Reaching Optimal Consensus for Multi-agent Systems Based on Approximate Projection*

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***Abstract*— In this paper, we propose an approximately projected consensus algorithm for a multi-agent system to cooperatively compute the intersection of several convex sets, each of which is known only to a particular node. Instead of assuming the exact convex projection, we allow each node to just compute an approximate projection. The communication graph is directed and time-varying, and nodes can only exchange information via averaging among local view. We present sufficient and/or necessary conditions for the considered algorithm on how much projection accuracy is required to ensure a global consensus within the intersection set, under the assumption that the communication graph is uniformly jointly strongly connected. A numerical example indicates that the approximately projected consensus algorithm achieves better performance than the exact projected consensus algorithm. The results add the understanding of the fundamentals of distributed convex intersection computation.**

***Index Terms*— Multi-agent systems, approximate projection, intersection computation, optimal consensus**

I. INTRODUCTION

In recent years, dynamics on large-scale networks has drawn various research attention in different areas including engineering, computer science, and social science. Cooperative control of a group of autonomous agents fully employs local information exchange and distributed protocol design to accomplish collective tasks such as agreement, formation, and aggregation [7], [8], [18], [15], [16], [33], [19], [17], [11], [12]. Moreover, in parallel computation, load-balance problems require realtime balance of the load from different computing resources [9], [10]. Additionally, a central problem of opinion dynamics in social networks is how the agreement is achieved via individual belief exchange processes [13], [14]. A fundamental question in these problems is, how consensus can be guaranteed based on local information exchange, time-varying node interconnections and limited knowledge of the global objective.

Various distributed optimization problems arise for consensus with particular optimization purpose in practice. Minimizing a sum of convex functions, where each component is known only to a particular node, has attracted much attention recently, due to its simple formulation and wide applications [22], [20], [21], [26], [27], [23], [31], [30], [28], [29], [32], [25], [24]. The key idea is that properly designed distributed control protocols or computation algorithms can lead to a collective optimization, based on simple exchanged information and individual optimum observation. Subgradient-based incremental methods were established via deterministic or randomized iteration, where each node is assumed to be able to compute a local subgradient value of its objective function [20], [21], [26], [22], [25], [24]. Non-subgradient-based methods also showed up in the literature. For instance, a non-gradient-based algorithm was proposed, where each node starts at its own optimal solution and updates using a pairwise equalizing protocol [28], [29], and later an augmented Lagrangian method was introduced in[32].

In particular, if the optimal solution set of its own objective can be obtained for each node, the considered optimization problem is then converted to a set intersection computation problem when we additionally assume there is a nonempty intersection among all solution sets [31], [30], [27]. In fact, convex intersection computation problem is a classical problem in the optimization study [34], [35], [36]. The so-called "alternating projection algorithm" was a standard centralized solution, where projection is carried out alternatively onto each set [34], [35], [36]. Then the "projected consensus algorithm" was presented as a decentralized version of alternating projection algorithm, where each node alternatively projects onto its own set and averages with its neighbors, and comprehensive convergence analysis was given for this projected algorithm under time-varying directed interconnections in [27]. Following this work, a flip-coin algorithm was introduced when each node randomly chooses projection or averaging by Bernoulli processes, and

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almost sure convergence was shown for the system to reach an optimal consensus in [31]. A dynamical system solution was given in [30], where the network reaches a global optimal consensus by a simple continuous-time control. In all these algorithms, each node needs to know the exact convex projection of its current state onto its objective set [31], [30], [27].

However, in practice, the exact convex projection is usually hard to compute due to the common environmental noise and computation inaccuracy. In this paper, we therefore propose an approximately projected consensus algorithm (APCA) to solve the convex intersection computation problem. Instead of assuming the exact convex projection, we allow each node to just compute an approximate projection point which locates in the intersection of the convex cone generated by the current state and all directions with the exact projection direction less than some angle and the half-space containing the current state with its boundary being a supporting hyperplane to its own set at its exact projection point onto its set. The communication graph is supposed to be directed and time-varying. With uniformly jointly strongly connected conditions, we show that the whole network can achieve a global consensus within the intersection of all convex sets when sufficient projection accuracy can be guaranteed. For a special approximate projection case when the nodes can get the exact direction of the projection, a necessary and sufficient condition is given on how much projection accuracy is critical to ensure a global intersection computation. A numerical example is also given, and surprisingly, the APCA sometimes achieves better performance for convergence than the exact projected consensus algorithm.

The paper is organized as follows. Section II gives some basic concepts on graph theory and convex analysis. Section III introduces the network model and formulates the problem of interest. Section IV presents the main results and convergence analysis for the APCA. Section V gives a numerical example and finally, Section VI shows some concluding remarks.

II. Preliminaries

In this section, we introduce preliminary knowledge on graph theory [5] and convex analysis [1].

