Chapter 9
Discrete-Time Networked Control Under Scheduling Protocols

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Abstract This chapter analyzes the exponential stability of discrete-time networked control systems viadelay-dependent Lyapunov-Krasovskii methods. The time-delay approach has been developed recently for the stabilization of continuous-time networked control systems under a Round-Robin protocol and a weighted Try-Once-Discard protocol, respectively. In the present chapter, the time-delay approach is extended to the stability analysis of discrete-time networked control systems under both these scheduling protocols. First, the closed-loop system is modeled as a discrete-time switched system with multiple and ordered time-varying delays under the Round-Robin protocol. Then, a discrete-time hybrid system model for the closed-loop system is presented under these protocols. It contains time-varying delays in the continuous dynamics and in the reset conditions. The communication delays are allowed to be larger than the sampling intervals. Polytopic uncertainties in the system model can be easily included in our analysis. The efficiency of the time-delay approach is illustrated in an example of a cart-pendulum system.

9.1 Introduction

Network control systems (NCSs) are spatially distributed systems in which the communication between sensors, actuators, and controllers occurs through a communication network [1, 22]. In many such systems, only one node is allowed to use

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the communication channel at each time instant. In the present chapter, we focus on the stability analysis of discrete-time NCSs with communication constraints. The scheduling of sensor information towards the controller is defined by a Round-Robin (RR) protocol or by a weighted Try-Once-Discard (TOD) protocol. A linear system with distributed sensors is considered. Three recent approaches for NCSs are based on discrete-time systems [2, 3, 7], impulsive/hybrid systems [10, 18, 19] and time-delay systems [4, 6, 8, 12, 21].

The time-delay approach was developed for the stabilization of continuous-time NCSs under a RR protocol in [15] and under a weighted TOD protocol in [16]. The closed-loop system was modeled as a switched system with multiple and ordered time-varying delays under RR protocol or as a hybrid system with time-varying delays in the dynamics and in the reset equations under TOD protocol. Differently from the existing hybrid and discrete-time approaches on the stabilization of NCS with scheduling protocols, the time-delay approach allows treating the case of large communication delays.

In the present chapter, the time-delay approach is extended to the stability analysis of discrete-time NCSs under RR and weighted TOD scheduling protocols. First, the closed-loop system is modeled as a discrete-time switched system with multiple and ordered time-varying delays under RR protocol. Then, a discrete-time hybrid system model for the closed-loop system is presented that contains time-varying delays in the continuous dynamics and in the reset conditions under TOD and RR protocols. Differently from [14], the same conditions are derived for the exponential stability of the resulting hybrid system model under both these scheduling protocols. The communication delays are allowed to be larger than the sampling intervals. The conditions are given in terms of Linear Matrix Inequalities (LMIs). Polytopic uncertainties in the system model can be easily included in the analysis. The efficiency of the presented approach is illustrated by a cart-pendulum system.

Notation: Throughout the chapter, the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. $\mathbb{Z}^+$, $\mathbb{N}$ and $\mathbb{R}^+$ denote the set of non-negative integers, positive integers integers and non-negative real numbers, respectively.

9.2 Discrete-Time Networked Control Systems Under Round-Robin Scheduling: A Switched System Model

9.2.1 Problem Formulation

Consider the system architecture in Fig. 9.1 with plant

$$x(t+1) = Ax(t) + Bu(t), \quad t \in \mathbb{Z}^+, \tag{9.1}$$

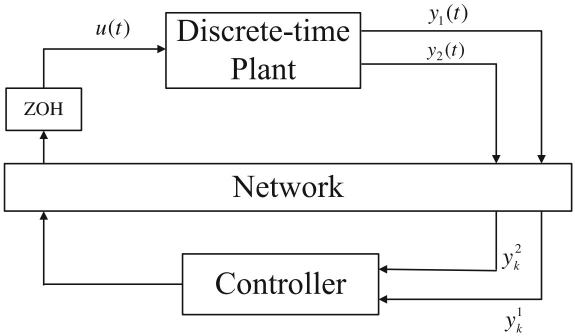

Fig. 9.1 Discrete-time NCSs under RR scheduling

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, A and B are system matrices with appropriate dimensions. The initial condition is given by $x(0) = x_0$.

The NCS has several nodes connected via networks. For the sake of simplicity, we consider two sensor nodes $y_i(t) = C_i x(t), i = 1, 2$ and we denote $C = \begin{bmatrix} C_1^T & C_2^T \end{bmatrix}^T$, $y(t) = \begin{bmatrix} y_1^T(t) & y_2^T(t) \end{bmatrix}^T \in \mathbb{R}^{n_y},\ t \in \mathbb{Z}^+$. We let s_k denote the unbounded and monotonously increasing sequence of sampling instants, i.e.,

$$0 = s_0 < s_1 < \cdots < s_k < \cdots,\ \lim_{k \to \infty} s_k = \infty,\ s_{k+1} - s_k \leq \text{MATI},\ k \in \mathbb{Z}^+, \tag{9.2}$$

