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Stochastic Control with Stale Information–Part I: Fully Observable Systems

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***Abstract*— Timeliness is an emerging requirement for cyber-physical systems, where the value of information can quickly diminish with time. Nevertheless, there are different imperfections and constraints that hinder the immediate access of decision makers to the latest states of such systems. This obliges the designers of these systems to study the impact of information staleness on the control performance. In this paper, we focus on control with stale information and study a trade-off between the information staleness and control performance. To this purpose, we design a test channel in which the staleness of observations is chosen deliberatively. This test channel should be regarded as an abstract model that allows us to obtain the achievable region in our trade-off analysis. Based on this trade-off, the performance of any communication channel with time-varying delay used for control applications can be assessed, and the maximum staleness that is tolerable for stability can be specified.**

***Index Terms*— age of information, communication channel, time-varying delay, estimation, freshness of information, optimal control, status update.**

I. Introduction

Timeliness is an emerging requirement for cyber-physical systems, where the *value of information* [1] can quickly diminish with time. It is not hard to see, for instance, that too stale information regarding the state of an unmanned vehicle at the corresponding decision maker can cause a catastrophic failure in an intelligent transportation system. Nevertheless, there are different imperfections and constraints that hinder the immediate access of decision makers to the latest states of such systems. This obliges the designers of these systems to study the impact of information staleness on the control performance. However, prior to any analysis in this regard, it is necessary that first the notion of staleness associated with an information flow is defined properly.

A measure of staleness, which has recently received a significant attention in the literature, is *age of information* (AoI) [2]. The main characteristic of age of information is that it captures the staleness from the perspective of the receiver from the generation until the delivery of observations. Formally speaking, age of information quantifies the time elapsed since the last received observation was generated. Fig. 1 schematically shows the evolution of the age of information as a function of time for a receiver. As it is seen, this quantity drops when an observation is delivered. Otherwise, it grows linearly.

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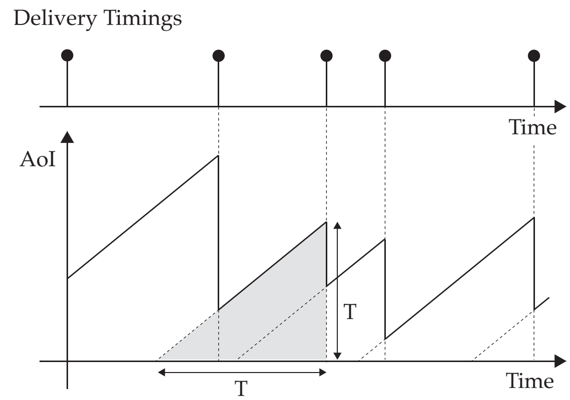

Fig. 1: Evolution of the age of information as a function of time with the corresponding delivery timings for a receiver.

Age of information was introduced by Kaul *et al.* [3] for quantifying the staleness in wireless networks where nodes periodically broadcast time-sensitive information. Age of information has since been adopted in various applications, and different related metrics such as average age of information, peak age of information, and nonlinear age of information have been proposed accordingly (see e.g., [4]–[9]). Despite extensive research on age of information, the fact is that the context, which determines the value of information, has largely been neglected.

In this paper, we contextually adopt age of information in a generic control task subject to stale information, and make a trade-off between the information staleness and control performance. To this purpose, we design a test channel in which the staleness of observations is chosen deliberatively. This test channel should be regarded as an abstract model that, as we shall show, allows us to obtain the achievable region in our trade-off analysis. Based on this trade-off, the performance of any communication channel with *time-varying delay* used for control applications can be assessed, and the maximum staleness that is tolerable for stability can be specified.