A. Graph Theory

A directed graph (digraph) $\mathcal{G}=(\mathcal{V},\mathcal{E},A)$ consists of node set $\mathcal{V}=\{1,2,...,n\}$, arc set $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ and an adjacency matrix $A=[a_{ij}]_{n\times n}$ with nonnegative adjacency elements a_{ij}. The element a_{ij} of matrix A associated with arc (i,j) is positive if and only if $(i,j)\in\mathcal{E}$. $\mathcal{N}_i$ denotes the set of neighbors of node i, that is, $\mathcal{N}_i=\{j\in\mathcal{V}|(i,j)\in\mathcal{E}\}$. In this paper, we assume $(i,i)\in\mathcal{E}$ for all i. A path from i to j in digraph $\mathcal{G}$ is a sequence $(i_0,i_1),(i_1,i_2),...,(i_{p-1},i_p)$ of arcs with $i_0=i$ and $i_p=j$. $\mathcal{G}$ is said to be strongly connected if there exists a path from i to j for each pair of nodes $i,j\in\mathcal{V}$.

B. Convex Analysis

A function $f(\cdot):R^m\to R$ is said to be convex if $f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$ for all $x,y\in R^m$ and $0<\lambda<1$. A function f is said to be concave if $-f$ is convex.

A set $K\subset R^m$ is said to be convex if $\lambda x+(1-\lambda)y\in K$ for any $x,y\in K$ and $0<\lambda<1$ and is said to be a convex cone if $\lambda_1 x+\lambda_2 y\in K$ for any $x,y\in K$ and $\lambda_1,\lambda_2\geq 0$. For a set K, $co(K)$ denotes the convex set consisting of all finite convex combinations of elements in K. For a closed convex set K in R^m, we can associate to any $x\in R^m$ a unique element $P_K(x)\in K$ satisfying $|x-P_K(x)|=\inf_{y\in K}|x-y|$, which is denoted as $|x|_K$, where $|\cdot|$ denotes the Euclidean norm and P_K is the projection operator onto K.

For a closed convex set K, if $x\notin K$, then by the supporting hyperplane theorem, there is a supporting hyperplane to K at $P_K(x)$. The angle between vectors a and b is denoted as $\mathrm{Ang}(a,b)\in[0,\pi]$ for which $\cos\mathrm{Ang}(a,b)=\langle a,b\rangle/(|a||b|)$, where $\langle a,b\rangle$ denotes the Euclidean inner product of vectors a and b.

We cite a lemma for the following analysis (see example 3.16 in [3] (pp. 88)).

Lemma 2.1: $f(z)=|z|_K$ is a convex function, where K is a closed convex set in R^m.

The following properties hold for the projection operator P_K. Here (i) is the standard non-expansiveness property for convex projection; (ii) comes from exercise 1.2 (c) in [2] (pp. 23) and (iii) is a special case of proposition 1.3 in [2] (pp. 24).

Lemma 2.2: K be a closed convex set in R^m. Then

$$(i)\quad |P_K(x)-P_K(y)|\leq|x-y|\ \ \forall\, x,\ y;$$
$$(ii)\quad \big||x|_K-|y|_K\big|\leq|x-y|\ \ \forall\, x,\ y;$$
$$(iii)\ P_K(\lambda x+(1-\lambda)P_K(x))=P_K(x)\ \ \forall\, x,\ \forall\, 0<\lambda<1.$$

The next lemma can be found in [31].

Lemma 2.3: Let K and $K_0\subseteq K$ be two closed convex sets. We have

$$|P_K(x)|_{K_0}^2+|x|_K^2\leq|x|_{K_0}^2\ \ \forall\, x.$$

III. Problem Formulation

In this section, we introduce the intersection computation problem and the approximately projected algorithm (APCA).

Consider a multi-agent system consisting of n agents with node set $\mathcal{V}=\{1,2,...,n\}$. Each node i is associated with a set $X_i\subseteq R^m$ and set X_i is known only by node i. The intersection of all these sets is nonempty, i.e., $\bigcap_{i=1}^n X_i\neq\emptyset$. Let us denote $X_0=\bigcap_{i=1}^n X_i$. The target of the system is to find a point in X_0 in a distributed way. For $X_i, i=1,...,n$, we use the following assumption:

A1 *(Convexity)* $X_i, i=1,...,n$, are closed convex sets.

A. Communication Graphs

The communication over the multi-agent system is modeled as a sequence of directed graphs, $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}(k), A(k)), k \geq 0$. We say node j is a neighbor of node i at time k if there is an arc $(i, j) \in \mathcal{E}(k)$, where $a_{ij}(k)$ represents its weight. Let $\mathcal{N}_i(k)$ denote the set of neighbors of agent i at time k. We introduce an assumption on the weights [26], [31].

A2 *(Weights Rule)* (i) $\sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) = 1$ for all i and k.

(ii) There exists a constant $0 < \eta < 1$ such that $a_{ij}(k) \geq \eta$ for all i, k and $j \in \mathcal{N}_i(k)$.

