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ROBUST STABILITY OF TIME-VARYING DELAY SYSTEMS: THE QUADRATIC SEPARATION APPROACH†

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ABSTRACT

In this article, we are interested in analysing stability of systems that incorporate time-varying delays in their dynamic. The Lyapunov-Krasovskii approach is definitely the most popular method to address this issue and many results have proposed new functionals and enhanced techniques for deriving less conservative stability conditions. In this present work, we propose an original approach: the quadratic separation. To this end, the delay operator properties are exploited to provide delay range stability conditions. In particular, L_2-norm of delay-dependent operators are computed so as to reduce the conservatism of the approach. Moreover, the main result is able to assess the stability of non-small delay systems, i.e, it can detect a stability interval for systems that are unstable without any delay. Several examples illustrate the benefit of our methodology.

I. INTRODUCTION

Time delay system is a subclass of infinite dimensional systems that has been frequently employed since it can model commonly arising transport and propagation phenomena. Delays can be encountered in many processes such as biology, chemistry, economics, population dynamics [2] as well as in networks [3]. However, delays are the origin of performance and stability degradation, which thus have motivated a lot of work. In the case of constant delay, and unperturbed linear systems, efficient criteria exist based on root loci techniques (see [4] for a recent review). For the case of uncertain linear systems, the problem has been partially solved, either by using Lyapunov functionals [5, 6, 7] or robustness tools such as small gain theorem, integral quadratic constraints [5] and quadratic separation [8]. The proposed conditions are often rather conservative since they produce inner approximations of the stability regions, although recent techniques [8], [9] reduce the conservatism by introducing redundant equations and new decision variables in the optimization problem. Then, these results have been extended to time varying delay systems either using adapted Lyapunov-Krasovskii [9, 10, 11, 12, 13, 14, 15, 16, 17] or robustness tools [18, 19, 20]. These latter methodologies often require, explicitly or implicitly, the delay-free system to be stable, which is a rather important restriction.

This paper aims at providing a novel approach to address time-varying delay system stability. More precisely, we propose criteria based on an extension of the quadratic separation principle [21], [22]. They are then expressed in terms of Linear Matrix Inequalities (LMIs) which may be solved efficiently with Semi-Definite Programming (SDP). In this method, the key idea is how to model the operators that define the system. At first, redundant equations are introduced to construct an augmented model that relates the state vector, its derivative and the delay. Then, a new operator is proposed to refine the modelling of the delay dynamic. At last, using the quadratic separation

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framework with an appropriate modelling, we provide the main result: a delay range stability condition (where the delay h is belonging to a prescribed interval $[h_{\min}, h_{\max}]$). Differently from most of papers on this topic [13, 23], this condition is able to detect pockets of stability even in case of unstable delay-free systems. We emphasize that we do not intend to present an additional less conservative criterion that outperforms all existing results but rather an original methodology to cope with systems which may be unstable for sufficiently small delays.

The outline of the paper is as follows. In Section II, some preliminaries are presented and we state our quadratic separation theorem as well as a set of useful operators. In section III, this latter prior result is exploited to derive a stability condition for time-varying delay systems. Then, an additional operator is appended for the conservatism reduction. At last, numerical examples that show the effectiveness of the proposed criterion are provided in section IV. Section V concludes the paper.

Notations: Throughout the paper, the following notations are used. The set L_2^n consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{C}^n$ such $\|f\|_{L_2} = \left(\int_0^\infty (f^*(t)f(t))\right)^{1/2} dt < \infty$. When context allows it, the superscript n of the dimension will be omitted. The set L_{2e}^n denotes the extended set of L_2^n which consists of the functions whose time truncation lies in L_2^n. For two symmetric matrices, A and B, $A > (\geq)\ B$ means that $A - B$ is (semi-)positive definite. A^T denotes the transpose of A. 1_n and $0_{m\times n}$ denote, respectively, the identity matrix of size n and null matrix of size $m \times n$. If the context allows it, the dimensions of these matrices are omitted. $\mathrm{diag}(A_1, \ldots, A_k)$ stands for the block diagonal matrix with $A_1, \ldots, A_k$ on the diagonal. Introduce as well the truncation operator $\mathbb{P}_T$ such that:

$$\mathbb{P}_T(f) = f_T = \begin{cases} f(t), & t \leq T, \\ 0, & t > T. \end{cases}$$

II. PRELIMINARIES

2.1. Problem statement

Consider the following time-varying delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)) & \forall t \geq 0, \\ x(t) = \phi(t) & \forall t \in [-h_{\max}, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, ϕ is the initial condition and A, $A_d \in \mathbb{R}^{n\times n}$ are constant matrices. The delay h is time varying with

$$h(t) \in [h_{\min}, h_{\max}] \quad \text{and} \quad |\dot{h}(t)| \leq d, \tag{2}$$

where $h_{\min}$, $h_{\max}$ and d are given positive constants. In this work, we aim at assessing the stability of system (1) via the quadratic separation principle originally developed for robust control in [24]. We will show that various criteria, related to the available informations on the delay, can be derived choosing appropriately a set of operators.

