Generalized PID Synchronization of Higher Order Nonlinear Systems With a Recursive Lyapunov Approach

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***Abstract*—This paper investigates the problem of synchronization for nonlinear systems. Following a Lyapunov approach, we first study the global synchronization of nonlinear systems in the canonical control form with both distributed proportional-derivative and proportional-integral-derivative control actions of any order. To do so, we develop a constructive methodology and generate in an iterative way inequality constraints on the coupling matrices that guarantee the solvability of the problem or, in a dual form, provide the nonlinear weights on the coupling links between the agents such that the network synchronizes. The same methodology allows us to include a possible distributed integral action of any order to enhance the rejection of heterogeneous disturbances. The considered approach does not require any dynamic cancellation, thus preserving the original nonlinear dynamics of the agents. The results are then extended to linear and nonlinear systems admitting a canonical control transformation. Numerical simulations validate the theoretical results.**

***Index Terms*—Distributed proportional-integral-derivative (PID) control, higher order synchronization, networked control of companion forms, networked nonlinear systems.**

I. Introduction

The synchronization of networked systems has been widely studied in the last decade by different research communities [1]–[3].

In the control system community, starting from the consensus problem for single-integrator nodes, the problem of synchronization has been gradually and extensively extended to linear systems, first with assumptions on the eigenvalues of the dynamical matrix or input matrix [4], [5] and later under the mild assumption on the controllability and detectability alone of the linear systems [6], [7]. So, for the class of linear systems, general results are currently available [8]. Also, research on the synchronization of nonlinear systems has generated many results. However, due to the intrinsic difficulty, the synchronization of nonlinear systems is still under active investigation.

These days, various methodologies aim at studying the synchronization for wide classes of nonlinear systems. Approaches include Lyapunov methods [9], [10]; contraction analysis [11], [12]; and passivity and incremental dissipativity [13]–[15].

Other authors focus on the synchronization of agents whose model appears in the canonical control form, also called companion form [16]. This class of results is known as higher order synchronization and explicitly exploits the structure of the dynamical model.

Specifically, Lyapunov methods are considered, among others, in [9], [10], and [17]–[22]. These papers offer a huge spectrum of approaches for the synchronization problem. Without going too much into details, these works explore the possibility of leveraging on bounded Jacobian assumption, linear systems with additional Lipschitz nonlinearity, and the existence of the solution of suitable linear matrix inequalities, hypothesis on inequalities constraints for the nonlinear dynamics, and external reference pinner nodes.

Specifically, consensus among second-order integrators and higher order integrators has been addressed [23]–[32], following different approaches, such as studying the determinant of the overall networked linear system or via ensuring that the polynomial obtained considering the eigenvalue problem on the companion dynamical systems' matrix and the coupling feedback are Hurwitz. One of the motivations behind these studies is related to the fact that several dynamical systems, for example, mechanical systems, are naturally described in canonical control form and, in particular, higher order integrators are a more realistic model of mobile robotic vehicles than the simple integrators.

The papers reviewed above strongly rely on tools for linear systems or on the specific structure of companion form of higher order integrators, and their extension to nonlinear systems appears to be a nontrivial task.

Lyapunov methods for second-order integrators are considered in [27] and [28], in which a Lyapunov function specific for the second-order case is adopted. A specific second-order integrator Lyapunov approach is also considered in [29], where the presence of an external pinner is also required, whereas in [30], the specific second-order consensus is considered when bounded control actions are required. The case of higher order systems with nonlinear dynamics is instead studied in [32]. In that paper, the specific cases of first-order and second-order

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nonlinear systems are considered and, for these two cases, two suitable Lyapunov functions are introduced to prove convergence. The extension to higher order nonlinear dynamics is not addressed in this paper. In general, although these papers allow us to consider nonlinear dynamics via a Lyapunov function, the results appear to be specific to the order and the problem considered and, therefore, not straightforward to scale to any arbitrary system's order.

In [33], synchronization of second-order nonlinear dynamics is addressed via a nonlinear compensation through a neural network and the presence of an external reference. This approach is further extended in [34]–[36] for higher order nonlinear systems. Although such results provide a suitable methodology for addressing the higher order nonlinear synchronization, the methodology is not applicable to the free synchronization problem where the aim is to preserve the original nonlinear dynamics of the agents while studying an emerging common behavior without permanently forcing the overall system.

Motivated by the need for providing a general framework for the free synchronization problem, in this paper, we study the higher order free synchronization for nonlinear systems of any degree considering local state feedback. Referring to the previous literature on this problem, we compare our results with the strategies given in [23]–[32]. In our case, nonlinear dynamics are allowed and, therefore, a Lyapunov approach is developed. However, different from what was done in [23]–[32], we do not focus our investigation on a specific system's order but instead derive results for general degree higher order systems. Also, compared to [34]–[36], no dynamic cancellation (i.e., reduction to a higher order consensus) is needed, thus preserving the free system motion.