For the connectivity of the communication graphs, we introduce the following definition [30], [27].

Definition 3.1: The communication graph is said to be uniformly jointly strongly connected $(UJSC)$ if there exists a positive integer T such that $\mathcal{G}([k, k + T))$ is strongly connected for all $k \geq 0$, where $\mathcal{G}([k, k + T))$ denotes the union graph with node set $\mathcal{V}$ and arc set $\bigcup_{k \leq s < k+T} \mathcal{E}(s)$.

B. Approximate Projection

Projection methods have been widely used to solve various problems, including projected consensus [27], the convex intersection computation [35], [36] and distributed computation [4]. In the most literature, the projection point $P_K(z)$ of z onto closed convex set K is required to achieve desired convergence, but in practice it is hard to be obtained and often is computed approximately. Here is the definition of approximate projection.

Definition 3.2: Suppose $K \subseteq R^m$ is a closed convex set and $0 < \theta < \pi/2$. If $v \in K$, $\mathscr{P}^a_K(v, \theta) = \{v\}$; if $v \notin K$, we define the approximate projection $\mathscr{P}^a_K(v, \theta)$ of point z onto K with approximate angle θ as the following set:

$$\mathscr{P}^a_K(v, \theta) = \mathbf{C}_K(v, \theta) \bigcap \mathbf{H}^+_K(v), \tag{1}$$

where

$$\mathbf{C}_K(v, \theta) = v + \big\{z| \ \langle z, P_K(v) - v\rangle \geq |z||v|_K \cos\theta\big\};$$
$$\mathbf{H}^+_K(v) = \big\{z| \ \langle v - P_K(v), z\rangle \geq \langle v - P_K(v), P_K(v)\rangle\big\}.$$

In fact, $\mathbf{C}_K(v, \theta)$ is the convex cone generated by point $v \notin K$ and all vectors having angle with $P_K(v) - v$ less than θ and $\mathbf{H}^+_K(v)$ is the half-space containing point v with

$$\mathbf{H}_K(v) := \big\{z| \ \langle v - P_K(v), z\rangle = \langle v - P_K(v), P_K(v)\rangle\big\}$$

being a supporting hyperplane to K at $P_K(v)$.

C. Distributed Iterative Algorithm

To solve the intersection computation problem, we propose the following approximately projected consensus algorithm (APCA):

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) P^a_j(k) \tag{2}$$

where $P^a_i(k) \in \mathscr{P}^a_{X_i}(x_i(k), \theta_k)$ for all i and k.

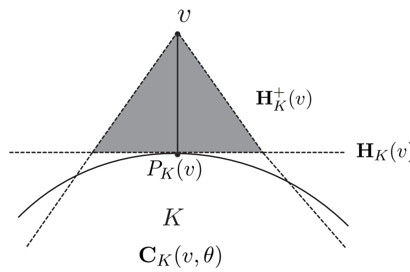


Fig. 1. The set marked by the shaded area is the approximate projection of point v onto closed convex K.

Denote $\widehat{P}^a_i(k)$ as the intersection point of the half-line $\{z| \ z = x_i(k) + r(P^a_i(k) - x_i(k)), r \geq 0\}$ and the hyperplane $\mathbf{H}_{X_i}(x_i(k))$ if $x_i(k) \notin X_i$. Therefore, it is easy to see that there exists $0 \leq \alpha_{i,k} \leq 1$ such that

$$P^a_i(k) = (1 - \alpha_{i,k}) x_i(k) + \alpha_{i,k} \widehat{P}^a_i(k). \tag{3}$$

Combining with (2) and (3), we have

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) \Big((1 - \alpha_{j,k}) x_j(k) + \alpha_{j,k} \widehat{P}^a_j(k) \Big), \tag{4}$$

where if $x_i(k) \notin X_i$, $\widehat{P}^a_i(k) \in \mathbf{H}_{X_i}(x_i(k))$ and $\mathrm{Ang}(\widehat{P}^a_i(k) - x_i(k), P_{X_i}(x_i(k)) - x_i(k)) \leq \theta_k$.

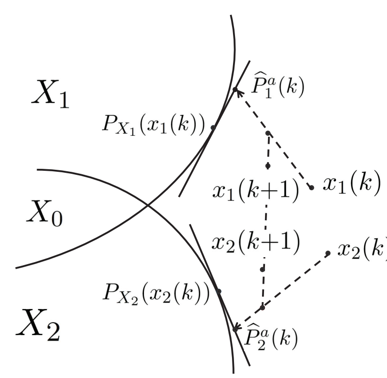


Fig. 2. Approximately projected consensus algorithm

We illustrate the iteration process of APCA in Figure 2. For the approximate angle θ_k, we use the following assumption.

A3 $0 \leq \theta_k \leq \theta^* < \pi/2$ for all k.

Our problem is introduced in the following definition.