In Part I of our study, we concentrate on fully observable linear systems. In Part II, which will be published elsewhere, we will address partially observable linear systems. The main contributions of the current paper are as follows: 1) Adopting a measure for staleness associated with an information flow in control systems, 2) Making a trade-off between the information staleness and control performance. This paper is organized in 6 sections. We formally state the problem in Section II. We provide the main results in Section III. Then,

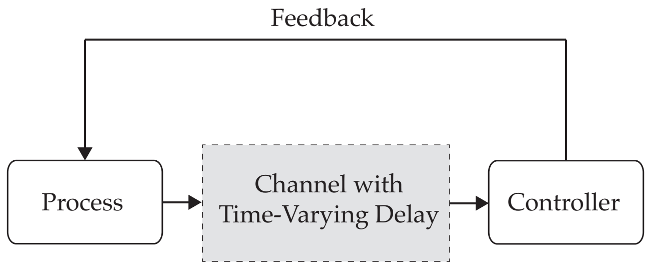

Fig. 2: A model for control with stale information where observations of the process are transmitted through a channel with time-varying delay.

we present the numerical results in Section IV. Finally, we conclude the paper in Section V.

A. Preliminaries

In the sequel, vectors, matrices, and sets are represented by lower case, upper case, and Calligraphic letters like x, X, and $\mathcal{X}$ respectively. The sequence of all vectors $x_t,\ t=0,\dots,k$, is represented by $\mathbf{x}_k$, and the sequence of all vectors $x_t,\ t=k,\dots,N$ for a specific time horizon N, is represented by $\mathbf{x}^k$. For matrices X and Y, the relations $X \succ 0$ and $Y \succeq 0$ denote that X and Y are positive definite and positive semi-definite respectively. The expected value and covariance of the random variable x are represented by $\mathsf{E}[x]$ and $\mathsf{cov}[x]$ respectively.

Consider a team game with two decision makers. Let $\gamma^1 \in \mathcal{G}^1$ and $\gamma^2 \in \mathcal{G}^2$ be the policies of the first and the second decision makers respectively where $\mathcal{G}^1$ and $\mathcal{G}^2$ are the sets of admissible policies, and $J(\gamma^1,\gamma^2)$ be the cost function. A policy profile $(\gamma^{1*},\gamma^{2*})$ represents a *Nash equilibrium* [10] if and only if

$$J(\gamma^{1*},\gamma^{2*}) \le J(\gamma^1,\gamma^{2*}),\ \text{for all } \gamma^1 \in \mathcal{G}^1,$$
$$J(\gamma^{1*},\gamma^{2*}) \le J(\gamma^{1*},\gamma^2),\ \text{for all } \gamma^2 \in \mathcal{G}^2.$$

The optimality considered in this study is in the above sense.

II. Problem Formulation

Consider a process with stochastic dynamics governed by the following linear discrete-time state system:

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

for $k \ge 0$ with initial condition x_0 where $x_k \in \mathbb{R}^n$ is the state of the system, $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, $u_k \in \mathbb{R}^m$ is the control input to be decided by a controller, and $w_k \in \mathbb{R}^n$ is a Gaussian white noise with zero mean and covariance $W_k \succ 0$. It is assumed that x_0 is a Gaussian vector with mean m_0 and covariance M_0, that x_0 and w_k are mutually independent for all k, and that the pair (A,B) is controllable. The process is fully observable, and observations are transmitted to the controller through a channel with time-varying delay (see Fig. 2). In this study, we design this channel (i.e. queue) such that the staleness of observations is chosen deliberatively.

Let $\mathbf{s}_N$ and $\mathbf{r}_N$ be two sequences that represent the inputs and outputs of the channel over the time horizon N. By assumption, $s_k = x_k$. However, we have $r_k = x_{k'}$ with $k' \le k$ if $x_{k'}$ is received by the controller at time k after a $(k-k')$-step delay, and $r_k = \varnothing$ if nothing is received by the controller at time k. For example, for $N=10$, we may have

$$\mathbf{s}_{10} = \Big\{x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}\Big\},$$
$$\mathbf{r}_{10} = \Big\{x_0,\varnothing,x_2,\varnothing,\varnothing,\varnothing,x_3,x_4,\varnothing,\varnothing,x_6\Big\}.$$

An observation x_k is said to be *informative* if the latest observation at the controller is $x_{k'}$ such that $k' < k$. Moreover, an observation x_k is said to be *obsolete* if there is at least one observation $x_{k'}$ that is transmitted to the controller such that $k' \ge k$. Clearly, an observation that is not informative is obsolete. Obsolete observations are discarded by the controller. Thus, for the above example, the following sequences are equivalent:

$$\mathbf{r}_{10} = \Big\{x_0,\varnothing,x_2,\varnothing,\varnothing,\varnothing,x_3,x_4,\varnothing,\varnothing,x_6\Big\},$$
$$\mathbf{r}'_{10} = \Big\{x_0,\varnothing,x_2,\varnothing,x_1,\varnothing,x_3,x_4,\varnothing,\varnothing,x_6\Big\}.$$