More specifically, we address the problem via finding a Lyapunov function whose structure is based on the system's order considered. Therefore, called n the order of the nonlinear agents, a Lyapunov function is derived via a suitable algorithm that generates, up to iteration n, a set of appropriate matrices. These matrices, blocked together in a specific way depending on the order n, will constitute the core of the Lyapunov function expression, which, in turn, will prove free synchronization. A key novelty of the approach followed in this paper, with respect to the literature, is that the conducted analysis is constructive, providing an iterative way inequality constraints on the coupling matrices that guarantee the solvability of the problem or, in a dual form, providing the nonlinear weights on the coupling links between the agents such that the network synchronizes. The given procedure relies on the iterative computation of the solution of a system of three second-order inequalities that for this reason are, contrary to other approaches in the literature (see, for example, [31] for the case of networked integrators), computable in an easier way.

Also, we believe that the analysis/synthesis method via a constructive Lyapunov function represents a relevant theoretical achievement due to its generality and scalability. Furthermore, the approach naturally encompasses the possibility to have distributed integral control actions of any order, that is, distributed PI^hD^{n-1} controllers, with $h \geq 0$ being the degree of the integral action, without any additional hypothesis. Such integral action can be used to attenuate possible distributed and heterogeneous disturbances acting on the interconnected plants. As shown in [37], an integral action significantly enhances the performances of the closed-loop system.

We note here that generalized PI^hD^{n-1} structures have already been introduced in the literature. Specifically, in [38] and [39], controllers with an analogous structure to the one proposed in this paper have been adopted for the flocking problem of a team of mobile robots following a polynomial reference trajectory. Such mobile agents are modeled with single [39] and higher order [38] integrators, and PI^n and $PI^{l_m-m}D^{m-1}$ containment controllers are, respectively, designed. To prove convergence, the adopted methodology exploits a pole-placement technique for the individual linear system and then solves a Lyapunov equation on the overall linear systems. Also, the proposed method can be adopted to the leader–follower control problem as in a particular case. In [38], a discrete-time version of the proposed strategies is also developed. Despite the analogy of the controllers' structure, however, these works differ from the results presented here in the control goal, the agents' model, and the analytical techniques adopted.

Relevant recent papers with generalized PID controllers can be found in the literature. Specifically, in [47], generalized PID controllers have been considered to synchronize a network of possibly heterogeneous scalar linear systems subjected to constant disturbances. The results have been extended in [48], where general linear systems and multiplex PI interactions are considered. Also, in [49] generalized P and PI controllers are considered to synchronize nonlinear agents.

The results in our paper, however, differ from these latter ones in the nonlinear systems considered and in the input channel chosen to control the network which, in our case, affects directly only one state component.

As a further contribution of our paper, the approach studied for higher order nonlinear systems is extended to the relevant class of interconnected nonlinear systems admitting a canonical control transformation, resulting in a distributed nonlinear control action that guarantees the synchronization of the network. Classes of the problem studied in the literature, such as second-order and higher order consensus, can be seen as special cases of such a general framework. The particular case of linear systems is also addressed as a corollary of such general framework, thus resulting in the sufficient condition of controllability of the linear systems, as already shown in a different way in [6]. However, it is worth noticing that also for the case of linear systems, the approach presented in the paper naturally allows us to explicitly consider integral control actions of any order for possible disturbances rejections.

This paper is organized in the following way. A mathematical background and the problem statement can be found in Sections II and III, respectively. In Section IV, the aforementioned iterative algorithms are presented. The synchronization of systems in a companion form is proved in Section V for both PD^{n-1} and PI^hD^{n-1} local control laws, whereas an extension to controllable systems is addressed in Section VI. Numerical examples are illustrated in Section VII, whereas concluding remarks and future work are given in Section VIII.

II. Mathematical Background

A. Matrix Analysis

Here, we report some concepts of matrix analysis, which will be useful in the rest of this paper [40].

Let us consider a generic square matrix $A \in \mathbb{R}^{n\times n}$. For any index $k \in \{1,\ldots,n\}$, the $k\times k$ top-left submatrix obtained from A, so considering the entries that lie in the first k rows and columns of A, is called a *leading principal submatrix* and its determinant is called *leading principal minor*. In an analogous way, the $k\times k$ bottom-right submatrix is called *trailing principal submatrix* and its determinant is called *trailing principal minor*.

Two matrices $A, B \in \mathbb{R}^{n\times n}$ are said to be *commutative* if $AB = BA$. Furthermore, they are said to be *simultaneously diagonalizable* if there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal. The following result holds.

Lemma 1: Let $A, B \in \mathbb{R}^{n\times n}$ be simultaneously diagonalizable. Then they are commutative.

Let $A \in \mathbb{R}^{n\times n}$ be any symmetric matrix, that is, $A = A^T$. Then, the eigenvalues of A are real and the eigenvectors constitute an orthonormal basis for A. We denote with $\mathrm{eig}(A)$ the set containing the eigenvalues of A and with $\lambda_{\min}(A) = \min_{\lambda_i\in \mathrm{eig}(A)} \lambda_i$, and $\lambda_{\max}(A) = \max_{\lambda_i\in \mathrm{eig}(A)} \lambda_i$ the minimum and maximum eigenvalue of A, respectively. For a symmetric matrix, the following results hold.