where $\{s_0, s_1, s_2, \dots\}$ is a subsequence of $\{0, 1, 2, \dots\}$ and MATI denotes the Maximum Allowable Transmission Interval. At each sampling instant s_k, one of the outputs $y_i(t) \in \mathbb{R}^{n_i} (n_1 + n_2 = n_y)$ is sampled and transmitted via the network. First, we consider the RR scheduling protocol for the choice of the active output node: the outputs are transmitted one after another, i.e., $y_i(t) = C_i x(t), t \in \mathbb{Z}^+$ is transmitted only at the sampling instant $t = s_{2p+i-1},\ p \in \mathbb{Z}^+,\ i = 1, 2$. After each transmission and reception, the values in $y_i(t)$ are updated with the newly received values, while the values of $y_j(t)$ for $j \neq i$ remain the same, as no additional information is received. This leads to the constrained data exchange expressed as

$$y_k^i = \begin{cases} y_i(s_k) = C_i x(s_k), & k = 2p + i - 1, \\ \qquad y_{k-1}^i, & k \neq 2p + i - 1, \end{cases} \quad p \in \mathbb{Z}^+.$$

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the network is subject to a variable delay $\eta_k \in \mathbb{Z}^+$, which is allowed to be larger than the sampling intervals. Then $t_k = s_k + \eta_k$ is the updating time instant of the ZOH device.

Assume that the network-induced delay η_k and the time span between the updating and the current sampling instants are bounded:

$$t_{k+1} - 1 - t_k + \eta_k \leq \tau_M,\ 0 \leq \eta_m \leq \eta_k \leq \eta_M,\ k \in \mathbb{Z}^+, \tag{9.3}$$

where τ_M, η_m and η_M are known non-negative integers. Then

$$\begin{aligned}
(t_{k+1}-1)-s_k &= s_{k+1}-s_k+\eta_{k+1}-1 \le \mathrm{MATI}+\eta_M-1=\tau_M,\\
(t_{k+1}-1)-s_{k-1} &= s_{k+1}-s_{k-1}+\eta_{k+1}-1 \le 2\,\mathrm{MATI}+\eta_M-1\\
&= 2\tau_M-\eta_M+1 \triangleq \bar{\tau}_M,\\
t_{k+1}-t_k &\le \tau_M-\eta_m+1.
\end{aligned} \tag{9.4}$$

Note that the first updating time t_0 corresponds to the first data received by the actuator, which means that $u(t)=0, t\in[0,t_0-1]$. Then for $t\in[0,t_0-1]$, (9.1) is given by

$$x(t+1)=Ax(t),\ t=0,1,\ldots,t_0-1,\quad t\in\mathbb{Z}^+. \tag{9.5}$$

In [15], a time-delay approach was developed for the stability and L_2-gain analysis of continuous-time NCSs with RR scheduling. In this section, we consider the stability analysis of discrete-time NCSs under RR scheduling protocol.

It is assumed that the controller and the actuator are event-driven. Suppose that there exists a matrix $K=\begin{bmatrix}K_1 & K_2\end{bmatrix}$, $K_1\in\mathscr{R}^{n_u\times n_1}$, $K_2\in\mathscr{R}^{n_u\times n_2}$ such that $A+BKC$ is Schur. Consider the static output feedback of the form:

$$u(t)=K_1y_k^1+K_2y_k^2,\ t\in\left[t_k,t_{k+1}-1\right],\ t\in\mathbb{N},\ k\in\mathbb{N},$$

Following [15], the closed-loop system with RR scheduling is modeled as a switched system:

$$\begin{aligned}
x(t+1)&=Ax(t)+A_1x(t_{k-1}-\eta_{k-1})+A_2x(t_k-\eta_k),\ t\in[t_k,t_{k+1}-1],\\
x(t+1)&=Ax(t)+A_1x(t_{k+1}-\eta_{k+1})+A_2x(t_k-\eta_k),\ t\in[t_{k+1},t_{k+2}-1],
\end{aligned} \tag{9.6}$$

where $k=2p-1,\ p\in\mathbb{N},\ A_i=BK_iC_i,\ i=1,2$. For $t\in[t_k,t_{k+1}-1]$, we can represent

$$t_k-\eta_k=t-\tau_1(t),\quad t_{k-1}-\eta_{k-1}=t-\tau_2(t),$$

where

$$\begin{aligned}
&\tau_1(t)=t-t_k+\eta_k<\tau_2(t)=t-t_{k-1}+\eta_{k-1},\\
&\tau_1(t)\in[\eta_m,\tau_M],\quad \tau_2(t)\in[\eta_m,\ \bar{\tau}_M],\quad t\in[t_k,t_{k+1}-1].
\end{aligned} \tag{9.7}$$

Therefore, (9.6) for $t\in[t_k,t_{k+1}-1]$ can be considered as a system with two time-varying interval delays, where $\tau_1(t)<\tau_2(t)$. Similarly, for $t\in[t_{k+1},t_{k+2}-1]$, (9.6) is a system with two time-varying delays, one of which is less than another.