2.2. Stability analysis via quadratic separation

The quadratic separation provides a fruitful framework to address stability of non-linear and uncertain systems [21], [22]. Recent studies [8] have shown that such a framework reduces significantly the conservatism of the stability analysis of time-delay systems with constant delay. In this paper we extend this method to time varying delay systems, which involves the development of results for a new set of operators. Consider the interconnection in Figure 1 where $\mathcal{E}$ and $\mathcal{A}$ are two, real valued, possibly non-square, matrices and ∇ is a linear operator from L_{2e} to L_{2e}. For simplicity, we assume that $\mathcal{E}$ is full column rank. Assuming well-posedness, we are interested in looking for conditions that ensure stability of the interconnection.

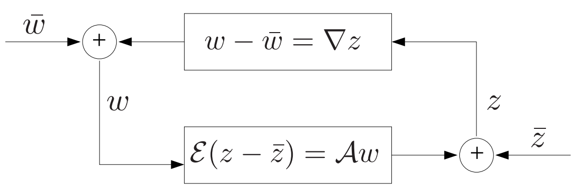

Fig. 1. Feedback system.

Theorem 1 *The interconnected system of Figure 1 is stable if there exists a symmetric matrix $\Theta = \Theta^T$ satisfying both conditions*

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp^T} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \tag{3}$$

and $\forall u \in L_{2e}, \forall T > 0,$

$$\langle \begin{bmatrix} 1 \\ \mathbb{P}_T\nabla \end{bmatrix} u_T, \Theta \begin{bmatrix} 1 \\ \mathbb{P}_T\nabla \end{bmatrix} u_T \rangle \leq 0 \tag{4}$$

PROOF : Inspired from [21], the proof is detailed in [19]. $\diamondsuit$

This result includes two conditions: a matrix inequality (3) related to the lower block of the feedback system and an inner product (4) that states an Integral Quadratic Constraint (IQC) on the upper block. It will be used throughout the paper to prove stability of systems under consideration.

2.3. Some suitable operators

It is required to define appropriate operators to model the time-delay system (1) as the feedback system in Figure 1. Clearly, two operators are essential, describing the dynamics and the delay: the integral operator

$$\begin{aligned} \mathcal{I}: \quad & L_{2e} \to L_{2e}, \\ & x(t) \to \int_0^t x(\theta)d\theta, \end{aligned} \tag{5}$$

and the delay operator (or shift operator)

$$\begin{aligned} \mathcal{D}: \quad & L_{2e} \to L_{2e}, \\ & x(t) \to x(t-h). \end{aligned} \tag{6}$$

The next step is to characterize the two operators by the use of IQCs introduced in the following two lemmas.

Lemma 1 *An IQC for the operator $\mathcal{I}$ is given by the following inequality: $\forall x \in L_{2e}^n$ and $\forall P > 0$,*

$$\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{I} 1_n \end{bmatrix} x_T, \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{I} 1_n \end{bmatrix} x_T \rangle \leq 0.$$

PROOF : Simple calculus shows that $\forall T > 0, \forall x \in L_{2e}^n$,

$$\begin{aligned} & \langle \begin{bmatrix} 1_n \\ \mathcal{I} 1_n \end{bmatrix} x, \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} 1_n \\ \mathcal{I} 1_n \end{bmatrix} x \rangle \\ & = -2 \int_0^T x(t)^T P \int_0^t x(s)ds\, dt \\ & = -2 \int_0^T \tfrac{d}{dt}(\mathcal{I}x)^T P(\mathcal{I}x)dt \\ & = -(\textstyle\int_0^T x(s)ds)^T P(\int_0^T x(s)ds) \leq 0 \end{aligned}$$

$\diamondsuit$

Lemma 2 *An IQC for the operator $\mathcal{D}$ is given by the following inequality: $\forall T > 0, \forall x \in L_{2e}^n$ and $\forall Q > 0$,*

$$\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{D} 1_n \end{bmatrix} x_T, \begin{bmatrix} -Q & 0 \\ 0 & \alpha Q \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{D} 1_n \end{bmatrix} x_T \rangle \leq 0. \tag{7}$$

with $\alpha = 1 - \dot{h}$.