Lemma 2: (Rayleigh) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then, for all $y \in \mathbb{R}^n$, it holds $\lambda_{\min} y^T y \le y^T Ay \le \lambda_{\max} y^T y$.

Lemma 3: (Sylvester's criterion) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then, A is positively defined iff every leading (respectively, trailing) principal minor of A is positive (including the determinant of A).

B. Lie Algebra and Weak-Lipschitz Functions

Here, we give some useful definitions and basic concepts on differential geometry (for more details, see also [16] and [41]) and the definition of weak-Lipschitz functions that will be useful in the rest of this paper.

Definition 1: A function $T(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined in a region $\Omega \subseteq \mathbb{R}^n$ is said to be a *diffeomorphism* if it is smooth and invertible, with inverse function $T^{-1}(x)$ smooth.

Given a smooth scalar function $h(x) : \mathbb{R}^n \mapsto \mathbb{R}$, its gradient will be denoted by the row vector $\frac{\partial}{\partial x}h(x) = [\frac{\partial}{\partial x_1}h(x),\ldots,\frac{\partial}{\partial x_n}h(x)]$. In the case of vector function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$, with the same notation $\frac{\partial}{\partial x}f(x)$, we denote the Jacobian matrix of $f(x)$. The following definitions can be now given.

Definition 2: Let us consider a smooth scalar function $h(x) : \mathbb{R}^n \mapsto \mathbb{R}$ and a smooth vector field $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$, the *Lie derivative of h with respect to f* is the scalar function defined as $\mathcal{L}_f h(x) := \frac{\partial}{\partial x}h(x)f(x)$.

Multiple Lie derivative can be easily written by recursively extending the notation as $\mathcal{L}_f^k h(x) = \mathcal{L}_f(\mathcal{L}_f^{k-1}h)$, for $k = 1, 2, \ldots$, and with $\mathcal{L}_f^0 h(x) = h$.

Definition 3: Let us consider two smooth vector fields $f(x), g(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$, the *Lie bracket of f and g* is the vector field defined as $\mathrm{ad}_f g(x) = \frac{\partial}{\partial x}g\, f - \frac{\partial}{\partial x}f\, g$.

Analogous to what is done for the Lie derivative, multiple Lie bracket can be defined as $\mathrm{ad}_f^k g = \mathrm{ad}_f(\mathrm{ad}_f^{k-1}g)$, for $k = 1, 2, \ldots$, with $\mathrm{ad}_f^0 g = g$.

Definition 4: A set of linearly independent vector fields $\{f_1(x),\ldots,f_m(x)\}$ is said to be *involutive* if and only if, for all i, j, there exist scalar functions $\alpha_{ijk}(x) : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\mathrm{ad}_{f_i} f_j(x) = \sum_{k=1}^m a_{ijk}(x)f_k(x)$.

Definition 5: A function $f(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be *globally Lipschitz* with respect to x if $\forall x, y \in \mathbb{R}^n$, $\forall t \ge 0$, there exists a constant $w > 0$ s.t. $\|f(t, x) - f(t, y)\| \le w\|x - y\|$.

Definition 6: A function $f(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ is said to be *globally weak-Lipschitz* with respect to x if $\forall x, y \in \mathbb{R}^n$, $\forall t \ge 0, \forall i \in \{1,\ldots,n\}$ there exists a constant $w > 0$ s.t. $(x_i - y_i)[f(t, x) - f(t, y)] \le w\|x - y\|^2$, with x_i and y_i being the ith element of vector x and y, respectively.

The following lemma points out a relation between Lipschitz and weak-Lipschitz functions.

Lemma 4: A Lipschitz function $f(t, x) = \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$, with Lipschitz constant w, is also weak-Lipschitz with the same constant w.

Proof: Let us introduce the function $F_i(t, x) \in \mathbb{R}^n$ whose ith entry is $f(t, x)$, whereas the others are null. It is immediate to observe that $\|F_i(t, x) - F_i(t, y)\| = \|f(t, x) - f(t, y)\|$. So, the lemma is proved considering, for all $i \in \{1,\ldots,n\}$, the following relation:

$$\begin{aligned}(x_i - y_i)[f(t, x) - f(t, y)] &= (x - y)^T [F_i(t, x) - F_i(t, y)]\\ &\le w\|x - y\|^2.\end{aligned}$$

■

Remark 1: In this paper, we will assume that the function $f(t, x^{(i)})$ of the dynamical model given later in (1) is weak-Lipschitz. However, as also reported in [42], in the presence of synchronization in a compact invariant set, this condition can be replaced by the assumption of locally Lipschitz $f(t, x^{(i)})$. Indeed, each locally Lipschitz function can be extended outside a compact set by appropriate extension theorems.

III. Problem Formulation

The aim of this paper is to study the free synchronization for multiagent systems whose dynamics can be expressed in the canonical control form.