For $t\in[t_0,t_1-1]$, the closed-loop system is reduced to the following form

$$x(t+1)=Ax(t)+A_1x(t_0-\eta_0),\ t\in[t_0,t_1-1].$$

9.2.2 Stability Analysis of the Switched System Model

Applying the following discrete-time Lyapunov-Krasovskii functional (LKF) to system (9.6) with time-varying delay from the maximum delay interval $[\eta_m, \bar{\tau}_M]$ [5]:

$$\begin{aligned}
V_{RR}(t) = x^T(t)Px(t) + &\sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) \\
&+\eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_0 \eta(s) + \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \lambda^{t-s-1} x^T(s) S_1 x(s) \\
&+(\bar{\tau}_M - \eta_m) \sum_{j=-\bar{\tau}_M}^{-\eta_m-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \eta^T(s) R_1 \eta(s), \\
\eta(t) = x(t+1) - x(t),\ P > 0,\ &S_i > 0,\ R_i > 0,\ i = 0, 1,\ 0 < \lambda < 1,\ t \geq 0,
\end{aligned}$$

where following [13], we define (for simplicity) $x(t) = x_0,\ t \leq 0$.

Similar to [15], by taking advantage of the ordered delays and using convex analysis of [20], we arrive to the following sufficient conditions for the stability of the switched system:

Theorem 1 *Given scalars* $0 < \lambda \leq 1$*, positive integers* $0 \leq \eta_m \leq \eta_M < \tau_M$*, and* K_1, K_2*, let there exist scalars* $n \times n$ *matrices* $P > 0$*,* $S_\vartheta > 0$*,* $R_\vartheta > 0$ $(\vartheta = 0, 1)$*,* G_1^i, G_2^i, G_3^i $(i = 1, 2)$ *such that the following matrix inequalities are feasible:*

$$\Omega_i = \begin{bmatrix} R_1 & G_1^i & G_2^i \\ * & R_1 & G_3^i \\ * & * & R_1 \end{bmatrix} \geq 0, \tag{9.8}$$

$$(F_0^i)^T P F_0^i + \Sigma + (F_{01}^i)^T H F_{01}^i - \lambda^{\eta_m} F_{12}^T R_0 F_{12} - \lambda^{\bar{\tau}_M} F^T \Omega_i F < 0, \tag{9.9}$$

where

$$\begin{aligned}
&F_0^i = [A\ 0\ A_i\ A_{3-i}\ 0], \\
&F_{01}^i = [A - I\ 0\ A_i\ A_{3-i}\ 0],\ i = 1, 2, \\
&F_{12} = [I\ -I\ 0\ 0\ 0], \\
&F = \begin{bmatrix} 0 & I & -I & 0 & 0 \\ 0 & 0 & I & -I & 0 \\ 0 & 0 & 0 & I & -I \end{bmatrix}, \\
&\Sigma = \mathrm{diag}\{S_0 - \lambda P, -\lambda^{\eta_m}(S_0 - S_1), 0, 0, -\lambda^{\bar{\tau}_M} S_1\}, \\
&H = \eta_m^2 R_0 + (\bar{\tau}_M - \eta_m)^2 R_1.
\end{aligned}$$

Then, the closed-loop system (9.6) is exponentially stable with the decay rate λ.

Proof Consider $t \in [t_k, t_{k+1} - 1]$ and define $\xi(t) = \mathrm{col}\{x(t), x(t - \eta_m), x(t_{k-1} - \eta_{k-1}), x(t_k - \eta_k), x(t - \bar{\tau}_M)\}$. Along (9.6), we have

$$\begin{aligned} V_{RR}(t+1) - \lambda V_{RR}(t) \leq \xi^T(t)[(F_0^i)^T P F_0^i + \Sigma + (F_{01}^i)^T H F_{01}^i]\xi(t) \\ -\lambda^{\eta_m}\eta_m \sum_{s=t-\eta_m}^{t-1} \eta^T(s) R_0 \eta(s) \\ -\lambda^{\bar{\tau}_M}(\bar{\tau}_M - \eta_m) \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s). \end{aligned}$$

By Jensen's inequality [9], we have

$$\begin{aligned} \eta_m \sum_{s=t-\eta_m}^{t-1} \eta^T(s) R_0 \eta(s) &\geq \sum_{s=t-\eta_m}^{t-1} \eta^T(s) R_0 \sum_{s=t-\eta_m}^{t-1} \eta(s) \\ &= \xi^T(t) F_{12}^T R_0 F_{12} \xi(t). \end{aligned}$$