In this study, we adopt *age of information* (AoI) to measure the staleness of $\mathbf{r}_N$ with respect to $\mathbf{s}_N$. Let us define AoI at time k by an integer variable η_k such that $\eta_k \in \{0,\dots,\eta_{k-1}+1\}$ for $k \ge 0$. Then, the latest observation at the controller at time k is specified by $x_{k-\eta_k}$. We have $\eta_k \le \eta_{k-1}$ if the observation $x_{k-\eta_k}$ is delivered to the controller at time k. However, we have $\eta_k = \eta_{k-1}+1$ if no observation is delivered at time k. We assume by convention that $\eta_0 = 0$.

The information set of the queue is defined by the set of the current and prior observations x_t for all $t \le k$:

$$\mathcal{I}_k^q = \Big\{x_t \ \Big|\ t \le k\Big\}, \tag{2}$$

and the information set of the controller is defined by the set of the delivered observations $x_{t-\eta_t}$ for all $t \le k$:

$$\mathcal{I}_k^c = \Big\{x_{t-\eta_t} \ \Big|\ t \le k\Big\}. \tag{3}$$

We assume that any information that η_k might carry about non-transmitted observations is neglected at the controller. Notice that neglecting this information will simplify the structure of the optimal estimator at the controller.

Let $\pi = \{\eta_0,\dots,\eta_N\}$ and $\mu = \{u_0,\dots,u_N\}$ be a queuing policy and a control policy respectively. We say that $\pi \in \mathcal{P}$ if η_k is a measurable function of $\mathcal{I}_k^q$ for all $0 \le k \le N$ and $\mu \in \mathcal{M}$ if u_k is a measurable function of $\mathcal{I}_k^c$ for all $0 \le k \le N$. Given these policies, we measure the information staleness by

$$A(\pi,\mu) = \tfrac{1}{N+1}\,\mathsf{E}\Big[\textstyle\sum_{k=0}^N f_k(\eta_k)\Big], \tag{4}$$

where $f_k(\eta_k)$ is a, possibly nonlinear, function of η_k, and the control performance by

$$J(\pi,\mu) = \tfrac{1}{N+1}\,\mathsf{E}\Big[x_{N+1}^T Q_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u_k^T R_k u_k\Big], \tag{5}$$

where $Q_k \succeq 0$ and $R_k \succ 0$ are weighting matrices.

We would like to make a trade-off between the information staleness and control performance. To do so, we define the following loss function

$$\chi_1(\pi,\mu) = \lambda J(\pi,\mu) - (1-\lambda)A(\pi,\mu) \tag{6}$$

where $\lambda > 0$ and aim at finding the optimal policies π^* and μ^* such that the policy profile (π^*,μ^*) represents a Nash equilibrium. Notice that optimization of $\chi_1(\pi,\mu)$ over π^* and μ^* is in general intractable.

III. Main Results

Due to the existence of the channel, the controller cannot access the current state of the process at each time, and hence is required to employ an estimator. The next proposition provides the optimal estimator at the controller.

Proposition 1: The optimal estimator minimizing the mean-square error at the controller with the information set $\mathcal{I}_k^c$ is given by

$$\hat{x}_k = A^{\eta_k} x_{k-\eta_k} + \textstyle\sum_{t=1}^{\eta_k} A^{t-1} B u_{k-t}, \tag{7}$$

where $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^c]$.