In further detail, a dynamical agent $\dot{x}^{(i)} = X(t, u^{(i)}, x^{(i)})$, with $x^{(i)} \in \mathbb{R}^n, u^{(i)} \in \mathbb{R}, t \in [0, +\infty)$, is said to be in the *canonical control form* or *companion form* [16] when it is in the following form:

$$\begin{aligned}\dot{x}_1^{(i)} &= x_2^{(i)}\\ &\vdots\\ \dot{x}_n^{(i)} &= f(t, x^{(i)}) + g(t, x^{(i)})u^{(i)} \qquad (1)\end{aligned}$$

where $x^{(i)} = [x_1^{(i)}, \ldots, x_n^{(i)}]^T$ and with $x^{(i)}(0) = x_0^{(i)}$. In this paper, we will consider the case of[1] $g(t, x^{(i)}(t)) \neq 0$, $\forall t \geq 0$, and so the control input can be rewritten as $u^{(i)} = 1/g(t, x^{(i)}(t))\tilde{u}^{(i)}$, with $\tilde{u}^{(i)} \in \mathbb{R}$.

The problem of the free synchronization of a multiagent system is formally defined in what follows.

Definition 7: A multiagent system of identical agents $\dot{x}^{(i)} = X(t, u^{(i)}, x^{(i)})$, with $i = 1, \ldots, N$, is *free synchronizable*, if for all of the agents, there exists a distributed control law $u_i = u_i(t, x_i, x_j)$ with $j \in \mathcal{N}_i$ such that

$$\lim_{t\to\infty} \|x^{(i)}(t) - x^{(j)}(t)\| = 0 \qquad \forall i, j = 1, \ldots, N \tag{2a}$$

$$\lim_{t\to\infty} \|u^{(i)}(t)\| = 0 \qquad \forall i = 1, \ldots, N. \tag{2b}$$

The goal of this paper is to study the free synchronization of a multiagent system with agents' dynamics expressed in the companion form (1) or that can be transformed in such canonical form. We will give conditions under which the problem of finding a distributed $u^{(i)}$ for each agent able to guarantee conditions (2a) and (2b) is solvable. Furthermore, our proofs will be based on a constructive method, so a proportional-derivative (PD^{n-1}) and proportional-integral-derivative (PI^hD^{n-1}) control law that is able to synchronize the agents will be explicitly given. Specifically, in Section V, the problem of synchronization of systems in the canonical control form will be addressed, whereas in Section VI, the results will be extended to the relevant case of systems admitting a canonical transformation. Defining the average state trajectory as $\bar{x}(t) := [\bar{x}_1^T(t), \ldots, \bar{x}_n^T(t)]^T \in \mathbb{R}^n$, with each $\bar{x}_k \in \mathbb{R}$ given by $\bar{x}_k(t) = \frac{1}{N}\sum_{j=1}^N x_k^{(j)}(t)$, we can define the stack error trajectory as $e := [e_1^T, \ldots, e_n^T]^T \in \mathbb{R}^{nN}$, and $e_k := [e_k^{(1)}, \ldots, e_k^{(N)}]^T = x_k - \bar{x}_k 1_N$, with 1_N vector of N unitary entries. It is easy to see that condition (2a) can be equivalently stated in the alternative way $\lim_{t\to\infty} \|e(t)\| = 0$.

IV. Synchronization Couplings Constraints

In this section, we identify, via an iterative procedure, a class of feedback gain matrices that suffices to achieve free synchronization for systems in the companion form. Specifically, instead of using a closed form for identifying the conditions on the feedback gains, which guarantee the synchronization, we will define it via such a procedure. The advantage is that, in this way, PI^hD^{n-1} controllers can be defined in a general way and the results can be proven considering any arbitrary degree.

When the case of a specific communication topology has to be considered, a second iterative procedure is also presented, which further imposes on the feedback gains the topology constraint. As we already said, our main purpose is to investigate the solvability of the higher order free synchronization problem. However, since the methodology is constructive, the derived conditions can also be used to either check if a given weighted topology allows synchronization or to synthesize distributed gains able to enforce synchronization.

We start giving the following definition.

[1]Notice that when a nonlinear system can be transformed in the companion form, this condition is always guaranteed by the transformation procedure itself [16].

Definition 8: A symmetric matrix $L \in \mathbb{R}^{N\times N}$ is said to be an $\mathfrak{L}_N$ *matrix* if $L1_N = 0_N$ and for its eigenvalues $\lambda_1, \ldots, \lambda_N$ it holds that $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$, where 1_N and 0_N are vectors of N unitary and null entries, respectively. Furthermore, we denote with $\mathfrak{L}_N$*-class* the set of all $\mathfrak{L}_N$ matrices.

Notice that the $N \times N$ Laplacian matrices [43] belong to the $\mathfrak{L}_N$*-class*. However, the $\mathfrak{L}_N$*-class* is more generic since we do not require the off-diagonal elements of the matrix to be nonpositive and, furthermore, no specific structure of the matrices is *a priori* assumed.