PROOF : We get that $\forall T > 0, \forall x \in L_{2e}^n$,

$$\begin{aligned} & \langle \begin{bmatrix} 1_n \\ \mathcal{D} 1_n \end{bmatrix} x, \begin{bmatrix} -Q & 0 \\ 0 & Q(1-\dot{h}) \end{bmatrix} \begin{bmatrix} 1_n \\ \mathcal{D} 1_n \end{bmatrix} x \rangle \\ & = -\textstyle\int_0^{+\infty} x^T(u)Qx(u)du + \int_0^{\infty} x_d^T(t)Qx_d(t)\alpha\, dt \\ & = -\textstyle\int_0^{+T} x^T(t)Qx(t)dt + \int_{-h(0)}^{T-h(T)} x^T(u)Qx(u)du \\ & = -\textstyle\int_{T-h(T)}^{T} x(u)^T Qx(u)du \leq 0 \end{aligned}$$

where $x_d(t) = x(t-h(t))$. $\diamondsuit$

Since the IQC for the delay operator $\mathcal{D}$ do not depend on h, it is clear that it will induce some conservatism. As an example, in the constant delay case, the IQC defined by Lemma 2 is equivalent to replace the delay by a norm-bounded uncertainty. The phase of e^{-hs} is not taken into account. This can be approached by the operator $\delta_1(s) = \frac{1-e^{-hs}}{s}$, [20]. It can be embedded as a norm bounded uncertainty:

$$\sup_{\omega} \left\| \frac{1-e^{-j\omega h}}{j\omega} \right\| \leq h_{\max}.$$

The operator can also be interpreted as the first-order Taylor remainder of the exponential function e^{-hs}:

$$e^{-hs} = 1 - hs\delta_1(s).$$

Following the same idea, we formulate now the time-varying counterpart by considering a new operator $\mathcal{F}$ defined as follows:

$$\begin{aligned} \mathcal{F}: \quad & L_{2e} \to L_{2e}, \\ & x(t) \to \int_{t-h(t)}^{t} x(s)ds. \end{aligned} \tag{8}$$

Its characterization through an IQC can be derived as follows:

Lemma 3 *An IQC for the operator $\mathcal{F} = (1-\mathcal{D}) \circ \mathcal{I}$ is given by the following inequality: $\forall x \in L_{2e}^n$,*

$$\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{F} 1_n \end{bmatrix} x_T, \begin{bmatrix} -h_{\max}^2 R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{F} 1_n \end{bmatrix} x_T \rangle \leq 0,$$

where $h_{\max}$ is the upperbound on the delay $h(t)$ and R is some positive definite matrix.

PROOF : See [20] or [19]. $\diamondsuit$

The operator $\mathcal{F} = (1-\mathcal{D}) \circ \mathcal{I}$ can be slightly transformed as $\bar{\mathcal{F}} = \frac{1}{h(t)}\mathcal{F}$. The corresponding integral constraint is then expressed as follows.

Lemma 4 *An IQC for the operator $\bar{\mathcal{F}}$ is given by the following inequality:* $\forall T > 0, \forall x \in L_{2e}^n, \forall R > 0$,

$$\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \bar{\mathcal{F}} 1_n \end{bmatrix} x_T, \begin{bmatrix} -h_{\max} R & 0 \\ 0 & h(t) R \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \bar{\mathcal{F}} 1_n \end{bmatrix} x_T \rangle \leq 0.$$

PROOF : Omitted. ♢

An interesting contribution of this work is then the introduction of a novel operator that improves the modelling of the delay dynamic. This operator is related to the Taylor remainder of order two:

$$\mathcal{H} = \frac{\mathcal{I}^2 - \mathcal{D}\mathcal{I}^2 - h(t)\mathcal{I}}{h(t)} : \; x(t) \to \frac{1}{h(t)} \int_{t-h(t)}^{t} \int_{s}^{t} x(\theta) d\theta ds. \tag{9}$$

The following lemma gives a parameterized constraint on $\mathcal{H}$.

Lemma 5 *An IQC for the operator $\mathcal{H}$ is given by the following inequality:* $\forall T > 0, \forall x \in L_{2e}^n, \forall S > 0$,

$$\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{H} 1_n \end{bmatrix} x_T, \begin{bmatrix} -\frac{h_{\max}^2}{2} S & 0 \\ 0 & 2S \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T \mathcal{H} 1_n \end{bmatrix} x_T \rangle \leq 0.$$

PROOF : Note that

$$\|\mathcal{H}x\|^2 = \frac{1}{h^2(t)} \left(\int_{t-h(t)}^{t} \int_{s}^{t} x(\theta) d\theta ds \right)^T \left(\int_{t-h(t)}^{t} \int_{s}^{t} x(\theta) d\theta ds \right).$$

