} + \sum_{k=0}^{N-1}\mathrm{tr}\{Q_1P_k^s\} + \mathrm{tr}\{Q_0P_N^s\} \\
&+ \sum_{k=0}^{N-1}\mathrm{E}[\mathrm{tr}\{S_{k+1}\,\mathrm{E}[\gamma_k^2 \sum_{s=1}^{k-\tau_k}\sum_{r=1}^{k-\tau_k} A^{s-1}K_{k+1-s}e_{k+1-s} \\
&\times e_{k+1-r}^T K_{k+1-r}^T A^{r-1^T}]\}] + \sum_{k=1}^{N-1}\mathrm{E}[\mathrm{tr}\{Q_1 \\
&\times \mathrm{E}[\bar{\gamma}_{k-1}^2 \sum_{s=1}^{k-1-\tau_{k-1}}\sum_{r=1}^{k-1-\tau_{k-1}} A^{s-1}K_{k-s}e_{k-s}e_{k-r}^TK_{k-r}^TA^{r-1^T}]\}] \\
&+ \mathrm{E}[\mathrm{tr}\{Q_0\,\mathrm{E}[\bar{\gamma}_{N-1}^2 \sum_{s=1}^{N-1-\tau_{N-1}}\sum_{r=1}^{N-1-\tau_{N-1}} A^{s-1}K_{N-s}e_{N-s} \\
&\quad \times e_{N-r}^TK_{N-r}^TA^{r-1^T}]\}]
\end{aligned}$$

Finally, the increase in cost is given by J_ε in (9). This concludes our proof of Theorem 1. □

It is easy to show that J_ε is always a positive quantity, and it takes the value 0 when we have channel access at every instant of time. When channel access is denied, the third and fourth terms in (9) become non-zero quantities as they relate to the increase in covariance of the estimate at the controller with respect to the estimate at the sensor. Eventually, when

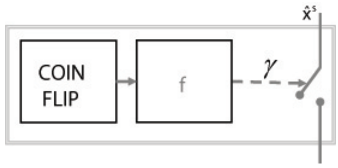

Fig. 2. A static medium access controller

the packet arrives, the first term exceeds the second term, as it reflects the unused information at the sensor until this time. We now derive corollaries for the three classes of medium access control methods defined earlier.

3.1 Static MAC

Recall that γ here is independent of the data, or the innovations set and the history of transmissions, or the information at the controller, as depicted in Fig. 2. Thus, in a static MAC, $p = 1-\bar{p}$ is the probability of accessing the channel. Then, the probability distribution of τ_k, p_{τ_k}, can be defined as

$$\Pr(\gamma = 1) = p \;\text{ and }\; \Pr(\gamma = 0) = \bar{p} = 1-p$$

$$\Pr(\tau_{k+1} = i) = \begin{cases} \bar{p}^{k-i}p & 0 \le i \le k \\ \bar{p}^{k+1} & i = -1 \end{cases}$$

We now state the corollary of Theorem 1 applicable to Static MACs.

Corollary 2. The condition for closed loop stability with the use of a Static MAC is $\bar{p}\rho(A)^2 < 1$. The increase in LQG cost over the nominal cost J_0 is $J_{\varepsilon S}$, which is given by

$$\begin{aligned}
J_{\varepsilon S} &= \sum_{k=0}^{N-1} p\,\mathrm{tr}\{S_{k+1}\sum_{i=-1}^{k-1}\Pr(\tau_k = i)\sum_{s=1}^{k-i}A^{s-1}K_{k+1-s} \\
&\times R_{e_{k+1-s}}K_{k+1-s}^TA^{s-1^T}]\}] - \sum_{k=0}^{N-1}\mathrm{tr}\{S_{k+1}K_kR_{e,k}K_k^T\} \\
&+ \bar{p}\,\mathrm{tr}\{Q_0\sum_{i=-1}^{N-2}\Pr(\tau_{N-1} = i)\sum_{s=1}^{N-1-i}A^{s-1}K_{N-s} \qquad (13)\\
&\times R_{e_{N-s}}K_{N-s}^TA^{s-1^T}]\}] + \sum_{k=1}^{N-1}\bar{p}\,\mathrm{tr}\{Q_1\sum_{i=-1}^{k-2}\Pr(\tau_{k-1} = i) \\
&\times \sum_{s=1}^{k-1-i}A^{s-1}KR_{e_{k-s}}K^TA^{s-1^T}]\}]
\end{aligned}$$