Given $n, N \in \mathbb{N}$ such that $n, N \geq 2$, let us consider the matrices $\{L_{n-k}\}_{k\in\mathcal{K}} \in \mathfrak{L}_N$*-class*, with $\mathcal{K} = \{0, \ldots, n-1\}$ and pairwise simultaneously diagonalizable. The orthonormal basis of the L_{n-k} matrices is denoted as $\{v^{(1)}, v^{(2)}, \ldots v^{(N)}\}$, with $v^{(1)} = \nu$ and $\nu = 1/N \cdot 1_N$, as stated in Section III. For each matrix L_{n-k}, we denote with $\lambda_{n-k}^{(i)}$ the eigenvalue corresponding to the eigenvector $v^{(i)}$, for all $i \in \{2, \ldots, N\}$, whereas $\lambda_{n-k}^{(1)} = 0$ by Definition 8. The algorithmic criteria we are going to give aim at identifying a class of synchronizing distributed feedback assigning spectral properties to the matrices $\{L_{n-k}\}_{k\in\mathcal{K}}$ and, thus, constraining their selection. In particular, for each eigenvalue $\lambda_{n-k}^{(i)}$ associated with eigenvector $v^{(i)}$, with $i \in \mathcal{I} = \{2, \ldots, N\}$, we consider inequality constraints via an iterative procedure.

First, let us consider the initialization $\lambda_0^{(i)} = 0$; $0 < \lambda_{n-1}^{(i)} < \lambda_n^{(i)^2}$; $\alpha_{n-1}^{(i)} = \min \operatorname{eig}\{A_{n-1}^{(i)}\}$; $\beta_{n-1}^{(i)} = \lambda_n^{(i)^2} - \lambda_{n-1}^{(i)}$; and $\gamma_{n-1}^{(i)} = 1$, with

$$A_{n-1}^{(i)} = \begin{bmatrix} 2\lambda_{n-1}^{(i)}\lambda_n^{(i)} & \lambda_{n-1}^{(i)} \\ \lambda_{n-1}^{(i)} & \lambda_n^{(i)} \end{bmatrix}.$$

It is easy to see that the coefficients $\alpha_{n-1}^{(i)}, \beta_{n-1}^{(i)}$, and $\gamma_{n-1}^{(i)}$ are strictly positive. Furthermore, for $k = 2, \ldots, n-1$, we define the iterative terms $\alpha_{n-k}^{(i)} = \min \operatorname{eig}\{A_{n-k}^{(i)}\}$; $\beta_{n-k}^{(i)} = \min \operatorname{eig}\{B_{n-k}^{(i)}\}$; and $\gamma_{n-k}^{(i)} = \gamma_{n-k+1}^{(i)} + 2\lambda_{n-k+2}^{(i)}$, with

$$A_{n-k}^{(i)} = \begin{bmatrix} 2\lambda_{n-k}^{(i)}\lambda_{n-k+1}^{(i)} & \gamma_{n-k}^{(i)}\lambda_{n-k}^{(i)} \\ \gamma_{n-k}^{(i)}\lambda_{n-k}^{(i)} & \alpha_{n-k+1}^{(i)} \end{bmatrix},$$

$$B_{n-k}^{(i)} = \begin{bmatrix} \lambda_{n-k+1}^{(i)^2} - 2\lambda_{n-k}^{(i)}\lambda_{n-k+2}^{(i)} & -\frac{1}{2}\gamma_{n-k+1}^{(i)}\lambda_{n-k}^{(i)} \\ -\frac{1}{2}\gamma_{n-k+1}^{(i)}\lambda_{n-k}^{(i)} & \beta_{n-k+1}^{(i)} \end{bmatrix}.$$

For convenience, we also define $B_0^{(i)}$ and $\beta_0^{(i)}$ by iterating the above $B_{n-k}^{(i)}$ and $\beta_{n-k}^{(i)}$ up to step $k = n$.

Taking into account the aforementioned definitions, Algorithm 1 considers for each eigenvector $v^{(i)}$, with $i \in \mathcal{I}$, a particular choice on the corresponding eigenvalues $\lambda_{n-k}^{(i)}$, with $i \in \mathcal{I}$ and $k \in \mathcal{K}$, in order to generate spectral constraints on the matrices $\{L_{n-k}\}_{k\in\mathcal{K}}$. In particular, each L_{n-k} is computed as $L_{n-k} = UD_{n-k}U^T$, with matrices $U = [\nu|v^{(2)}|\ldots|v^{(n)}]$ and $D_{n-k} = \operatorname{diag}\{0, \lambda_{n-k}^{(2)}, \ldots, \lambda_{n-k}^{(N)}\}$.

Notice that inequalities (3a)–(3c) are always feasible, since the right-hand side of (3b) is strictly positive and the second-order equation associated with (3c) has one strictly negative and

Algorithm 1: Spectral constraints assignment.

for all i = 2,..., N **do**
  for k = 2,..., n − 1 **do**
    Compute $\alpha^{(i)}_{n-k+1}$
    Compute $\gamma^{(i)}_{n-k}$
    Choose a $\lambda^{(i)}_{n-k}$ satisfying the following inequalities

$$\lambda^{(i)}_{n-k} > 0, \tag{3a}$$

$$\lambda^{(i)}_{n-k} < \frac{2\lambda^{(i)}_{n-k+1}\alpha^{(i)}_{n-k+1}}{\gamma^{(i)^2}_{n-k}}, \tag{3b}$$

$$\gamma^{(i)^2}_{n-k+1}\lambda^{(i)^2}_{n-k} + 8\lambda^{(i)}_{n-k+2}\beta^{(i)}_{n-k+1}\lambda^{(i)}_{n-k} - 4\lambda^{(i)^2}_{n-k+1}\beta^{(i)}_{n-k+1} < 0. \tag{3c}$$