Definition 3.3: A global optimal consensus is achieved for the APCA if, for any initial condition $x(0) \in R^{nm}$, there exists $x^* \in X_0$ such that

$$\lim_{k \to \infty} x_i(k) = x^*, \ i = 1, ..., n.$$

IV. Main Results and Convergence Analysis

In this section, we obtain the results on APCA as follows. Denote $\alpha_k^- = \min_{1\le i\le n}\alpha_{i,k}$ and $\alpha_k^+ = \max_{1\le i\le n}\alpha_{i,k}$.

Theorem 4.1: Suppose **A1**-**A3** hold. Global optimal consensus is achieved for the APCA if

(i) the communication graph is UJSC;

(ii) $\sum_{k=0}^{\infty}\alpha_k^- = \infty$;

(iii) $\sum_{k=0}^{\infty}\alpha_k^+\theta_k < \infty$.

To investigate the necessity of divergent projection accuracy sum, we impose another assumption on the boundedness of the n sets $X_i, i = 1, ..., n$.

A4 *(Compact Sets)* $X_i, i = 1, ..., n$, are bounded.

Theorem 4.2: Suppose **A1**-**A4** hold and the communication graph is UJSC. Let $\theta_k \equiv 0$. Global optimal consensus is achieved for the APCA if $\sum_{k=0}^{\infty}\alpha_k^- = \infty$ and only if $\sum_{k=0}^{\infty}\alpha_k^+ = \infty$. Particularly, if there exist $0 < \alpha_k < 0$ such that $\alpha_{i,k} = \alpha_{j,k} = \alpha_k$ for all i, j and k, then global optimal consensus is achieved for the APCA if and only if $\sum_{k=0}^{\infty}\alpha_k = \infty$.

A. Lemmas

We establish several useful lemmas in this subsection, some proofs are omitted due to space limitations.

Lemma 4.3: For all i and $k \ge s$, we have

$$|x_i(k+1)|_{X_0} \le \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)\Big((1-\alpha_{j,k})|x_j(k)|_{X_0} + \alpha_{j,k} \sqrt{|x_j(k)|^2_{X_0} - |x_j(k)|^2_{X_j}} + \tan\theta_k\alpha_{j,k}|x_j(k)|_{X_0}\Big). \quad (5)$$

Proof. By Lemma 2.2 (ii), we have

$$|\widehat{P}^a_j(k)|_{X_0} \le \big|\widehat{P}^a_j(k) - P_{X_j}(x_j(k))\big| + |P_{X_j}(x_j(k))|_{X_0}. \quad (6)$$

The definition of $\widehat{P}^a_j(k)$ ensures that

$$\big|\widehat{P}^a_j(k) - P_{X_j}(x_j(k))\big| \le \tan\theta_k|x_j(k)|_{X_j}. \quad (7)$$

Moreover, it follows from Lemma 2.3 that for any $j\in\mathcal{V}$,

$$|P_{X_j}(x_j(k))|_{X_0} \le \sqrt{|x_j(k)|^2_{X_0} - |x_j(k)|^2_{X_j}}. \quad (8)$$

By applying Lemma 2.1 for (4) and noting inequalities (6), (7) and (8), the conclusion follows. ■

Lemma 4.4: For any $z \in X_0$, we have for all k,

$$\max_{1\le i\le n}|x_i(k+1) - z| \le e^{\sum_{l=0}^{\infty}\alpha_l^+\tan\theta_l}\max_{1\le i\le n}|x_i(0) - z|.$$

The next lemma is a special case of various random versions, for example, see Lemma 11 in [6] (pp. 50).

Lemma 4.5: Let $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ be non-negative sequences with $\sum_{k=0}^{\infty} b_k < \infty$. Suppose

$$a_{k+1} \le a_k + b_k \text{ for all } k$$

Then $\lim_{k\to\infty} a_k$ is a finite number.

It is easy to see that $\tan\theta \le (\tan\theta^*/\theta^*)\theta$ for $0 \le \theta \le \theta^*$. Thus, if $\sum_{k=0}^{\infty}\alpha_k^+\theta_k < \infty$, $\sum_{k=0}^{\infty}\alpha_k^+\tan\theta_k < \infty$ and then $\{x_i(k), i\in\mathcal{V}\}_{k=0}^{\infty}$ is bounded by Lemma 4.4. By Lemmas 4.3, 4.4 and 4.5, we have the following lemma.

Lemma 4.6: If $\sum_{k=0}^{\infty}\alpha_k^+\theta_k < \infty$, the following limit exists

$$\vartheta := \lim_{k\to\infty}\max_{1\le i\le n}|x_i(k)|_{X_0}.$$

Denote

$$\eta_i^+ = \limsup_{k\to\infty}|x_i(k)|_{X_0},\ \ \eta_i^- = \liminf_{k\to\infty}|x_i(k)|_{X_0},\ i\in\mathcal{V}.$$

Obviously, $0 \le \eta_i^- \le \eta_i^+ \le \vartheta$ for all i.