Taking into account that $t_{k-1} - \eta_{k-1} < t_k - \eta_k$ (i.e. that the delays satisfy the relation (9.7)) and applying further Jensen's inequality, we obtain

$$\begin{aligned} &-(\bar{\tau}_M - \eta_m) \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s) \\ &= -(\bar{\tau}_M - \eta_m)\Big[\sum_{s=t_k-\eta_k}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s) + \sum_{s=t_{k-1}-\eta_{k-1}}^{t_k-\eta_k-1} \eta^T(s) R_1 \eta(s) \\ &\qquad + \sum_{s=t-\bar{\tau}_M}^{t_{k-1}-\eta_{k-1}-1} \eta^T(s) R_1 \eta(s) \Big] \\ &\leq -\frac{1}{\alpha_1} f_1(t) - \frac{1}{\alpha_2} f_2(t) - \frac{1}{\alpha_3} f_3(t), \end{aligned}$$

where

$$\begin{aligned} &\alpha_1 = \frac{t - \eta_m - t_k + \eta_k}{\bar{\tau}_M - \eta_m},\ \alpha_2 = \frac{t_k - \eta_k - t_{k-1} + \eta_{k-1}}{\bar{\tau}_M - \eta_m},\ \alpha_3 = \frac{\bar{\tau}_M - t + t_{k-1} - \eta_{k-1}}{\bar{\tau}_M - \eta_m}, \\ &f_1(t) = [x(t-\eta_m) - x(t_k - \eta_k)]^T R_1 [x(t-\eta_m) - x(t_k - \eta_k)], \\ &f_2(t) = [x(t_k - \eta_k) - x(t_{k-1} - \eta_{k-1})]^T R_1 [x(t_k - \eta_k) - x(t_{k-1} - \eta_{k-1})], \\ &f_3(t) = [x(t_{k-1} - \eta_{k-1}) - x(t - \bar{\tau}_M)]^T R_1 [x(t_{k-1} - \eta_{k-1}) - x(t - \bar{\tau}_M)]. \end{aligned}$$

Denote

$$\begin{aligned} g_{1,2}(t) &= [x(t-\eta_m) - x(t_k - \eta_k)]^T G_1^1 [x(t_k - \eta_k) - x(t_{k-1} - \eta_{k-1})], \\ g_{1,3}(t) &= [x(t-\eta_m) - x(t_k - \eta_k)]^T G_2^1 [x(t_{k-1} - \eta_{k-1}) - x(t - \bar{\tau}_M)], \\ g_{2,3}(t) &= [x(t_k - \eta_k) - x(t_{k-1} - \eta_{k-1})]^T G_3^1 [x(t_{k-1} - \eta_{k-1}) - x(t - \bar{\tau}_M)]. \end{aligned}$$

Note that (9.8) with $i = 1$ guarantees $\begin{bmatrix} R_1 & G_j^1 \\ * & R_1 \end{bmatrix} \geq 0$ $(j = 1, 2, 3)$, and, thus,

$$\begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{i,j}(t) & f_j(t) \end{bmatrix} \geq 0, \ i \neq j, \ i = 1, 2, \ j = 2, 3.$$

Then, we arrive to

$$-(\bar{\tau}_M - \eta_m) \sum_{s=t-\bar{\tau}_M}^{t-\eta_m-1} \eta^T(s) R_1 \eta(s) \leq -\frac{1}{\alpha_1} f_1(t) - \frac{1}{\alpha_2} f_2(t) - \frac{1}{\alpha_3} f_3(t)$$
$$\leq -f_1(t) - f_2(t) - f_3(t) - 2g_{1,2}(t) - 2g_{1,3}(t) - 2g_{2,3}(t) = -\xi^T(t) F^T \Omega_1 F \xi(t),$$

where Ω_1 is given by (9.8) with $i = 1$. The latter inequality holds if (9.8) is feasible [20]. Hence, (9.9) with $i = 1$ guarantees that $V_{RR}(t+1) - \lambda V_{RR}(t) \leq 0$ for $t \in [t_k, t_{k+1} - 1]$.

Similarly, for $t \in [t_{k+1}, t_{k+2} - 1]$, (9.8) and (9.9) with $i = 2$ guarantee $V_{RR}(t + 1) - \lambda V_{RR}(t) \leq 0$. Thus, (9.6) is exponentially stable with the decay rate λ.

9.3 Discrete-Time Networked Systems Under the Try-Once-Discard and Round-Robin Scheduling: A Hybrid Time-Delay Model

9.3.1 Problem Formulation

In [16], a weighted TOD protocol was analyzed for the stabilization of continuous-time NCSs. In this section, we consider discrete-time NCSs under TOD and RR scheduling via a hybrid delayed model.

Consider (9.1) with two sensor nodes $y_i(t) = C_i x(t), i = 1, 2$ and a sequence of sampling instants (9.2). At each sampling instant s_k, one of the outputs $y_i(t) \in \mathbb{R}^{n_i}$ $(n_1 + n_2 = n_y)$ is sampled and transmitted via the network. Denote by $\hat{y}(s_k) = \left[\, \hat{y}_1^T(s_k) \ \hat{y}_2^T(s_k) \,\right]^T \in \mathbb{R}^{n_y}$ the output information submitted to the scheduling protocol. At each sampling instant s_k, one of $\hat{y}_i(s_k)$ values is updated with the recent output $y_i(s_k)$.