    Define $B^{(i)}_{n-k}$
    Compute $\beta^{(i)}_{n-k}$
  end for
end for
for k = 0,..., n − 1 **do**
  Set $D_{n-k} \leftarrow diag\{0, \lambda^{(2)}_{n-k}, \ldots, \lambda^{(N)}_{n-k}\}$
  Set $L_{n-k} \leftarrow UD_{n-k}U^T$
end for

one strictly positive root. Furthermore, notice also that matrices $\{L_{n-k}\}_{k\in\mathcal{K}} \in \mathfrak{L}_N$-class and, as said before, in general, they are not Laplacian matrices of any graph $\mathcal{G}$. The collection of pairwise simultaneously diagonalizable matrices obtained by imposing the iterative constraints (3a)–(3c) is formalized in the following definition.

Definition 9: Given two integers $N, n \in \mathbb{N}$, with $n, N \geq 2$, the collection of matrices $\{L_{n-k}\}_{k\in\mathcal{K}} \in \mathfrak{L}_N$*-class*, with $\mathcal{K} = \{0, \ldots, n-1\}$, is said to be a (N, n)*-collection* if the matrices are pairwise simultaneously diagonalizable and satisfy the iterative spectrum constraints (3a)–(3c) of Algorithm 1.

Notice that since inequalities (3a)–(3c) are always feasible, such collection is never empty.

When a specific interconnection topology $\mathcal{G}$ needs to be taken into account, the more restrictive $(\mathcal{G}, n)$*-collection* can be considered, as it is clear from the following definition.

Definition 10: Given a connected graph $\mathcal{G}$ of N nodes and an integer $n \in \mathbb{N}$, with $n, N \geq 2$, the collection of matrices $\{L_{n-k}\}_{k\in\mathcal{K}} \in \mathfrak{L}_N$*-class*, with $\mathcal{K} = \{0, \ldots, n-1\}$, is said to be a $(\mathcal{G}, n)$*-collection* if they are a (N, n)*-collection* and $\{L_{n-k}\}_{k\in\mathcal{K}}$ are weighted Laplacian matrices of the graph $\mathcal{G}$.

For the existence of a $(\mathcal{G}, n)$-collection associated with a given connected graph $\mathcal{G}$, the following lemma can be given.

Lemma 5: Given a connected graph $\mathcal{G}$ of N nodes and an integer $n \in \mathbb{N}$, with $n, N \geq 2$, there always exists an associated $(\mathcal{G}, n)$*-collection*.

Proof: The existence of a $(\mathcal{G}, n)$-collection can be proved in a constructive way via Algorithm 2. ■

Roughly speaking, the procedure described in Algorithm 2 allows us to obtain $\{L_{n-k}\}_{k\in\mathcal{K}}$, which are weighted Laplacian for any arbitrary connected graph $\mathcal{G}$. Their expression is $L_{n-k} = l_{n-k}L$, where $L = L(\mathcal{G})$ and l_{n-k} is a positive gain defined by the recursive formula $l_{n-k} = \bar{\rho}_{n-k}l_{n-k+1}$, with $l_n = 1$. Furthermore, the fact that such matrices are also a (N, n)-collection can be trivially shown by noticing that the spectral constraints (3a)–(3c) are satisfied.

Algorithm 2: Spectral constraints assignment for constrained topologies.

Choose any $L(\mathcal{G})$ which is a compatible weighted Laplacian of any desired connected graph $\mathcal{G}$.
Set $L_n \leftarrow L$
Set $\{\lambda^{(1)}_n, \lambda^{(2)}_n, \ldots, \lambda^{(N)}_n\} \leftarrow \mathrm{eig}\{L_n\}$
for i = 2,..., N **do**
  Set $s^{(i)}_{n-1} \leftarrow \lambda^{(i)^2}_n$
  Set $\rho^{(i)}_{n-1} \leftarrow \frac{s^{(i)}_{n-1}}{\lambda^{(i)}_n}$
end for
Choose $0 < \bar{\rho}_{n-1} < \min_{i=2,\ldots,N} \rho^{(i)}_{n-1}$
Set $L_{n-1} \leftarrow \bar{\rho}_{n-1}L_n$
for k = 2,..., n − 1 **do**
  Set $\{\lambda^{(1)}_{n-k+1}, \lambda^{(2)}_{n-k+1}, \ldots, \lambda^{(N)}_{n-k+1}\} \leftarrow \mathrm{eig}\{L_{n-k+1}\}$
  for i = 2, ..., N **do**
    Compute $\beta^{(i)}_{n-k+1}$
    Compute $\alpha^{(i)}_{n-k+1}$
    Compute $\gamma^{(i)}_{n-k}$
    Set $s^{(i)}_{n-k} \leftarrow \min\{r^{(i)}_{n-k,1}, r^{(i)}_{n-k,2}\}$, with

$$r^{(i)}_{n-k,1} = \frac{2\lambda^{(i)}_{n-k+1}\alpha^{(i)}_{n-k+1}}{\gamma^{(i)^2}_{n-k}},$$

$$r^{(i)}_{n-k,2} = \sup_{r\in\mathbb{R}} \left\{\gamma^{(i)^2}_{n-k+1}r^2 + 8\lambda^{(i)}_{n-k+2}\beta^{(i)}_{n-k+1}r - 4\lambda^{(i)^2}_{n-k+1}\beta^{(i)}_{n-k+1} < 0\right\}.$$