Proof. As γ is independent of the innovations and τ, we can write

$$\begin{aligned}
&\mathrm{E}[\bar{\gamma}_{k-1}\sum_{s=1}^{k-1-\tau_{k-1}}A^{s-1}K_{k-s}e_{k-s}] \\
&= \mathrm{E}[\bar{\gamma}_{k-1}]\sum_{s=1}^{k-1-\tau_{k-1}}A^{s-1}K_{k-s}\,\mathrm{E}[e_{k-s}] = 0
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathrm{E}[\hat{x}_{k+1}^c|\hat{x}_k^c,\tau_k] = A\hat{x}_k^c + Bu_k \\
&\mathrm{E}[\gamma_k^2\sum_{s=1}^{k-\tau_k}\sum_{r=1}^{k-\tau_k}A^{s-1}K_{k+1-s}e_{k+1-s}e_{k+1-r}^TK_{k+1-s}^TA^{r-1^T}|\hat{x}_k^c,\tau_k] \\
&= \gamma_k\sum_{s=1}^{k-\tau_k}A^{s-1}K_{k+1-s}R_{e_{k+1-s}}K_{k+1-s}^TA^{s-1^T}
\end{aligned}$$

Substituting these terms in the expression for J_ε (9), we get $J_{\varepsilon S}$ (13), which is the increase in the LQG cost due to a Static MAC.

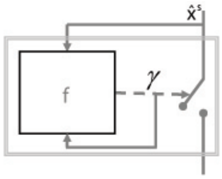

Fig. 3. A dynamic medium access controller

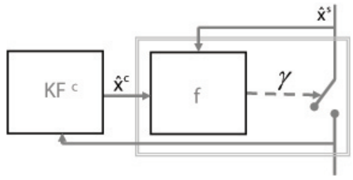

Fig. 4. An adaptive medium access controller

The condition for stability is obtained from the Discrete Lyapunov recursion, where $\rho(A)$ is the spectral radius of the state transition matrix.

$$P^c_k - P^s_k = \bar{p}\sum_{i=-1}^{k-2}\Pr(\tau_{k-1}=i)\sum_{s=1}^{k-1-i}A^{s-1}K_{k-s}R_{e_{k-s}}K^T_{k-s}A^{s-1^T}$$

$$P^c_{k+1} - P^s_{k+1} = \bar{p}A(P^c_k - P^s_k)A^T + \bar{p}K_kR_{e_k}K^T_k$$

The difference between P^c and P^s remains bounded, if $\bar{p}\rho(A)^2 < 1$. P^s is the error covariance of a Kalman filter that receives measurements at every sampling instant from a linear plant driven by a known input. Thus, P^s is bounded, and stability is ensured. This concludes our proof of Corollary 2. □

Note that the results derived here are also applicable to an IID Binary Erasure channel with packet loss probability $\bar{p}$. This case has been dealt with in Sinopoli et al. (2005).

3.2 Dynamic MAC

Here, γ_k depends on τ_k, or the history of transmissions, as depicted in Fig. 3. If p_i is the probability of accessing the channel after i attempts in a Dynamic MAC with memory N_M, then $p_{\tau_{k+1}}$ is given by

$$p_i = \Pr(\gamma_k = 1|\tau_k = k-i)$$

$$p_{\tau_{k+1}} = \begin{cases}\sum_{i=1}^{N_M} p_i\Pr(\tau_k = k-i) & \tau_{k+1} = k\\ \bar{p}_i\Pr(\tau_k = k-i) & \tau_{k+1} = k-i\end{cases} \quad (14)$$

where $\bar{p}_i = 1-p_i$ for $1 \le i \le N_M$

We now state the Corollary of Theorem 1 applicable to Dynamic MACs.

Corollary 3. The condition for closed loop stability with a Dynamic MAC is $\bar{p}_i\rho(A)^2 < 1$ for $2 \le i \le N_M$ and the increase in LQG cost over the nominal cost J_0 is given by

$$\begin{aligned}J_{\varepsilon D} = &\sum_{k=0}^{N-1}\sum_{i=1}^{N_M} p_i\Pr(\tau_k = k-i)\,\mathrm{tr}\{S_{k+1}\sum_{s=1}^{i}A^{s-1}K_{k+1-s}\\ &\times R_{e_{k+1-s}}K^T_{k+1-s}A^{s-1^T}]\}] - \sum_{k=0}^{N-1}\mathrm{tr}\{S_{k+1}K_kR_{e_k}K^T_k\}\\ &+\sum_{i=1}^{N_M}\bar{p}_i\Pr(\tau_{N-1} = N-1-i)\,\mathrm{tr}\{Q_0\sum_{s=1}^{i}A^{s-1}K_{N-s}R_{e_{N-s}}\\ &\times K^T_{N-s}A^{s-1^T}]\}] + \sum_{k=1}^{N-1}\sum_{i=1}^{N_M}\bar{p}_i\Pr(\tau_{k-1} = k-1-i)\\ &\times \mathrm{tr}\{Q_1\sum_{s=1}^{i}A^{s-1}K_{k-s}R_{e_{k-s}}K^T_{k-s}A^{s-1^T}]\}]\\ &\text{for } 1 \le i \le N_M\end{aligned} \quad (15)$$