Proof: First of all, it is clear that given the information set $\mathcal{I}_k^c$, the conditional expectation $\hat{x}_k = \mathsf{E}[x_k|\mathcal{I}_k^c]$ minimizes the mean-square error. From the definition, $x_{k-\eta_k}$ represents the latest observation at the controller. Given η_k, we can write x_k in terms of $x_{k-\eta_k}$ as follows:

$$x_k = A^{\eta_k} x_{k-\eta_k} + \textstyle\sum_{t=1}^{\eta_k} A^{t-1} B u_{k-t} + A^{t-1} w_{k-t}.$$

Taking conditional expectation given $\mathcal{I}_k^c$, we get

$$\begin{aligned}\mathsf{E}[x_k|\mathcal{I}_k^c] &= \mathsf{E}\Big[A^{\eta_k} x_{k-\eta_k} + \textstyle\sum_{t=1}^{\eta_k} A^{t-1} B u_{k-t} + A^{t-1} w_{k-t}\Big|\mathcal{I}_k^c\Big]\\ &= A^{\eta_k} x_{k-\eta_k} + \textstyle\sum_{t=1}^{\eta_k} A^{t-1} B u_{k-t},\end{aligned}$$

where we used the fact that $x_{k-\eta_k}$ and u_t for $k-\eta_k \le t \le k-1$ are $\mathcal{I}_k^c$-measurable and $\mathsf{E}[w_t|\mathcal{I}_k^c] = 0$ for all $k-\eta_k \le t \le k-1$. ∎

The following theorem presents the main result of this paper. It characterizes the optimal policies π^* and μ^* such that (π^*,μ^*) represents a Nash equilibrium.

Theorem 1: Let $S_k \succeq 0$ be the solution of the following Riccati equation:

$$S_k = Q_k + A^T S_{k+1} A - A^T S_{k+1} B (R_k + B^T S_{k+1} B)^{-1} B^T S_{k+1} A, \tag{8}$$

for all k with initial condition $S_{N+1} = Q_{N+1}$. Then, the optimal queuing policy and optimal control policy are determined respectively by

$$\eta_k^* = \underset{i\in\{0,\ldots,\eta_{k-1}+1\}}{\operatorname{argmax}} \mathrm{VoI}_k^i, \tag{9}$$

$$u_k^* = -K_k \hat{x}_k, \tag{10}$$

where VoI_k^i is the value of information (associated with the observation x_{k-i}) and K_k is the control gain defined respectively as

$$\begin{aligned}\mathrm{VoI}_k^i =& \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t}\Big)^T \Gamma_k \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t}\Big)\\ &+ 2\Big(\textstyle\sum_{t=1}^{i} A^{t-1} w_{k-t}\Big)^T \Gamma_k \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t}\Big)\\ &- \theta\big(f_k(\eta_{k-1}+1) - f_k(i)\big) + c_k^i,\end{aligned} \tag{11}$$

$$K_k = (R_k + B^T S_{k+1} B)^{-1} B^T S_{k+1} A, \tag{12}$$

where $\theta = (1-\lambda)/\lambda$, $\Gamma_k = K_k^T (B^T S_{k+1} B + R_k) K_k$, and c_k^i is a variable related to cost-to-go with $c_k^{\eta_{k-1}+1} = 0$.

Proof: Using (1) and (8) and after few algebraic operations, one can show that the following identity holds:

$$\begin{aligned}&x_{N+1}^T S_{N+1} x_{N+1} + \textstyle\sum_{k=0}^{N} x_k^T Q_k x_k + u^T R_k u_k\\ &= x_0^T S_0 x_0 + \textstyle\sum_{k=0}^{N} w_k^T S_{k+1} w_k + 2(Ax_k + Bu_k)^T S_{k+1} w_k\\ &\quad + (u_k + K_k x_k)^T (B^T S_{k+1} B + R_k)(u_k + K_k x_k).\end{aligned}$$

Subtracting the term $\sum_{k=0}^{N} \theta f_k(\eta_k)$ from both sides of the above identity and taking expectation, we obtain the loss function:

$$\begin{aligned}\chi_2(\pi,\mu) &= \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^{N} -\theta f_k(\eta_k) + w_k^T S_{k+1} w_k\\ &\quad + 2(Ax_k + Bu_k)^T S_{k+1} w_k\\ &\quad + (u_k + K_k x_k)^T (B^T S_{k+1} B + R_k)(u_k + K_k x_k)\Big]\\ &= \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^{N} -\theta f_k(\eta_k) + w_k^T S_{k+1} w_k\\ &\quad + (u_k + K_k x_k)^T (B^T S_{k+1} B + R_k)(u_k + K_k x_k)\Big],\end{aligned}$$

where in the second equality we used the fact that w_k is independent of x_k and u_k. Clearly, optimizing $\chi_2(\pi,\mu)$ is equivalent to optimizing $\chi_1(\pi,\mu)$ as $\lambda\chi_2(\pi,\mu) = (N+1)\chi_1(\pi,\mu)$.