It is assumed that no packet dropouts and no packet disorders will happen during the data transmission over the network. The transmission of the information over the network is subject to a variable delay η_k. Then $t_k = s_k + \eta_k$ is the updating time instant. As in the previous section, we allow the delays to be large provided that the old sample cannot get to the destination (to the controller or to the actuator) after the current one. Assume that the network-induced delay η_k and the time span between the updating and the current sampling instants satisfy (9.3).

Following [16], consider the error between the system output $y(s_k)$ and the last available information $\hat{y}(s_{k-1})$:

$$\begin{array}{l} e(t) = \mathrm{col}\{e_1(t), e_2(t)\} \equiv \hat{y}(s_{k-1}) - y(s_k),\ t \in [t_k, t_{k+1} - 1], \\ t \in \mathbb{Z}^+, k \in \mathbb{Z}^+,\ \hat{y}(s_{-1}) \stackrel{\Delta}{=} 0,\ e(t) \in \mathbb{R}^{n_y}. \end{array}$$

The control signal to be applied to the system (9.1) is given by

$$u(t) = K_{i_k^*} y_{i_k^*}(t_k - \eta_k) + K_i \hat{y}_i(t_{k-1} - \eta_{k-1})_{|i \neq i_k^*},\ t \in [t_k, t_{k+1} - 1],$$

where $K = [K_1\ K_2]$, $K_1 \in \mathbb{R}^{n_u \times n_1}$, $K_2 \in \mathbb{R}^{n_u \times n_2}$ such that $A + BKC$ is Schur. The closed-loop system can be presented as

$$\begin{array}{l} x(t+1) = Ax(t) + A_1 x(t_k - \eta_k) + B_i e_i(t)_{|i \neq i_k^*}, \\ e(t+1) = e(t),\ t \in [t_k, t_{k+1} - 2],\ t \in \mathbb{Z}^+, \end{array} \tag{9.10}$$

with the delayed reset system for $t = t_{k+1} - 1$

$$\begin{array}{l} x(t_{k+1}) = Ax(t_{k+1} - 1) + A_1 x(t_k - \eta_k) + B_i e_i(t_k)_{|i \neq i_k^*}, \\ e_i(t_{k+1}) = C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})],\ i = i_k^*, \\ e_i(t_{k+1}) = e_i(t_k) + C_i[x(t_k - \eta_k) - x(t_{k+1} - \eta_{k+1})],\ i \neq i_k^*, \end{array} \tag{9.11}$$

where $A_1 = BKC,\ B_i = BK_i,\ K = [K_1\ K_2],\ i = 1, 2$. The initial condition for (9.10), (9.11) has the form of $e(t_0) = -Cx(t_0 - \eta_0) = -Cx_0$ and (9.5). We will consider stability analysis of the discrete-time hybrid system (9.10), (9.11) under TOD and RR protocols described next.

9.3.2 Scheduling Protocols

9.3.2.1 TOD Protocol

Let $Q_1 > 0,\ Q_2 > 0$ be some weighting matrices. At the sampling instant s_k, the weighted TOD protocol is a protocol for which the active output node is defined as any index i_k^* that satisfies

$$|\sqrt{Q}_{i_k^*} e_{i_k^*}(t)|^2 \geq |\sqrt{Q}_i e_i(t)|^2,\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}^+,\ i = 1, 2. \tag{9.12}$$

A possible choice of i_k^* is given by

$$i_k^* = \min\{\arg \max_{i \in \{1,2\}} |\sqrt{Q}_i \left(\hat{y}_i(s_{k-1}) - y_i(s_k)\right)|^2\}.$$

9.3.2.2 RR Protocol

The active output node is chosen periodically:

$$i_k^* = i_{k+2}^*, \text{ for all } k \in \mathbb{Z}^+, \; i_0^* \neq i_1^*. \tag{9.13}$$

9.3.3 Stability Analysis of Discrete-Time Hybrid Delayed System Under TOD and RR Protocols

Consider the LKF of the form:

$$V_e(t) = V(t) + \frac{t_{k+1} - t}{\tau_M - \eta_m + 1} e_i^T(t_k) Q_i e_i(t_k)_{|i \neq i_k^*},$$

where

$$\begin{aligned}
V(t) &= \tilde{V}(t) + V_Q(t),\\
V_Q(t) &= (\tau_M - \eta_m) \sum_{s=t_k-\eta_k}^{t-1} \lambda^{t-s-1} \zeta^T(s) Q \zeta(s),\\
\tilde{V}(t) &= x^T(t) P x(t) + \sum_{s=t-\eta_m}^{t-1} \lambda^{t-s-1} x^T(s) S_0 x(s) + \sum_{s=t-\tau_M}^{t-\eta_m-1} \lambda^{t-s-1} x^T(s) S_1 x(s)\\
&\quad + \eta_m \sum_{j=-\eta_m}^{-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \zeta^T(s) R_0 \zeta(s)\\
&\quad + (\tau_M - \eta_m) \sum_{j=-\tau_M}^{-\eta_m-1} \sum_{s=t+j}^{t-1} \lambda^{t-s-1} \zeta^T(s) R_1 \zeta(s),
\end{aligned}$$