Lemma 4.7: Suppose the communication graph is UJSC, **A2** holds, $\sum_{k=0}^{\infty}\alpha_k^+\theta_k < \infty$ and there exists some agent $i_0\in\mathcal{V}$ such that $\eta_{i_0}^- < \vartheta$. Then $\vartheta = 0$.

The next lemma can be obtained by combining Lemma 2 in [26].

Lemma 4.8: If the communication graph is UJSC and **A2** holds, then every entry of $\Phi(k, s)$ is not less than $\eta^{\hat{T}}$ for all s and $k \ge s + \hat{T} - 1$, where $\hat{T} = (n-1)T$, T is the constant in Definition 3.1 and η is the lower bound of weights in **A2**.

Lemma 4.9:

$$\frac{1}{n}\sum_{i=1}^{n}\sqrt{\bar{v}^2 - v_i^2} \le \sqrt{\bar{v}^2 - \Big(\frac{\sum_{i=1}^{n} v_i}{n}\Big)^2},$$

where $\bar{v} \ge v_i \ge 0$ for all i.

Consider the following consensus model with noise w_i,

$$z_i(k+1) = \sum_{j\in\mathcal{N}_i(k)} b_{ij}(k)z_j(k) + w_i(k),\ i = 1, ..., n, \quad (9)$$

where $\{b_{ij}(k), i, j\in\mathcal{V}, k\ge 0\}$ satisfy **A2**. The next lemma can be obtained from Theorem 1 in [33].

Lemma 4.10: If the communication graph of system (9) is UJSC with $\lim_{k\to\infty} w_i(k) = 0$ for all i, then consensus is achieved for system (9).

B. Proofs

In this subsection, we present the proofs of Theorems 4.1 and 4.2.

1) Proof of Theorem 4.1: Rewrite (4) as

$$x_i(k+1) = \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)x_j(k) + \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)\alpha_{j,k} \Big(\big(P_{X_j}(x_j(k)) - x_j(k)\big) + \big(\widehat{P}^a_j(k) - P_{X_j}(x_j(k))\big)\Big). \quad (10)$$

Based on (7), the second term in last equality is not greater than

$$\max_{1\le i\le n}\alpha_{i,k}|x_i(k)|_{X_i} + \alpha_k^+\tan\theta_k\max_{1\le i\le n}|x_i(k)|_{X_i}. \quad (11)$$

Note that $\vartheta = 0$ leads to $\lim_{k\to\infty}\max_{1\le i\le n}|x_i(k)|_{X_i} \le \lim_{k\to\infty}\max_{1\le i\le n}|x_i(k)|_{X_0} = 0$ and then the term in (11) tends to zero as $k\to\infty$. Therefore, by applying Lemma 4.10 for (10), we have that if $\vartheta = 0$, then the consensus is achieved.

Moreover, we claim that if $\vartheta=0$ and the consensus is achieved, then all agents will converge to a point in X_0. Since $\{x_i(k), i\in\mathcal{V}\}_{k=0}^{\infty}$ is bounded by Lemma 4.4 and the consensus is achieved, there is $x^*\in X_0$ and a subsequence $\{k_l\}_{l=1}^{\infty}$ such that $\lim_{l\to\infty} x_i(k_l)=x^*$ for all i. Similar with Lemma 4.4, we have

$$\max_{1\le i\le n}|x_i(k)-x^*|\le e^{\sum_{p=0}^{\infty}\alpha_p^+\tan\theta_p}\max_{1\le i\le n}|x_i(k_l)-x^*|$$

for $k\ge k_l$, which implies $\lim_{k\to\infty}x_i(k)=x^*$ for all i.

If there exists some agent i_0 such that $\eta_{i_0}^-<\vartheta$, then by Lemma 4.7, $\vartheta=0$. Therefore, we only need to prove $\vartheta=0$ when $\eta_i^+=\eta_i^-=\vartheta$ for all i, which shall be proven by contradiction. If $\vartheta>0$, then for any $\varepsilon>0$, there exist $K_0=K_0(\varepsilon)$ such that $|x_i(k)|_{X_0}\le\vartheta+\varepsilon$ and $d_0\alpha_k^+\theta_k\le\varepsilon$ for $k\ge K_0$ and all i, where $d_0=(\tan\theta^*/\theta^*)\sup_{1\le i\le n,k\ge 0}|x_i(k)|_{X_0}$. We complete the proof by the following two steps.

(**i**) Suppose $\eta_i^+=\eta_i^-=\vartheta$ for all i. The consensus is achieved: $\lim_{k\to\infty}|x_i(k)-x_j(k)|=0$ for all i,j.