Using Cauchy-Schwartz inequality and setting $\tilde{\mathcal{H}} = \mathcal{H}h(t)$, $\forall T > 0, \forall x \in L_{2e}^n$, we get the following inequality,

$$\|\tilde{\mathcal{H}}x\|^2 \leq \left(\int_{t-h(t)}^{t} \int_{s}^{t} d\theta ds \right) \left(\int_{t-h(t)}^{t} \int_{s}^{t} \|x(\theta)\|^2 d\theta ds \right),$$

$$\frac{\|\tilde{\mathcal{H}}x\|^2}{h^2(t)/2} \leq \int_{t-h(t)}^{t} \int_{s}^{t} \|x(\theta)\|^2 d\theta ds,$$

$$\int_0^\infty \frac{2}{h^2(t)} \|\tilde{\mathcal{H}}x\|^2 dt \leq \int_0^\infty \int_{-h_{\max}}^{0} \int_{s}^{0} \|x_t(\theta)\|^2 d\theta ds dt,$$

$$\int_0^\infty \frac{2}{h^2(t)} \|\tilde{\mathcal{H}}x\|^2 dt \leq \frac{h_{\max}^2}{2} \int_0^\infty \|x(t)\|^2 dt.$$

Hence, we get

$$\int_0^\infty \frac{2}{h^2(t)} \|\tilde{\mathcal{H}}x\|^2 - \frac{h_{\max}^2}{2} \|x(t)\|^2 dt \leq 0,$$

which concludes the proof. ♢

This result is a key result for the main theorem because it allows to build a stability condition that does not require the system to be stable for small delays as we will see in section IV.

III. MAIN RESULTS

We present in this section the main results of the article which is based on the quadratic separation framework already used for constant delay systems. This approach allows us to establish the main theorem for the robust delay range stability analysis.

3.1. Methodology

To illustrate the proposed methodology, let us reformulate the system (1) as the feedback in Figure 1. As a first modelling, the system (1) can be described as the feedback

$$w(t) = \underbrace{\begin{bmatrix} \mathcal{I}1_n & 0 & 0 \\ 0 & \mathcal{D}1_n & 0 \\ 0 & 0 & \mathcal{F}1_n \end{bmatrix}}_{\nabla} \underbrace{\begin{bmatrix} \dot{x}(t) \\ x(t) \\ \dot{x}(t) \end{bmatrix}}_{z(t)}, \tag{10}$$

with

$$w(t) = \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t) - x(t-h(t)) \end{bmatrix}$$

and

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{E}} z(t) = \underbrace{\begin{bmatrix} A & A_d & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix}}_{\mathcal{A}} w(t). \tag{11}$$

Then, according to Theorem 1, we have to find a separator Θ that fulfills both inequalities (3)-(4). Combining the three constraints related to the different operators (stated by the lemmas in Section 2.3), a global (conservative) constraint on ∇ is deduced. Hence, the matrix

$$\Theta = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & -P & 0 & 0 \\ 0 & -Q & 0 & 0 & 0 & 0 \\ 0 & 0 & -h_{\max}^2 R & 0 & 0 & 0 \\ \hline -P & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha Q & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{array}\right] \tag{12}$$

where P, Q and R are $n \times n$ positive definite matrices, satisfies the inequality (4). The interconnected system (10)-(11) (and therefore system (1)) is thus stable if the matrix inequality (3), with $\mathcal{E}$, $\mathcal{A}$ and Θ defined as (11) and (12), holds. Because of the occurrences of $h_{\max}$ and $\dot{h}(t)$ in the criterion, it is refered to as *delay and rate dependent*. Setting $\dot{h}(t) = d$ in the separator, the condition becomes a single LMI that can be easily solved via SDP.

Remark 1 *It has been shown in [19] that the above criterion, based on the three operators, provides the same results in terms of conservativeness as several classical results of the literature [6, 25]. Indeed, such a particular choice of operators and separator amounts to choosing a Lyapunov-Krasovskii functional candidate of the form:*

$$V(x_t) = x_t^T(0)Px_t(0) + \int_{-h(t)}^{0} x_t^T(\theta)Qx_t(\theta)d\theta + \int_{t-h_m}^{0}\int_{\theta}^{0} \dot{x}_t^T(s)R\dot{x}_t(s)dsd\theta.$$

Further discussions on the quadratic separation method and the Lyapunov-Krasovskii counterpart for the constant delay case can be found in [8]. Other authors have emphasized the links between the Lyapunov method and the robust analysis in general, e.g., [11, 24, 26].