Incorporating the optimal control policy in $\chi_2(\pi,\mu)$, we obtain

$$\chi_2(\pi,\mu^*) = \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^{N} -\theta f_k(\eta_k) + w_k^T S_{k+1} w_k + e_k^T K_k^T (B^T S_{k+1} B + R_k) K_k e_k\Big].$$

Following the fact that the terms x_0 and w_k are independent of the queuing policy, related with $\chi_2(\pi,\mu^*)$, we define the value function V_k^q as

$$V_k^q = \min_{\boldsymbol{\eta}^k} \mathsf{E}\Big[\textstyle\sum_{t=k}^{N} -\theta f_t(\eta_t) + e_t^T \Gamma_t e_t \Big|\mathcal{I}_k^q\Big].$$

Consequently, we have

$$V_k^q = \min_{\eta_k} \mathsf{E}\Big[-\theta f_k(\eta_k) + e_k^T \Gamma_k e_k + V_{k+1}^q \Big| \mathcal{I}_k^q \Big]$$
$$= \min_{\eta_k} \Big\{ -\theta f_k(\eta_k) + e_k^T \Gamma_k e_k + \mathsf{E}[V_{k+1}^q | \mathcal{I}_k^q] \Big\},$$

where in the first equality we used the linearity of the value function V_k^q with $V_{N+1}^q = 0$ and in the second equality the fact that e_k is $\mathcal{I}_k^q$-measurable. From Proposition 1, we can obtain the estimation error:

$$\begin{aligned} e_k &= x_k - \hat{x}_k \\ &= x_k - A^{\eta_k} x_{k-\eta_k} - \textstyle\sum_{t=1}^{\eta_k} A^{t-1} B u_{k-t} \\ &= \textstyle\sum_{t=1}^{\eta_k} A^{t-1} w_{k-t}. \end{aligned} \tag{13}$$

Let us define

$$\begin{aligned} \mathrm{VoI}_k^i &= \Big(\textstyle\sum_{t=1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big) \Gamma_k \Big(\textstyle\sum_{t=1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big) \\ &\quad - \theta f_k(\eta_{k-1}+1) + \mathsf{E}[V_{k+1}^q | \mathcal{I}_k^q, \eta_k = \eta_{k-1}+1] \\ &\quad - \Big(\textstyle\sum_{t=1}^{i} A^{t-1} w_{k-t} \Big) \Gamma_k \Big(\textstyle\sum_{t=1}^{i} A^{t-1} w_{k-t} \Big) \\ &\quad + \theta f_k(i) - \mathsf{E}[V_{k+1}^q | \mathcal{I}_k^q, \eta_k = i] \\ &= \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big)^T \Gamma_k \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big) \\ &\quad + 2 \Big(\textstyle\sum_{t=1}^{i} A^{t-1} w_{k-t} \Big)^T \Gamma_k \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big) \\ &\quad - \theta \big(f_k(\eta_{k-1}+1) - f_k(i) \big) + c_k^i, \end{aligned} \tag{14}$$

where $c_k = \mathsf{E}[V_{k+1}^q | \mathcal{I}_k^q, \eta_k = \eta_{k-1}+1] - \mathsf{E}[V_{k+1}^q | \mathcal{I}_k^q, \eta_k = i]$. Then, the optimal age of information at time k is given by

$$\eta_k^* = \operatorname*{argmax}_{i \in \{0, \dots, \eta_{k-1}+1\}} \mathrm{VoI}_k^i. \tag{15}$$

Hence, η_k^* is a function of w_t for $k - \eta_{k-1} - 1 \le t \le k-1$.