    Set $\rho^{(i)}_{n-k} \leftarrow \frac{s^{(i)}_{n-k}}{\lambda^{(i)}_{n-k+1}}$
  end for
  Choose $0 < \bar{\rho}_{n-k} < \min_{i=2,\ldots,N} \rho^{(i)}_{n-k}$
  Set $L_{n-k} \leftarrow \bar{\rho}_{n-k}L_{n-k+1}$
end for

Remark 2: It is worth noticing that Algorithm 1 has been introduced specifically to define a (N, n)*-collection* (and so also the special case of $(\mathcal{G}, n)$*-collection*). The spectral constraints assigned in such an iterative way to the matrices in the collection will be shown to be sufficient for the network synchronization. Notice also that in several papers in the literature, sufficient conditions on the spectrum of the Laplacian matrix of the graph are given in order to prove synchronization, and the same happens in this paper. However, due to the fact that any possible system degree is here considered, the conditions are given through an iterative procedure rather than using a closed expression.

It is also worth noticing the fact that a $(\mathcal{G}, n)$*-collection* is never empty, for any connected graph $\mathcal{G}$. This will ensure the solvability of the higher order free synchronization problem with local controllers.

V. Synchronization of Systems in Companion Form

In this section, we give the main results of this paper, that is, proving that local controllers are able to synchronize a network of nonlinear systems in the companion form of any given order, as stated in Section III. Specifically, here, we propose a generalized proportional-derivative and a generalized integral-proportional-derivative controller. It is worth noticing that in our approach, the analytic expression of the Lyapunov function that allows us to prove the results is parametrized by the system order n. Indeed, its expression will be obtained by means of the (N, n)-collection generated with Algorithm 1 for any given system order.

A. Synchronization With PD^{n-1} Controllers

The following theorem gives conditions on the existence of a solution for the free synchronization problem of dynamical systems in the companion form.

Theorem 1: Let us consider N dynamical agents in companion form (1) and suppose that $f(t, x^{(i)})$ is weak-Lipschitz with constant w. Let us consider a (N, n)*-collection* $\{L_1, \ldots, L_n\}$ (or, more specifically, a $(\mathcal{G}, n)$*-collection* associated with a connected graph $\mathcal{G}$). Then, the free synchronization problem stated in Section III is solvable with the following PD controllers:

$$\tilde{u}^{(i)}(t) = l \sum_{k=1}^{n} \sum_{j=1}^{N} l_{kij} \left(x_k^{(j)}(t) - x_k^{(i)}(t) \right), \quad i = 1, \ldots, N$$

with l_{kij} being the elements of the matrices $L_k = [l_{kij}]$, with $k = 1, \ldots, n$, and $l > 1$ being a scalar gain satisfying

$$l > \frac{1}{\tilde{\beta}} \left(w \bar{\lambda}_{\max} + \tilde{\beta} - \bar{\beta} \right) \tag{4}$$

where in the above expression $\bar{\beta}$, $\bar{\lambda}_{\max}$, and $\tilde{\beta}$ are positive scalars defined, respectively, as $\bar{\beta} = \min_{i=2,\ldots,N} \beta_0^{(i)}$, $\bar{\lambda}_{\max} = \max \operatorname{eig}\{\bar{L}\}$, with $\bar{L} = \sum_{k=1}^{n} L_k$, and $\tilde{\beta} = \min_{i=2,\ldots,N} \{\bar{\beta}, {\lambda_n^{(i)}}^2\}$.

Proof: The proof of the aforementioned result is obtained by constructing a suitable Lyapunov function for the synchronization error trajectory that is able to exploit the specific canonical structure. To do so, we will divide the proof in two steps. In the first one, we will define appropriate matrices upon which we will derive a candidate Lyapunov function. In the second part, we will define the stack error system and we will prove the stability by means of such an obtained function.

Part 1: Definition of appropriate matrices. Let us denote for convenience $L_{n+1} = 1/2 \cdot I_N$ and $L_0 = O_N$, and let us consider the positions $\lambda_{n+1}^{(i)} = 1/2$ and $\lambda_0^{(i)} = 0$. We define the matrices $\{M_{n-k}\}_{k \in \mathcal{K}}$, with $M_{n-k} \in \mathbb{R}^{(k+1)N \times (k+1)N}$, in the following recursive way:

$$M_{n-k} = \begin{bmatrix} M_{\varphi,n-k} & M_{\psi,n-k} \\ M_{\psi,n-k}^T & M_{n-k+1} \end{bmatrix} \tag{5}$$

with $M_{\varphi,n-k} = 2L_{n-k}L_{n-k+1}$ and $M_{\psi,n-k} = [2L_{n-k} L_{n-k+2}, \ldots, 2L_{n-k}L_n, 2L_{n-k}L_{n+1}]$, and where as terminal condition of the recursion we define $M_n = L_n$. It is easy to notice from the above definition that matrices $\{M_{n-k}\}_{k \in \mathcal{K}}$ are $(k+1) \times (k+1)$ symmetric block matrices.