Remark 2 *A simpler criterion can be derived by removing $\mathcal{F}$ from ∇. In that case, the stability condition is independent of the delay because no information on the size of $h(t)$ (for instance, $h_{\max}$) appears in the matrices $\mathcal{E}$, $\mathcal{A}$ and Θ. However, a bound on $\dot{h}$ is still required.*

Remark 3 *Because the inequality in the Lemma 2 imposes a constraint on the delay variation $\dot{h}(t)$, a rate independent condition can be obtained if the system (1) is represented only through the first and the third operators of (10).*

In the next sections, we investigate new operators for the delayed dynamics. The objective is to reduce the conservatism of the stability analysis by taking into account some further informations on the delay. Throughout the paper we will apply the following procedure:

(a) Rewrite the delay system (1) as an interconnected feedback.
(b) Embed the integrator, the delay and other auxiliary operators into the matrix ∇.
(c) Construct IQCs for ∇.
(d) Establish the LMIs of Theorem 1 and compute the separator Θ.

3.2. Model extension

By extending the dynamics of the time-delay system, it is possible to achieve less conservative results, see [27, 8, 28]. An augmented state is composed of the original state vector and its derivative. By defining relationship between augmented state $\dot{x}$, $\ddot{x}$, the delay h and its derivative $\dot{h}$, an enhanced stability condition is provided. Differentiating the system (1), we get:

$$\ddot{x}(t) = A\dot{x}(t) + (1 - \dot{h}(t))A_d\dot{x}(t - h(t)).$$

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)), \\ \ddot{x}(t) = A\dot{x}(t) + (1 - \dot{h}(t))A_d\dot{x}(t - h(t)). \end{cases} \tag{13}$$

Introduce the augmented state

$$\varsigma(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}, \tag{14}$$

so by specifying the relationship between the two components of $\varsigma(t)$ with the equality $[0 \quad 1]\dot{\varsigma}(t) = [1 \quad 0]\varsigma(t)$, we have the descriptor system

$$E\dot{\varsigma}(t) = \bar{A}\varsigma(t) + \bar{A}_d\varsigma(t - h(t)), \tag{15}$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \\ 0 & 1 \end{bmatrix},$$

$$\bar{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & (1 - \dot{h}(t))A_d \\ 0 & 0 \end{bmatrix}.$$

3.3. Delay range stability condition

We now model the augmented time-varying delay system (15) through the new set of operators:

$$\underbrace{\begin{bmatrix} \varsigma(t) \\ \varsigma_d(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}}_{w(t)} = \underbrace{\begin{bmatrix} \mathcal{I}1_{2n} & 0 & 0 & 0 \\ 0 & \mathcal{D}1_{2n} & 0 & 0 \\ 0 & 0 & \bar{\mathcal{F}}1_{2n} & 0 \\ 0 & 0 & 0 & \mathcal{H}1_n \end{bmatrix}}_{\nabla} \underbrace{\begin{bmatrix} \dot{\varsigma}(t) \\ \varsigma(t) \\ \dot{\varsigma}(t) \\ \ddot{x}(t) \end{bmatrix}}_{z(t)} \tag{16}$$

with

$$
\begin{aligned}
\varsigma_d(t) &= \varsigma(t-h(t)),\\
w_1(t) &= \frac{\varsigma(t)-\varsigma(t-h(t))}{h(t)},\\
w_2(t) &= \dot{x}(t) - \frac{x(t)-x(t-h(t))}{h(t)} = E_1\varsigma(t) - E_2 w_1(t)
\end{aligned}
$$

and $E_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Then, according to the Lemmas (1)-(5), the separator

$$\Theta = \left[\begin{array}{c|c} \Theta_{11} & \Theta_{12} \\ \hline * & \Theta_{22} \end{array}\right], \tag{17}$$

$$
\begin{aligned}
\Theta_{11} &= \mathrm{diag}\big(0_{2n}, -Q, -h_{\max}R, -\frac{h_{\max}^2}{2}S\big),\\
\Theta_{12} &= \mathrm{diag}\big(-P, 0_{5n}\big),\\
\Theta_{22} &= \mathrm{diag}\big(0_{2n}, (1-\dot{h}(t))Q, h(t)R, 2S\big),
\end{aligned}
$$

with some positive definite matrices P, Q, $R \in \mathbb{R}^{2n\times 2n}$ and $S \in \mathbb{R}^{n\times n}$, fulfils the requirement (4) of Theorem 1. Consequently, the stability of (15) (and thus (1)) holds if

$$\xi^T(t)\Theta(h(t),\dot{h}(t))\xi(t) > 0 \tag{18}$$

such that $\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}\xi(t) = 0$ with $\xi = \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}$. This condition is equivalent to (3) of Theorem 1. Condition (18) can be rewritten as another equivalent condition