In addition, using the optimal queuing policy and the identity $x_k = \hat{x}_k + e_k$ in $\chi_2(\pi, \mu)$, we obtain

$$\begin{aligned} \chi_2(\pi^*, \mu) &= \mathsf{E}\Big[x_0^T S_0 x_0 + \textstyle\sum_{k=0}^{N} -\theta f_k \big(\operatorname*{argmax}_{i \in \{0, \dots, \eta_{k-1}+1\}} \mathrm{VoI}_k^i \big) \\ &\quad + w_k^T S_{k+1} w_k + (u_k + K_k \hat{x}_k + K_k e_k)^T \\ &\quad \times (B^T S_{k+1} B + R_k)(u_k + K_k \hat{x}_k + K_k e_k) \Big]. \end{aligned}$$

Following the fact that the terms x_0, VoI_k^i, and w_k are independent of the control policy, related with $\chi_2(\pi^*, \mu)$, we define the value function V_k^c as

$$\begin{aligned} V_k^c &= \min_{\mathbf{u}^k} \mathsf{E}\Big[\textstyle\sum_{t=k}^{N} (u_t + K_t \hat{x}_t + K_t e_t)^T \\ &\quad \times (B^T S_{t+1} B + R_t)(u_t + K_t \hat{x}_k + K_t e_t) \Big| \mathcal{I}_k^c \Big]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} V_k^c &= \min_{\mathbf{u}^k} \mathsf{E}\Big[\textstyle\sum_{t=k}^{N} (u_t + K_t \hat{x}_t)^T \\ &\quad \times (B^T S_{t+1} B + R_t)(u_t + K_t \hat{x}_t) \\ &\quad + e_t^T K_t^T (B^T S_{t+1} B + R_t) K_t e_t \\ &\quad + 2(u_t + K_t \hat{x}_t)^T (B^T S_{t+1} B + R_t) K_t e_t \Big| \mathcal{I}_k^c \Big] \\ &= \min_{\mathbf{u}^k} \Big\{ \textstyle\sum_{t=k}^{N} (u_t + K_t \hat{x}_t)^T \\ &\quad \times (B^T S_{t+1} B + R_t)(u_t + K_t \hat{x}_t) \\ &\quad + \mathsf{E}\big[e_t^T K_t^T (B^T S_{t+1} B + R_t) K_t e_t \big| \mathcal{I}_k^c \big] \Big\}, \end{aligned}$$

where in the second equality we used the fact that $\hat{x}_k$ is $\mathcal{I}_k^c$-measurable and $\mathsf{E}[e_k | \mathcal{I}_k^c] = 0$. Now, we can conclude that $u_k^* = -K_k \hat{x}_k$. This completes the proof. ∎

The optimal policies characterized in Theorem 1 allow us to make the trade-off between the information staleness and control performance. We showed that the optimal queuing policy depends on the value of information VoI_k^i. In general, one can solve the related optimality equation backward in time and obtain the exact value of information VoI_k^i.

The following proposition can be used for finding an approximation of the value of information VoI_k^i with low computational complexity and with a performance guarantee.

Proposition 2: A suboptimal queuing policy that outperforms the zero-wait policy (i.e., the queuing policy with $\eta_k = 0$ for all k) is given by

$$\eta_k^+ = \operatorname*{argmax}_{i \in \{0, \dots, \eta_{k-1}+1\}} \mathrm{VoI}_k^i, \tag{16}$$

where VoI_k^i is the value of information (associated with the observation x_{k-i}) defined as

$$\begin{aligned} \mathrm{VoI}_k^i &= \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big)^T \Gamma_k \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big) \\ &\quad + 2 \Big(\textstyle\sum_{t=1}^{i} A^{t-1} w_{k-t} \Big)^T \Gamma_k \Big(\textstyle\sum_{t=i+1}^{\eta_{k-1}+1} A^{t-1} w_{k-t} \Big) \\ &\quad - \theta \big(f_k(\eta_{k-1}+1) - f_k(i) \big), \end{aligned} \tag{17}$$

where $\theta = (1-\lambda)/\lambda$ and $\Gamma_k = K_k^T (B^T S_{k+1} B + R_k) K_k$.