Denote

$$\varsigma_i=\limsup_{k\to\infty}\alpha_{i,k}|x_i(k)|_{X_i},\ i\in\mathcal{V}.$$

We prove $\varsigma_i=0$ for all i by contradiction. If there exists some agent i_0 such that $\varsigma_{i_0}>0$, then there is an increasing time subsequence $\{k_l\}_{l=1}^{\infty}$ with $k_1\ge K_0$ such that $\alpha_{i_0,k_l}|x_{i_0}(k_l)|_{X_{i_0}}\ge c\varsigma_{i_0}$ for all l and some $0<c<1$. Therefore, by (5) we have

$$\begin{aligned}|x_{i_0}(k_l+1)|_{X_0}\le&(1-\eta\alpha_{i_0,k_l})(\vartheta+\varepsilon)\\&+\eta\sqrt{\alpha_{i_0,k_l}^2(\vartheta+\varepsilon)^2-c^2\varsigma_{i_0}^2}+\varepsilon,\end{aligned}\quad(12)$$

which yields a contradiction since the right hand side of (12) is less than ϑ for sufficiently small ε and sufficiently large l.

Thus, $\lim_{k\to\infty}\alpha_{i,k}|x_i(k)|_{X_i}=0$ for all i. Moreover, since $\sum_{k=0}^{\infty}\alpha_k^+\theta_k<\infty$, $\lim_{k\to\infty}\alpha_k^+\tan\theta_k\le(\tan\theta^*/\theta^*)\lim_{k\to\infty}\alpha_k^+\theta_k=0$. The two preceding conclusions and the boundedness of $\{x_i(k), i\in\mathcal{V}\}_{k=0}^{\infty}$ imply that the term in (11) tends to zero and then the consensus is achieved by applying Lemma 4.10 for (10) again.

(**ii**) Suppose $\eta_i^+=\eta_i^-=\vartheta$ for all i. All agents converge to the optimal set: $\lim_{k\to\infty}|x_i(k)|_{X_0}=0$ for all i.

Denote

$$\delta=\liminf_{k\to\infty}\sum_{i=1}^n|x_i(k)|_{X_i}.$$

We prove that $\delta=0$ by contradiction. Otherwise, suppose $\delta>0$.

Denote $D_k=\mathrm{diag}\{\alpha_{1,k},\alpha_{2,k},...,\alpha_{n,k}\}$, $|x(k)|_{X_0}=(|x_1(k)|_{X_0},...,|x_n(k)|_{X_0})^T$ and $y(k)=(y_1(k),...,y_n(k))^T$,

$$y_i(k)=|x_i(k)|_{X_0}-\sqrt{|x_i(k)|_{X_0}^2-|x_i(k)|_{X_i}^2},\ i\in\mathcal{V}.$$

From inequality (5), we have for $k\ge s$,

$$\begin{aligned}|x(k+1)|_{X_0}\le&\Phi(k,s)|x(s)|_{X_0}\\&-\sum_{l=s}^{k-\hat{T}-1}\Phi(k,l)D_ly(l)+d_0\sum_{l=s}^{k}\alpha_l^+\theta_l,\end{aligned}\quad(13)$$

where $\hat{T}=(n-1)T$ and $\Phi(k,s)=A(k)\cdots A(s+1)A(s)$.

For $\bar{\varepsilon}=\delta^2/(4n^2\vartheta+2\delta)$, there exists sufficiently large K_1 such that $\sum_{i=1}^n|x_i(k)|_{X_i}>\delta-\bar{\varepsilon}$ and $\vartheta-\bar{\varepsilon}\le|x_i(k)|_{X_0}\le\vartheta+\bar{\varepsilon}$ for $k\ge K_1$. For $k\ge K_1$, from Lemma 4.9 we have

$$\begin{aligned}&\sum_{i=1}^n\Big(|x_i(k)|_{X_0}-\sqrt{|x_i(k)|_{X_0}^2-|x_i(k)|_{X_i}^2}\Big)\\&\ge n\Big(\vartheta-\bar{\varepsilon}-\sqrt{(\vartheta+\bar{\varepsilon})^2-\big((\delta-\bar{\varepsilon})/n\big)^2}\Big):=\zeta>0.\end{aligned}$$

Namely, $\sum_{i=1}^n y_i(l)\ge\zeta$ for $l\ge K_1$. Combining the preceding inequality with Lemma 4.8 yields that every component of $\Phi(k,l)D_ly(l)$ is not less than $\eta^{\hat{T}}\zeta\alpha_l^-$ for $K_1\le l\le k-\hat{T}-1$. Then by (13) with taking $s=K_1$, we obtain

$$\begin{aligned}|x(k+1)|_{X_0}\le&\Phi(k,K_1)|x(K_1)|_{X_0}\\&-\eta^{\hat{T}}\zeta\sum_{l=K_1}^{k-\hat{T}-1}\alpha_l^-\mathbf{1}+d_0\sum_{l=K_1}^{k}\alpha_l^+\theta_l,\end{aligned}\quad(14)$$

where $\mathbf{1}$ is the vector of all ones. Note that $\sum_{l=K_1}^{\infty}\alpha_l^-=\infty$, $\sum_{l=K_1}^{\infty}\alpha_l^+\theta_l<\infty$ and $\lim_{k\to\infty}|x(k)|_{X_0}=\vartheta\mathbf{1}$, a contradiction will yield by taking the limit as $k\to\infty$ in (14).