$$\psi^T(t)N^T(\dot{h}(t))\Theta(\dot{h}(t))N(\dot{h}(t))\psi(t) > 0, \tag{19}$$

where $\psi = \begin{bmatrix} x(t) \\ \varsigma(t-h(t)) \\ w_1(t) \\ w_2(t) \end{bmatrix}$, such that $S(h(t))\psi(t) = 0$ with

$$S = \begin{bmatrix} A & -1 & A_d & -1h(t) & 0 & 0 \\ 1 & 0 & -1 & 0 & -1h(t) & 0 \\ A & 0 & A_d & 0 & -1 & -1 \end{bmatrix} \tag{20}$$

and

$$N = \left[\begin{array}{ccc|c} AA & A_d(1-\dot{h}) & AA_d & \\ A & 0 & A_d & \\ A & 0 & A_d & \\ 1 & 0 & 0 & \\ AA & A_d(1-\dot{h}) & AA_d & 0_{8n\times 3n} \\ A & 0 & A_d & \\ AA & A_d(1-\dot{h}) & AA_d & \\ A & 0 & A_d & \\ \hline \multicolumn{4}{c}{1_{6n}} \end{array}\right]. \tag{21}$$

Applying Finsler's lemma, we note that condition (19) is equivalent to

$$N^T(\dot{h}(t))\Theta N(\dot{h}(t)) + XS(h(t)) + S^T(h(t))X^T > 0. \tag{22}$$

It is easy to show that $N^T(\dot{h}(t))\Theta N(\dot{h}(t))$ is affine, and thus convex, in h and $\dot{h}$. So condition (22) has to be assessed only at the four vertices of the polytop generated by the intervals of $h(t)$ and $\dot{h}(t)$. We are now in a position to state our main result.

Theorem 2 *For given positive scalars d, $h_{\min}$ and $h_{\max}$, if there exist positive definite matrices P, Q, $R \in \mathbb{R}^{2n\times 2n}$, a positive definite matrix $S \in \mathbb{R}^{n\times n}$ and a matrix $X \in \mathbb{R}^{6n\times 3n}$, then the system (1) with a time-varying delay constrained by (2) is asymptotically stable if the LMI (22) holds for $\dot{h}(t) \in \{-d, d\}$ and $h(t) \in \{h_{\min}, h_{\max}\}$.*

Remark 4 *Most of the papers in the literature provide the so-called delay dependent stability condition using the Lyapunov-Krasovskii method (see for example [6, 25, 29, 30]). Basically, a stable delay-free system is considered and the maximal value of the delay that preserves the stability is looked for. Recently, some papers have studied the problem of finding the largest delay interval $[h_{\min}, h_{\max}]$ for which the delay system is stable. In that case, the Lyapunov-Krasovskii functionnal depends explicitly on the delay $h(t)$, but also on the lower and upper bound [31, 13, 14, 23]. In these papers, they explore tightly the relations between $x(t-h_{\min})$ and $x(t-h_{\max})$ through the use of well-fitted Lyapunov-Krasovskii functionals. Nevertheless, their results are restricted to the case of a stable delay free system, i.e. a stable matrix $A + A_d$. We address in this paper the tricky case of the delay range condition where the delay belongs to an interval ($h(t) \in [h_{\min}, h_{\max}]$) and the system may be unstable for small delays.*

3.4. Robust stability

Quadratic separation provides a suitable framework for stability analysis of uncertain delay systems:

$$\dot{x}(t) = A(\Delta)x(t) + A_d(\Delta)x(t-h(t)) \tag{23}$$

where

$$\begin{bmatrix} A(\Delta) & A_d(\Delta) \end{bmatrix} = \begin{bmatrix} A & A_d \end{bmatrix} + B\Delta\begin{bmatrix} C & C_d \end{bmatrix}.$$

The second term of the right hand side describes the uncertainty characterizing system (23). The uncertain time-varying matrix $\Delta(t)$ satisfies

$$\Delta^T(t)\Delta(t) \leq 1,\ \forall t \geq 0,\ \forall \Delta \in \Omega, \tag{24}$$

and models non-linear and neglected dynamics as well as parametric uncertainties. The matrices B, C and C_d are constant and of appropriate dimensions. According to the set of admissible uncertainties and (24), we have to find a separator U such that

$$\langle \begin{bmatrix} 1 \\ \Delta \end{bmatrix} x, \underbrace{\begin{bmatrix} U_1 & U_2 \\ U_2^* & U_3 \end{bmatrix}}_{U} \begin{bmatrix} 1 \\ \Delta \end{bmatrix} x \rangle < 0,\ \forall \Delta \in \Omega. \tag{25}$$

For instance, assume Ω is a set of diagonal real-valued matrices with bounded uncertainties:

$$\Omega = \left\{ \Delta = \operatorname{diag}(\delta_1, ..., \delta_N) \mid |\delta_i| \leq \bar{\delta}_i \right\}.$$

Then, inequality (25) holds with

$$U = \operatorname{diag}\left(-\bar{\delta}_1^2 u_1, \ldots, -\bar{\delta}_N^2 u_N, u_1, \ldots, u_N\right)$$

where u_i, $i = 1, ..., N$, are scalar decision variables. We propose to analyze the robust stability of system (23) with the following theorem.