Proof. As γ is independent of the innovations, we can write

$$\mathrm{E}[\bar{\gamma}_{k-1}\sum_{s=1}^{k-1-\tau_{k-1}}A^{s-1}K_{k-s}e_{k-s}]$$

$$= \mathrm{E}\left[\mathrm{E}[\bar{\gamma}_{k-1}|\tau_{k-1}]\sum_{s=1}^{k-1-\tau_{k-1}}A^{s-1}K_{k-s}E[e_{k-s}]\right] = 0$$

Then, we have

$$E[\hat{x}^c_{k+1}|\hat{x}^c_k, \tau_k] = A\hat{x}^c_k + Bu_k$$

Substituting these in the expression for J_ε, we get $J_{\varepsilon D}$, which is the increase in the LQG cost due to a Dynamic MAC. The stability condition is derived from the following Discrete Lyapunov recursion.

$$P^c_k - P^s_k = \sum_{i=1}^{N_M}(P^c_k - P^s_k)_i$$

$$(P^c_k - P^s_k)_i = \bar{p}_i\Pr(\tau_{k-1} = k-1-i)\sum_{s=1}^{i}A^{s-1}K_{k-s}R_{e_{k-s}}K^T_{k-s}A^{s-1^T}$$

$$(P^c_{k+1} - P^s_{k+1})_i = \bar{p}_{i+1}A(P^c_k - P^s_k)_iA^T$$

$$P^c_{k+1} - P^s_{k+1} = \sum_{i=1}^{N_M}(P^c_{k+1} - P^s_{k+1})_i + \Pr(\gamma_k = 0)K_kR_{e_k}K^T_k$$

This concludes our proof of Corollary 3. □

Note that the results in this section are also applicable to a Binary Erasure channel with memory.

3.3 Adaptive MAC

For the most general case, γ depends on the data in the packet to be transmitted, or the innovations set and the history of transmissions, or the information at the controller, as depicted in Fig. 4. J_ε was derived for the most general case, which is the Adaptive MACs. We would like to design an Adaptive MAC that reduces J_ε and somehow stabilizes the closed loop system as well. We can rewrite P^c_k from (10) as

$$P^c_k = P^s_k + \mathrm{E}[(\hat{x}^s_k - \hat{x}^c_k)(\hat{x}^s_k - \hat{x}^c_k)^T]$$

By choosing to transmit when the error in the predicted estimates exceeds a threshold ε, we achieve stability in the closed loop. Thus, our decision criterion is

$$\gamma_k = 1 : |\hat{x}^s_{k+1} - \hat{x}^c_{k+1|\tau_k}|^2 \ge \varepsilon;$$

$$|\hat{x}^s_{k+1} - \hat{x}^c_{k+1|\tau_k}|^2 = \mathrm{tr}\{(\hat{x}^s_{k+1} - \hat{x}^c_{k+1|\tau_k})(\hat{x}^s_{k+1} - \hat{x}^c_{k+1|\tau_k})^T\}$$

$$= \mathrm{tr}\{\sum_{s=1}^{k-\tau_k}\sum_{r=1}^{k-\tau_k}A^{s-1}Ke_{k+1-s}e_{k+1-r}K^TA^{r-1^T}\}$$

This decision criterion also reduces J_ε. Note that this criterion satisfies both the constraints mentioned in the statement of Theorem 1. Then, the term in the last equation above approximately takes the value ε when a packet gains channel access, and is always less than ε when it does not. Thus, we can approximately upper bound J_ε when $x \in \mathbb{R}$.

$$J_\varepsilon \lesssim \sum_{k=1}^{N-1}Q_1\varepsilon\Pr(\gamma_{k-1} = 0) + Q_0\varepsilon\Pr(\gamma_{N-1} = 0)$$

$$+\sum_{k=0}^{N-1}S_{k+1}\varepsilon\Pr(\gamma_k = 1) - \sum_{k=0}^{N-1}S_{k+1}K_kR_{e_k}K^T_k$$

Also, for $\varepsilon < K_kR_{e_k}K^T_k$, the Adaptive MAC permits channel access at every sampling instant, and $J_\varepsilon = 0$.