and

$$\begin{array}{l}
\zeta(t+1) = x(t+1) - x(t), \; P > 0, \; S_i > 0, \; R_i > 0, \; Q > 0, \; Q_j > 0,\\
0 < \lambda < 1, \; i = 0, 1, \; j = 1, 2, \; t \in [t_k, t_{k+1} - 1], \; t \in \mathbb{Z}^+, \; k \in \mathbb{Z}^+,
\end{array}$$

where we define $x(t) = x_0, \; t \leq 0$. Our objective is to guarantee that

$$V_e(t+1) - \lambda V_e(t) \leq 0, \; t \in [t_k, t_{k+1} - 1], \; t \in \mathbb{Z}^+ \tag{9.14}$$

holds along (9.10), (9.11). The inequality (9.14) implies the following bound

$$\begin{array}{l}
V(t) \leq V_e(t) \leq \lambda^{t-t_0} V_e(t_0), \; t \geq t_0, \; t \in \mathbb{Z}^+,\\
V_e(t_0) \leq V(t_0) + \min_{i=1,2} \{|\sqrt{Q_i} e_i(t_0)|^2\},
\end{array} \tag{9.15}$$

for the solution of (9.10), (9.11) with the initial condition (9.5) and $e(t_0) \in \mathbb{R}^{n_y}$. Here we took into account that for the case of two sensor nodes

$$|\sqrt{Q_i}e_i(t_0)|^2_{|i\neq i_k^*} = \min_{i=1,2}\{|\sqrt{Q_i}e_i(t_0)|^2\}.$$

From (9.15), it follows that the system (9.10), (9.11) is exponentially stable with respect to x. The novel term $V_Q(t)$ of the LKF is inserted to cope with the delays in the reset conditions

$$\begin{aligned}
&V_Q(t_{k+1}) - \lambda V_Q(t_{k+1}-1)\\
&= (\tau_M - \eta_m)[\sum_{s=t_{k+1}-\eta_{k+1}}^{t_{k+1}-1} \lambda^{t_{k+1}-s-1}\zeta^T(s)Q\zeta(s) - \sum_{s=t_k-\eta_k}^{t_{k+1}-2} \lambda^{t_{k+1}-s-1}\zeta^T(s)Q\zeta(s)]\\
&\leq (\tau_M - \eta_m)[\zeta^T(t_{k+1}-1)Q\zeta(t_{k+1}-1) - \lambda^{\tau_M}\sum_{s=t_k-\eta_k}^{t_{k+1}-\eta_{k+1}-1}\zeta^T(s)Q\zeta(s)]\\
&\leq (\tau_M - \eta_m)\zeta^T(t_{k+1}-1)Q\zeta(t_{k+1}-1) - \lambda^{\tau_M}|\sqrt{Q}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]|^2,
\end{aligned}$$

where we applied Jensen's inequality (see e.g., [9]). The term $\frac{t_{k+1}-t}{\tau_M - \eta_m + 1}e_i^T(t_k)$ $Q_i e_i(t_k)$ is inspired by the similar construction of the LKF for the sampled-data systems [4]. We have

$$\begin{aligned}
&V_e(t_{k+1}) - \lambda V_e(t_{k+1}-1)\\
&= \tilde{V}(t_{k+1}) - \lambda\tilde{V}(t_{k+1}-1) + \frac{t_{k+2}-t_{k+1}}{\tau_M - \eta_m + 1}e_i^T(t_{k+1})Q_i e_i(t_{k+1})_{|i\neq i_{k+1}^*}\\
&\quad - \frac{\lambda}{\tau_M - \eta_m + 1}e_i^T(t_k)Q_i e_i(t_k)_{|i\neq i_k^*} + (\tau_M - \eta_m)\zeta^T(t_{k+1}-1)Q\zeta(t_{k+1}-1)\\
&\quad - \lambda^{\tau_M}|\sqrt{Q}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]|^2.
\end{aligned}$$

Note that under TOD protocol for $i_{k+1}^* = i_k^*$

$$e_i^T(t_{k+1})Q_i e_i(t_{k+1})_{|i\neq i_{k+1}^*} \leq e_{i_k^*}^T(t_{k+1})Q_{i_k^*}e_{i_k^*}(t_{k+1}), \tag{9.16}$$

whereas for $i_{k+1}^* \neq i_k^*$ the latter relation holds with equality. Under RR protocol we have $i_{k+1}^* \neq i_k^*$. Hence

$$\begin{aligned}
\frac{t_{k+2}-t_{k+1}}{\tau_M - \eta_m + 1}e_i^T(t_{k+1})Q_i e_i(t_{k+1})_{|i\neq i_{k+1}^*} &\leq e_{i_k^*}^T(t_{k+1})Q_i e_{i_k^*}(t_{k+1})\\
&= |\sqrt{Q_{i_k^*}}C_{i_k^*}[x(t_{k+1}-\eta_{k+1}) - x(t_k-\eta_k)]|^2.
\end{aligned}$$