Therefore, $\delta=\liminf_{k\to\infty}\sum_{i=1}^n|x_i(k)|_{X_i}=0$, that is, there is a subsequence $\{k_l\}_{l=0}^{\infty}$ such that $\lim_{l\to\infty}\sum_{i=1}^n|x_i(k_l)|_{X_i}=0$. Since the consensus is achieved by what we have proven in the first step (**i**), we have

$$\lim_{l\to\infty}\sum_{i=1}^n|x_i(k_l)|_{X_j}=0\text{ for all }j\in\mathcal{V},$$

which implies $\vartheta=\lim_{l\to\infty}\max_{1\le i\le n}|x_i(k_l)|_{X_0}=0$. ■

2) Proof of Theorem 4.2: The sufficiency has been obtained in Theorem 4.1, here we focus on the necessity. It is easy to find that if $\theta_k\equiv 0$, the intersection set in (1) is the line segment from $x_i(k)$ to $P_{X_i}(x_i(k))$ and then $\widehat{P}_i^{\alpha}(k)=P_{X_i}(x_i(k))$.

Denote $d^*:=\sup_{y_1,y_2\in\bigcup_{i=1}^n X_i}|y_1-y_2|$, which is finite since $X_i, i=1,...,n$ are bounded. We next prove that if $\sum_{k=0}^{\infty}\alpha_k^+<\infty$, then there exist initial conditions from which all agents will not converge to set X_0. Let $\bar{x}\in R^m$, which will be selected later, and $x_i(0)=\bar{x}$ for all $i\in\mathcal{V}$.

By (4), $x_i(1)$ can be rewritten as

$$\begin{aligned} x_i(1) &= \sum_{j\in\mathcal{N}_i(0)} a_{ij}(0)\Big((1-\alpha_{j,0})x_j(0)+\alpha_{j,0}P_{X_j}(x_j(0))\Big) \\ &= \sum_{j\in\mathcal{N}_i(0)} a_{ij}(0)\big((1-\alpha_{j,0})\bar{x}+\alpha_{j,0}P_{X_0}(\bar{x})\big)+\Delta_{i0}, \\ &= (1-\beta_{i,0})\bar{x}+\beta_{i,0}P_{X_0}(\bar{x})+\Delta_{i0}, \end{aligned}$$

where $1-\beta_{i,0} = \sum_{j\in\mathcal{N}_i(0)} a_{ij}(0)(1-\alpha_{j,0})$ and $\Delta_{i0} = \sum_{j\in\mathcal{N}_i(0)} a_{ij}(0)\alpha_{j,0}(P_{X_j}(\bar{x})-P_{X_0}(\bar{x}))$ with $|\Delta_{i0}|\le \alpha_0^+ d^*$ for all i.

We also have

$$\begin{aligned} x_i(2) &= \sum_{j\in\mathcal{N}_i(1)} a_{ij}(1)\Big((1-\alpha_{j,1})x_j(1)+\alpha_{j,1}P_{X_j}(x_j(1))\Big) \\ &= \sum_{j\in\mathcal{N}_i(1)} a_{ij}(1)(1-\alpha_{j,1})\Big((1-\beta_{j,0})\bar{x}+\beta_{j,0}P_{X_0}(\bar{x})\Big) \\ &+\Delta_{i1}+\sum_{j\in\mathcal{N}_i(1)} a_{ij}(1)\alpha_{j,1}P_{X_0}\Big((1-\beta_{j,0})\bar{x}+\beta_{j,0}P_{X_0}(\bar{x})\Big) \\ &= (1-\beta_{i,1})\bar{x}+\beta_{i,1}P_{X_0}(\bar{x})+\Delta_{i1}, \end{aligned}$$

where $1-\beta_{i,1} = \sum_{j\in\mathcal{N}_i(1)} a_{ij}(1)(1-\alpha_{j,1})(1-\beta_{j,0})$, the third equality follows from Lemma 2.2 (iii) and $\Delta_{i1} = \Delta_{i1}^1 + \Delta_{i1}^2 + \Delta_{i1}^3$ with $\Delta_{i1}^1 = \sum_{j\in\mathcal{N}_i(1)} a_{ij}(1)(1-\alpha_{j,1})\Delta_{j0}$;

$$\begin{aligned} \Delta_{i1}^2 &= \sum_{j\in\mathcal{N}_i(1)} a_{ij}(1)\alpha_{j,1}\Big(P_{X_j}(x_j(1))-P_{X_0}(x_j(1))\Big); \\ \Delta_{i1}^3 &= \sum_{j\in\mathcal{N}_i(1)} a_{ij}(1)\alpha_{j,1}\Big(P_{X_0}(x_j(1)) \\ &\qquad - P_{X_0}\big((1-\beta_{j,0})\bar{x}+\beta_{j,0}P_{X_0}(\bar{x})\big)\Big). \end{aligned}$$

Lemma 2.2 (i) implies that $|\Delta_{i1}^1+\Delta_{i1}^3|\le \max_{1\le i\le n}|\Delta_{i0}|\le \alpha_0^+ d^*$ and then $|\Delta_{i1}|\le|\Delta_{i1}^1+\Delta_{i1}^3|+|\Delta_{i1}^2|\le(\alpha_0^++\alpha_1^+)d^*$ for all i.