Theorem 3 *For given positive scalars $h_{\max}$ and d, if there exists positive definite matrices P, Q, $R \in \mathbb{R}^{n\times n}$ and matrices U_1, U_2, U_3 such that (25) holds, then system (23) with a time-varying delay constrained by (2) is asymptotically stable for any uncertainty $\Delta \in \Omega$ if the LMI condition (3) holds with Θ, $\mathcal{E}$ and $\mathcal{A}$ defined as follows:*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{22} \end{bmatrix}, \quad \begin{aligned} \Theta_{11} &= \operatorname{diag}(0_n, -Q, -h_{\max}^2 R, U_1), \\ \Theta_{12} &= \operatorname{diag}(-P, 0_{2n}, U_2), \\ \Theta_{22} &= \operatorname{diag}(0_n, (1-d)Q, R, U_3), \end{aligned} \tag{26}$$

$$\mathcal{E} = \begin{bmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ -1_n & 0 & 1_n & 0 \\ 0 & 0 & 0 & 1_n \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & A_d & 0 & B \\ 1_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C & C_d & 0 & 0 \\ 1_n & -1 & -1 & 0 \end{bmatrix}. \tag{27}$$

PROOF : First, introducing the exogenous signals

$$w_\Delta = \Delta z_\Delta, \text{ with } z_\Delta = Cx(t) + C_d x(t-h(t)),$$

we rewrite system (23) as the interconnection of

$$\underbrace{\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t) - x(t-h(t)) \\ w_\Delta(t) \end{bmatrix}}_{w(t)} = \nabla \underbrace{\begin{bmatrix} \dot{x}(t) \\ x(t) \\ \dot{x}(t) \\ z_\Delta(t) \end{bmatrix}}_{z(t)}$$

with $\nabla = (\mathcal{I}1_n, \mathcal{D}1_n, \mathcal{F}1_n, \Delta)$ and

$$\mathcal{E}z(t) = \mathcal{A}w(t), \tag{28}$$

where $\mathcal{E}$ and $\mathcal{A}$ are defined in (27). Combining every IQC related to each operators defined by lemmas and the struture of the uncertainty leading to (25), a separator of the form of (26) fulfills the requirement (4). Finally, condition (3) provides the robust (with respect to the uncertain set Ω) stability criterion. $\diamondsuit$

For the sake of simplicity, Theorem 3 is given only with the two operators $\mathcal{D}$ and $\mathcal{F}$. In the case of time invariant uncertainties, it is easy to extend to the third operator $\mathcal{H}$. If Δ is time-dependent, the model extension (Subsection 3.2) is however more tricky to apply and a good knowledge of the uncertainty is required.

IV. NUMERICAL EXAMPLES

We illustrate the developed theory through three examples.

4.1. First example: delay dependent case

Consider

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h(t)). \tag{29}$$

First, let us remark that the delay-free case is stable. Next, the maximal allowable delay, $h_{\max}$, is computed. To demonstrate the effectiveness of our approach, results are compared to the literature. All papers, except [18, 20, 32], use Lyapunov theory in order to derive stability criteria. In [18], [20], the stability problem is solved in an IQC framework. The results are shown in Table 1.

In [20] and [32], the delay is modeled as an uncertain parameter and appropriate weighting filters are used to bound it. Their methodologies provide very

Table 1. The maximal allowable delays $h_{\max}$ for system (29)

d	0	0.1	0.2	0.5	0.8	1	$\forall d > 1$
Fridman et al (2002) [6]	4.472	3.604	3.033	2.008	1.364	0.999	0.999
Wu et al (2004) [25]	4.472	3.604	3.033	2.008	1.364	-	-
Kao et al (2005) [18]	4.472	3.604	3.033	2.008	1.364	0.999	-
Fridman et al (2006) [11]	1.632	1.632	1.632	1.632	1.632	1.632	1.632
Kao et al (2007) [20]	6.117	4.714	3.807	2.280	1.608	1.360	-
He et al (2007) [29]	4.472	3.605	3.039	2.043	1.492	1.345	1.345
He et al (2007) [9]	4.472	3.605	3.039	2.043	1.492	1.345	1.345
Ariba et al (2009) [32]	6.117	4.794	3.995	2.682	1.957	1.602	1.345
Sun et al (2010) [14]	4.476	3.611	3.047	2.072	1.590	1.529	-
Theorem 2	5.120	4.081	3.448	2.528	2.152	1.991	-

good results, however, they are restricted to time-delay system that are stable without delay. Although Theorem 2 does not provide the best condition, it provides conservatism reduction compared to many conditions from the literature. Besides, Theorem 2 addresses the stability of systems with *interval delays*, which may be unstable for small delays (or without delays).