Assume that

$$\lambda^{\tau_M}Q > C_i^T Q_i C_i,\ i = 1, 2. \tag{9.17}$$

Then for $t = t_{k+1} - 1$, we obtain

$$V_e(t+1) - \lambda V_e(t) \le \tilde{V}(t+1) - \lambda \tilde{V}(t) + (\tau_M - \eta_m)\zeta^T(t)Q\zeta(t) \\ - \frac{\lambda}{\tau_M - \eta_m + 1} e_i^T(t_k)Q_i e_i(t_k)_{|i \ne i_k^*}.$$

Furthermore, due to

$$-\frac{\lambda}{\tau_M - \eta_m + 1} = -\frac{1}{\tau_M - \eta_m + 1} + \frac{1-\lambda}{\tau_M - \eta_m + 1} \le -\frac{1}{\tau_M - \eta_m + 1} + 1 - \lambda,$$

for $t = t_{k+1} - 1$, we arrive at

$$V_e(t+1) - \lambda V_e(t) \le \tilde{V}(t+1) - \lambda \tilde{V}(t) + (\tau_M - \eta_m)\zeta^T(t)Q\zeta(t) \\ -[\frac{1}{\tau_M - \eta_m + 1} - (1-\lambda)] e_i^T(t_k)Q_i e_i(t_k)_{|i \ne i_k^*} \overset{\Delta}{=} \Psi(t). \tag{9.18}$$

For $t \in [t_k, t_{k+1} - 2]$, we have

$$V_e(t+1) - \lambda V_e(t) \le \tilde{V}(t+1) - \lambda \tilde{V}(t) + (\tau_M - \eta_m)\zeta^T(t)Q\zeta(t) \\ + \left[\frac{t_{k+1} - t - 1}{\tau_M - \eta_m + 1} - \lambda \frac{t_{k+1} - t}{\tau_M - \eta_m + 1}\right] e_i^T(t_k)Q_i e_i(t_k)_{|i \ne i_k^*}.$$

Since

$$\frac{t_{k+1} - t - 1}{\tau_M - \eta_m + 1} - \lambda \frac{t_{k+1} - t}{\tau_M - \eta_m + 1} = -\frac{1}{\tau_M - \eta_m + 1} + (1-\lambda)\frac{t_{k+1} - t}{\tau_M - \eta_m + 1} \\ \le -\frac{1}{\tau_M - \eta_m + 1} + 1 - \lambda,$$

we conclude that (9.18) is valid also for $t \in [t_k, t_{k+1} - 2]$. Therefore, (9.14) holds if

$$\Psi(t) \le 0, \quad t \in [t_k, t_{k+1} - 1]. \tag{9.19}$$

Note that $i \ne i_k^*$ for $i = 1, 2$ is the same as $i = 3 - i_k^*$. By using the standard arguments for the delay-dependent analysis [20], we derive the following conditions for (9.19) (and, thus for (9.15)):

Theorem 2 *Given scalar* $0 < \lambda < 1$, *positive integers* $0 \le \eta_m \le \eta_M < \tau_M$, *and* K_1, K_2, *if there exist* $n \times n$ *matrices* $P > 0$, $Q > 0$, $S_j > 0$, $R_j > 0$ $(j = 0, 1)$, S_{12}, $n_i \times n_i$ *matrices* $Q_i > 0$ $(i = 1, 2)$ *such that (9.17) and*

$$\Omega = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \ge 0,$$

$$\hat{F}_0^T P \hat{F}_0 + \hat{\Sigma} + \hat{F}_{01}^T W \hat{F}_{01} - \lambda^{\eta_m} F_{12}^T R_0 F_{12} - \lambda^{\tau_M} \hat{F}^T \Omega \hat{F} < 0,$$

are feasible, where

$$\begin{aligned}
&\hat{F}_0 = [A\ 0\ A_1\ 0\ B_{3-i}],\\
&\hat{F}_{01} = [A - I\ 0\ A_1\ 0\ B_{3-i}],\\
&\hat{F} = \begin{bmatrix} 0 & I & -I & 0 & 0 \\ 0 & 0 & I & -I & 0 \end{bmatrix},\\
&\hat{\Sigma} = \mathrm{diag}\{S_0 - \lambda P, -\lambda^{\eta_m}(S_0 - S_1), 0, -\lambda^{\tau_M} S_1, \varphi\},\\
&\varphi = -[\frac{1}{\tau_M - \eta_m + 1} - (1-\lambda)]Q_{3-i},\\
&W = \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1 + (\tau_M - \eta_m)Q, \quad i = 1, 2.
\end{aligned}$$

Then the solutions of the hybrid system (9.10), (9.11) satisfy the bound (9.15), implying exponential stability of (9.10), (9.11) with respect to x. Moreover, if the aforementioned inequalities are feasible with $\lambda = 1$, then the bound (9.15) holds with $\lambda = 1 - \varepsilon$, where $\varepsilon > 0$ is small enough.