Proof: Let π' denote the suboptimal queuing policy given by (16) and π'' denote the zero-wait policy. Clearly, in order to prove that $\chi_2(\pi', \mu^*) \le \chi_2(\pi'', \mu^*)$ it is enough to show that $V_k^{q'} \le V_k^{q''}$. This holds at time $N+1$ as $V_{N+1}^{q'} = V_{N+1}^{q''} = 0$. Assume that the claim holds at time $k+1$, we prove that it also holds at time k. We have

$$\begin{aligned} V_k^{q'} &= -\theta f_k(\eta_k') + {e_k'}^T \Gamma_k e_k' + \mathsf{E}[V_{k+1}^{q'} | \mathcal{I}_k^q] \\ &\le -\theta f_k(\eta_k') + {e_k'}^T \Gamma_k e_k' + \mathsf{E}[V_{k+1}^{q''} | \mathcal{I}_k^q] \\ &\le -\theta f_k(\eta_k'') + {e_k''}^T \Gamma_k e_k'' + \mathsf{E}[V_{k+1}^{q''} | \mathcal{I}_k^q] = V_k^{q''}, \end{aligned}$$

where e_k' and e_k'' are the estimation errors associated with η_k' and η_k'' respectively. Moreover, for the zero-wait policy we

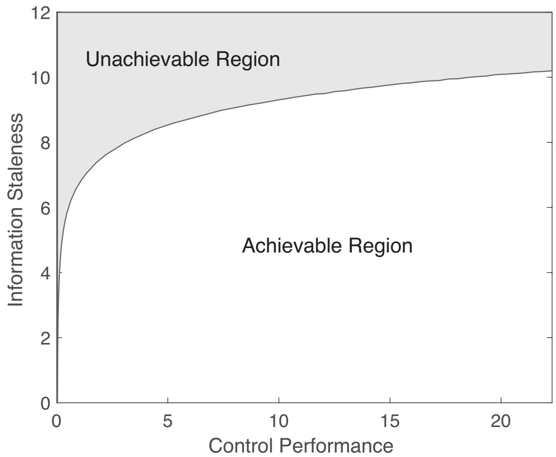

Fig. 3: Trade-off curve between the information staleness and control performance. The control performance is scaled by one thousandth. The area below the trade-off curve represents the achievable region in control with stale information.

have

$$\mathsf{E}[V_{k+1}^{q''}|\mathcal{I}_k^q] = \mathsf{E}\left[\sum_{t=k+1}^{N} -\theta f_t(\eta_t'') + e_t''^T \Gamma_t e_t'' \middle| \eta_t'' = 0, \mathcal{I}_k^q\right]$$
$$= 0.$$

This completes the proof. ■

IV. Example

Consider a scalar stochastic process:

$$x_{k+1} = 1.5x_k + 0.5u_k + w_k, \tag{18}$$

with initial condition $x_0 = 0$ and noise variance $W_k = 4$ for all k. The function $f_k(\eta_k) = \eta_k$. The weighting coefficients are $Q_k = 5$, $R_k = 0.1$ for all k, and $Q_{N+1} = 10$. The terminal time is $N = 100$. Following Proposition 1, the optimal estimator can be constructed as

$$\hat{x}_k = (1.5)^{\eta_k} x_{k-\eta_k} + \sum_{t=1}^{\eta_k} (1.5)^{t-1} 0.5 u_{k-t}. \tag{19}$$

For this system, we obtained the optimal control policy and the suboptimal queuing policy based on Theorem 1 and Proposition 2.

The trade-off curve, depicted in Fig. 3, was obtained by using different values of λ. The area below the trade-off curve represents the achievable region in control with stale information. Note that the obtained trade-off curve should be conceived as an approximate bound due to the assumptions used. Under these assumptions, there exists no channel with time-varying delay that can achieve an operating point in the unachievable region. An interesting point regarding the trade-off curve is its asymptote, which implies that there exists critical information staleness above which the system cannot be stabilized.

V. Conclusion

In this paper, we studied the trade-off between the information staleness and control performance for fully observable linear systems. We adopted the notion of age of information for measuring staleness associated with an information flow. We formulated an optimization problem, and obtained the optimal policies. Critical information staleness, partially observable linear systems, optimality gap, and more complex examples will be addressed in future research.

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