Similarly, we can show by induction that for all i and k, $x_i(k+1)$ can be expressed as

$$x_i(k+1) = (1-\beta_{i,k})\bar{x}+\beta_{i,k}P_{X_0}(\bar{x})+\Delta_{ik}, \tag{15}$$

where $|\Delta_{ik}|\le\sum_{l=0}^k \alpha_l^+ d^*$ and $\{\beta_{i,k}, i\in\mathcal{V}\}_{k=0}^\infty$ satisfy

$$1-\beta_{i,k} = \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)(1-\alpha_{j,k})(1-\beta_{j,k-1}). \tag{16}$$

Based on (16), we can show by induction that

$$1-\beta_{i,k}\ge\prod_{l=0}^k(1-\alpha_l^+)\ \text{ for all } i \text{ and } k. \tag{17}$$

It follows from (15), Lemma 2.2 (ii), (iii) and (17) that

$$|x_i(k+1)|_{X_0}\ge\prod_{l=0}^k(1-\alpha_l^+)|\bar{x}|_{X_0}-|\Delta_{ik}|. \tag{18}$$

Taking the inferior limit on the two sides in (18), we have

$$\liminf_{k\to\infty}|x_i(k)|_{X_0}\ge\prod_{l=0}^\infty(1-\alpha_l^+)|\bar{x}|_{X_0}-\sum_{l=0}^\infty\alpha_l^+ d^*,$$

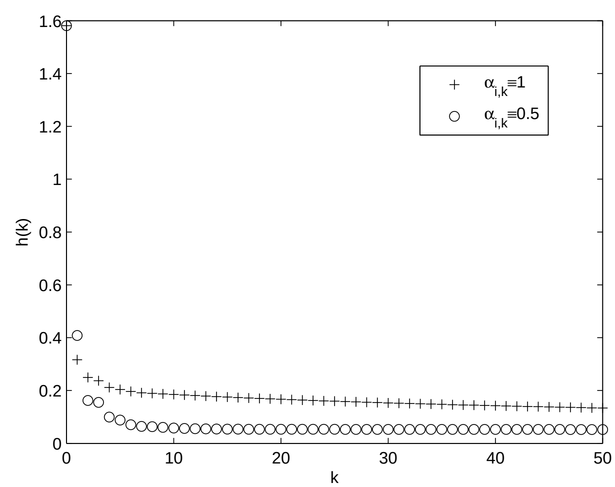
which is positive provided that

$$|\bar{x}|_{X_0} > \frac{\sum_{l=0}^\infty \alpha_l^+ d^*}{\prod_{l=0}^\infty(1-\alpha_l^+)}, \tag{19}$$

where $\prod_{l=0}^\infty(1-\alpha_l^+)>0$ since $\sum_{l=0}^\infty\alpha_l^+<\infty$. Thus, all agents can not achieve an optimal consensus for all initial conditions satisfying (19). We complete the proof. ■

V. A NUMERICAL EXAMPLE

Example 5.1: The multi-agent system consists of three agents 1, 2 and 3 in R^2 with fixed graph, where X_1, X_2 and X_3 are three balls with centers $(1,0),(-1,0),(0,-1)$ and radius 1; $X_0 = \{(0,0)\}$; $\theta_k\equiv 0$; weights $a_{11} = a_{12} = a_{22} = a_{23} = a_{31} = a_{33} = 0.5$; the initial condition $x_1(0) = (-0.5,-0.5)$, $x_2(0) = (0.5,-0.5)$ and $x_3(0) = (0.5,1.5)$. Here $h(k) = \max_{1\le i\le 3}|x_i(k)|_{X_0}$. The following figure shows that the APCA ($\alpha_{i,k} = 0.5$ for all i and k) converges faster than the exact projected consensus algorithm ($\alpha_{i,k} = 1$ for all i and k).



VI. CONCLUSIONS

In this paper, we presented an approximately projected consensus algorithm (APCA) for a multi-agent system to cooperatively compute the intersection of a serial of convex sets, each of which is known only to a particular node. We allowed each node to only compute an approximate projection. Sufficient and/or necessary conditions were obtained for the considered algorithm on how much projection accuracy is required to ensure a global consensus within the intersection set, under the assumption that the communication graph is uniformly jointly strongly connected. A numerical example was also given indicating that the APCA sometimes achieves better performance than the exact projected consensus algorithm. This implied that, individual optimum seeking may not be so important for optimizing the collective objective.

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