Remark 1 The inequality $V_e(t) \le \lambda^{t-t_0} V_e(t_0),\ t \ge t_0,\ t \in \mathbb{Z}^+$ in (9.15) guarantees that

$$\frac{t_{k+1} - t_k}{\tau_M - \eta_m + 1} e_i^T(t_k) Q_i e_i(t_k)_{|i \ne i_k^*}$$

is bounded, and it does not guarantee that $e(t_k)$ is bounded. That is why (9.15) implies only partial stability with respect to x.

Remark 2 Note that for the stability analysis of discrete-time systems with time-varying delay in the state, a switched system transformation approach can be used in addition to a Lyapunov-Krasovskii method. See more details in [11].

Remark 3 Application of Schur complement leads the matrix inequalities of Theorems 1 and 2 to LMIs that are affine in the system matrices. Therefore, for the case of system matrices from the uncertain time-varying polytope

$$\begin{aligned}
&\tilde{\Theta} = \sum_{j=1}^{N} \tilde{g}_j(t)\tilde{\Theta}_j,\ 0 \le \tilde{g}_j(t) \le 1,\\
&\sum_{j=1}^{N} \tilde{g}_j(t) = 1,\ \tilde{\Theta}_j = \left[\, A^{(j)}\ B^{(j)} \,\right],
\end{aligned}$$

the LMIs need to be solved simultaneously for all N vertices $\tilde{\Theta}_j$, using the same decision matrices.

9.4 Example: Discrete-Time Cart-Pendulum

Consider the following linearized model of the inverted pendulum on a cart [15]:

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{Ml} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{a}{M} \\ 0 \\ \dfrac{-a}{Ml} \end{bmatrix} u(t), \ t \in \mathbb{R}^+$$

with $M = 3.9249\,\mathrm{kg}, m = 0.2047\,\mathrm{kg}, l = 0.2302\,\mathrm{m}, g = 9.81\,\mathrm{N/kg}, a = 25.3\,\mathrm{N/V}$. In the model, x and θ represent cart position coordinate and pendulum angle from vertical, respectively. Such a model discretized with a sampling time $\mathrm{T_s} = 0.001$ s:

$$\begin{bmatrix} x(t+1) \\ \Delta x(t+1) \\ \theta(t+1) \\ \Delta\theta(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.001 & 0 & 0 \\ 0 & 1 & -0.0005 & 0 \\ 0 & 0 & 1.00 & 0.001 \\ 0 & 0 & 0.0448 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ \Delta x(t) \\ \theta(t) \\ \Delta\theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0064 \\ 0 \\ -0.0280 \end{bmatrix} u(t), \ t \in \mathbb{Z}^+.$$

The pendulum can be stabilized by a state feedback $u(t) = K \begin{bmatrix} x \ \Delta x \ \theta \ \Delta\theta \end{bmatrix}^T$ with the gain $K = [K_1 \ \ K_2]$

$$K_1 = \begin{bmatrix} 5.825 \ 5.883 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 24.941 \ 5.140 \end{bmatrix},$$

which leads to the closed-loop system eigenvalues $\{0.8997, 0.9980 + 0.0020i, 0.9980 - 0.0020i, 0.9980\}$. Suppose the variables θ, $\Delta\theta$ and x, Δx are not accessible simultaneously. We consider measurements $y_i(t) = C_i x(t), t \in \mathbb{Z}^+$, where

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the values of η_m given in Table 9.1, we apply Theorem 2 with $\lambda = 1$ and find the maximum values of MATI that preserve the stability of hybrid system (9.10), (9.11) with respect to x (see Table 9.1). From Table 9.1, it is observed that the presented TOD protocol, which possesses less decision variables in the LMI conditions, stabilizes the system for larger MATI than the RR protocol in Theorem 1. Moreover, when $\eta_m >$ MATI ($\eta_m \geq 4$), our method is still feasible (communication delays are larger than the sampling intervals).

Table 9.1 Example: maximum values of MATI for different $\eta_m = \eta_M$

MATI $\backslash \eta_m = \eta_M$	0	2	4	5	8	11	Decision variables
Theorem 1 (RR)	4	3	3	2	1	Infeasible	146
Theorem 2 (TOD/RR)	4	4	3	3	2	1	82

9.5 Conclusions

In this chapter, a time-delay approach has been developed for the stability analysis of discrete-time NCSs under the RR or under a weighted TOD scheduling. Polytopic uncertainties in the system model can be easily included in the analysis. A numerical example illustrated the efficiency of our method. It was assumed that no packet dropouts will happen during the data transmission over the network. Note that the time-delay approach has been developed for NCSs under stochastic protocols in [17], where the network-induced delays are allowed to be larger than the sampling intervals in the presence of collisions. For application of the presented approach in this chapter to discrete-time NCSs under scheduling protocols and actuator constraints, see [14].

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