Table 2. Interval of stabilizing delays for system (30)

d	$h_{\min}$	$h_{\max}$
0	0.102	1.424
0.1	0.102	1.424
0.2	0.103	1.423
0.5	0.104	1.421
0.8	0.105	1.419
1.0	0.105	1.418
0 (analytical)	0.10016826	1.7178

4.2. Second example: delay range case

Consider $\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = u(t)$, with a static delayed output feedback $u(t) = ky(t - h(t))$. Choosing $k = 1$, we get:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - h(t)). \tag{30}$$

In order to assess the interval of the delay such that system (30) is stable, Theorem 2 is applied with given $h_{\min}$ and $h_{\max}$. Then, a sliding window principle is performed to stretch the bounds. The results are presented in Table 2.

Theorem 2 allows us to assess a conservative region of stability w.r.t. k and $h(t)$ (for $d = 1$). It provides a set of values of k that ensures a stabilizing delayed output feedback for $\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = u(t)$ as shown in Figure 2.

4.3. Third example: robust stability

Consider

$$\dot{x}(t) = \begin{bmatrix} -2 + \delta_1 \cos t & 0 \\ 0 & -1 + \delta_2 \sin t \end{bmatrix} x(t) + \begin{bmatrix} -1 + \gamma_1 \cos t & 0 \\ -1 & -1 + \gamma_2 \sin t \end{bmatrix} x(t - h(t)), \tag{31}$$

extracted from [33]. δ_i and γ_i are uncertain bounded parameters:

$$|\delta_1| \le 1.6, \ |\delta_2| \le 0.05, \ |\gamma_1| \le 0.1, \ |\gamma_2| \le 0.3.$$

Let us rewrite the system as in (23) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \ A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$B = \mathsf{1}_2, \ \mathsf{C} = \mathrm{diag}(1.6, \ 0.05), \ \mathsf{C_d} = \mathrm{diag}(0.1, \ 0.3).$$

Simulation results are gathered in the Table 3.

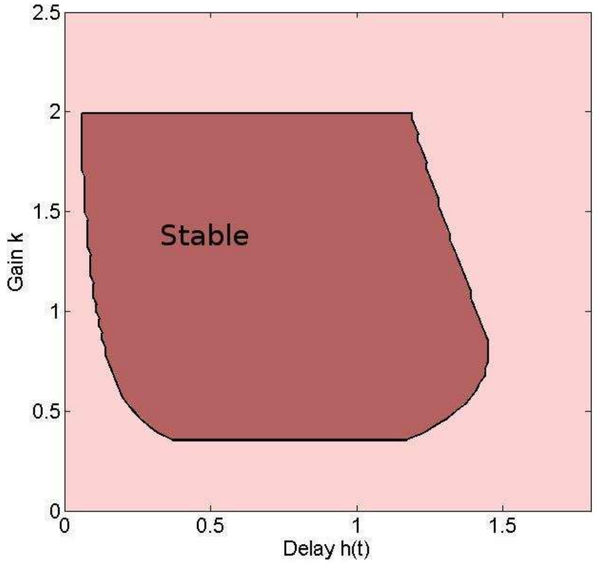

Fig. 2. Stability region of $\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = ky(t - h(t))$ w.r.t. k and $h(t)$.

Table 3. Maximal allowable delay $h_{\max}$ for the system (31) w.r.t. d.

d	0	0.1	0.5	0.8
[33]	0.241	0.234	0.188	0.110
[25]	1.149	1.106	0.924	0.760
[20]	1.416	1.302	0.974	0.829
Theorem 3	1.515	1.422	1.105	0.910

V. CONCLUSION

In this paper, stability analysis of a time-varying delay system has been studied by means of quadratic separation. Inspired from previous work on time-delay systems with constant delay [8], novel stability criteria for time-varying delay system were provided. Using an augmented state, which emphasizes the relation between $\dot{h}$ and $(\dot{x}, \ddot{x})$, the resulting criteria have been expressed in terms of a convex optimization problem with LMI constraints. Finally, numerical examples show that this method reduced conservatism and improve the maximal allowable interval on the delay.

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