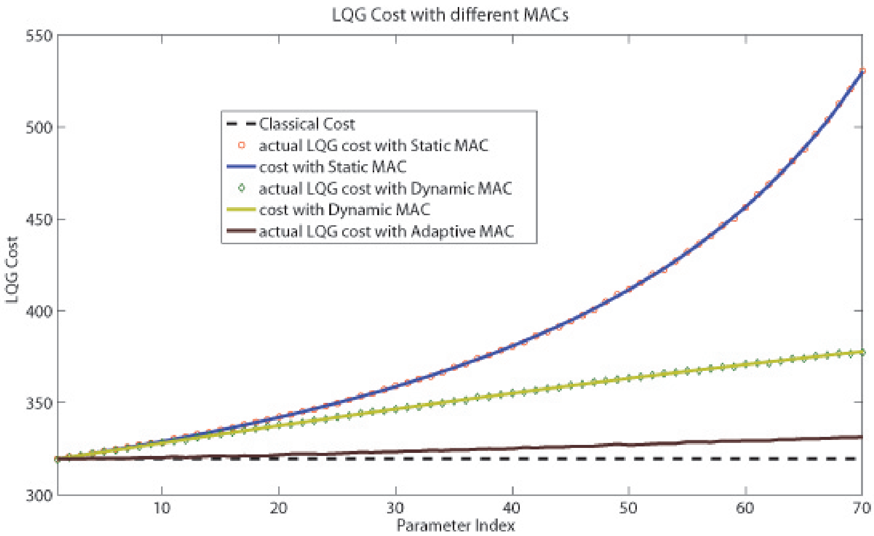

Fig. 5. A comparison of the control cost with different MACs against the conventional cost. The parameter index refers to varying the probability of channel access in the first attempt for all three MACs. The Adaptive MAC results in a significant reduction in the cost of controlling the same plant over a network as against using a Static or Dynamic MAC.

For the Adaptive MAC designed above, the probability of channel access at any time k is given by

$$\Pr(\gamma_k = 1) = \Pr(\tau_{k+1} = k) = \sum_{i=1}^{k+1} p_i \Pr(\tau_k = k - i)$$

$$\Pr(\gamma_k = 0) = \Pr(\tau_{k+1} \neq k) = \sum_{i=1}^{k+1} \bar{p}_i \Pr(\tau_k = k - i)$$

where p_i is the probability of accessing the channel after i attempts. In the above equation, p_{τ_k} is the probability distribution of τ, which evolves as shown below.

$$p_{\tau_k} = \begin{cases} \sum_{i=1}^{k} p_i \Pr(\tau_{k-1} = k - 1 - i) & \tau_k = k - 1 \\ \bar{p}_{i-1} \Pr(\tau_{k-1} = k - i) & \tau_k = k - i \\ \vdots & \\ \bar{p}_k \Pr(\tau_{k-1} = -1) & \tau_k = -1 \end{cases}$$

where $\bar{p}_i = 1 - p_i$

Finally, p_i is defined as

$$p_i = \Pr(|\hat{x}^s_{k+1} - \hat{x}^c_{k+1|k-i}|^2 \geq \varepsilon)$$

$$= 1 - \phi_{\chi^2_1}\left(\frac{\varepsilon}{\sum_{s=1}^{i} A^{s-1} K_{k+1-s} R_{e_{k+1-s}} K^T_{k+1-s} A^{s-1^T}}\right)$$

where $\phi_{\chi^2_1}$ is the cumulative distribution function of a Chi-squared distribution with one degree of freedom.

4. SIMULATION

Fig. 5 shows the results of a simulation of a scalar plant with different MACs. For the static MAC, the cost of controlling the plant increases drastically over the conventional cost as p varies from 1 to 0.3. For the dynamic MAC, the increase in cost is considerably reduced for a similar variation in p_1 as the maximum delay is bounded by the memory of the channel. For the adaptive MAC, ε varies from 0 to 0.7, which results in nearly the same p_1. However, the performance improvement is dramatic, and such methods show considerable potential in reducing the cost of controlling a plant over an unreliable network.

5. CONCLUSION

We have analyzed the effect of the medium access controller on a quadratic cost function, and find that there is an additive increase in cost associated with the decision statistics of the MAC. Furthermore, we were able to establish that the separation principle holds for decision criteria based on (a symmetric function of) the innovation in the data to be transmitted. An adaptive MAC, which uses a threshold based decision criterion, has been designed, and verified through simulations, to reduce the increase in cost associated with MAC. But, finding the optimal adaptive method remains a challenge, especially when the problem is extended to include multiple loops over a shared network.

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