Analogously, we consider the $\{M_{n-k}^{(i)}\}_{(i,k) \in \mathcal{I} \times \mathcal{K}}$ matrices, with $M_{n-k}^{(i)} \in \mathbb{R}^{(k+1)\times(k+1)}$ and with $\mathcal{I} = \{2, \ldots, N\}$, recursively defined as

$$M_{n-k}^{(i)} = \begin{bmatrix} M_{\varphi,n-k}^{(i)} & M_{\psi,n-k}^{(i)} \\ {M^T}_{\psi,n-k}^{(i)} & M_{n-k+1}^{(i)} \end{bmatrix} \tag{6}$$

where $M_{\varphi,n-k}^{(i)} = 2\lambda_{n-k}^{(i)}\lambda_{n-k+1}^{(i)}$, $M_{\psi,n-k}^{(i)} = [2\lambda_{n-k}^{(i)}\lambda_{n-k+2}^{(i)}, \ldots, 2\lambda_{n-k}^{(i)}\lambda_n^{(i)}, 2\lambda_{n-k}^{(i)}\lambda_{n+1}^{(i)}]$, and with $M_n^{(i)} = \lambda_n^{(i)}$.

Together with matrices $\{M_{n-k}\}_{k \in \mathcal{K}}$ and $\{M_{n-k}^{(i)}\}_{(i,k) \in \mathcal{I} \times \mathcal{K}}$, we also define the symmetric matrices $\{H_{n-k}\}_{k \in \mathcal{K}}$, with $H_{n-k} \in \mathbb{R}^{(k+1)N \times (k+1)N}$ and $\{H_{n-k}^{(i)}\}_{(i,k) \in \mathcal{I} \times \mathcal{K}}$, with $H_{n-k}^{(i)} \in \mathbb{R}^{(k+1)\times(k+1)}$. Specifically

$$H_{n-k} = \begin{bmatrix} H_{\varphi,n-k} & H_{\psi,n-k} \\ H_{\psi,n-k}^T & H_{n-k+1} \end{bmatrix} \tag{7}$$

where $H_{\varphi,n-k} = L_{n-k}^2 - 2L_{n-k-1}L_{n-k+1}$, $H_{\psi,n-k} = [-L_{n-k-1}L_{n-k+2}, \ldots, -L_{n-k-1}L_n, -L_{n-k-1}L_{n+1}]$, and with $H_n = L_n^2 - L_{n-1}$, whereas $H_{n-k}^{(i)}$ is defined as

$$H_{n-k}^{(i)} = \begin{bmatrix} H_{\varphi,n-k}^{(i)} & H_{\psi,n-k}^{(i)} \\ {H^T}_{\psi,n-k}^{(i)} & H_{n-k+1}^{(i)} \end{bmatrix} \tag{8}$$

where $H_{\varphi,n-k}^{(i)} = {\lambda_{n-k}^{(i)}}^2 - 2\lambda_{n-k-1}^{(i)}$, $H_{\psi,n-k}^{(i)} = [-\lambda_{n-k-1}^{(i)} \lambda_{n-k+2}^{(i)}, \ldots, -\lambda_{n-k-1}^{(i)}\lambda_n^{(i)}, -\lambda_{n-k-1}^{(i)}\lambda_{n+1}^{(i)}]$, and with $H_n^{(i)} = {\lambda_n^{(i)}}^2 - \lambda_{n-1}^{(i)}$.

From the above definitions, it is immediate to see that $y^T M_1 y = 0$ and $y^T H_1 y = 0$, for all $y \in \Delta$. We are now going to prove that for all $y \in \Delta^\perp - \{0\}$, that is, for all the vector orthogonal to the synchronization manifold, we have $y^T M_1 y > 0$ and $y^T H_1 y > 0$. This fact will be a key aspect later, where we will derive a Lyapunov function for the system.

First, let us consider the following set of vectors:

$$S_{\Delta^\perp} = \Big\{ \varepsilon_1 \otimes v^{(2)}, \ldots, \varepsilon_1 \otimes v^{(N)}, \varepsilon_2 \otimes v^{(2)}, \ldots, \varepsilon_2 \otimes v^{(N)}, \\ \ldots, \varepsilon_n \otimes v^{(2)}, \ldots, \varepsilon_n \otimes v^{(N)} \Big\}$$

with $\varepsilon_i \in \mathbb{R}^n$ being the vector with a unitary entry in the ith position and all other entries null.

It is easy to see that $S_{\Delta^\perp} \subset \mathbb{R}^{nN}$ is a set of orthogonal unitary vectors and that $\Delta^\perp = \operatorname{span}\{S_{\Delta^\perp}\}$. Hence, any vector $y \in \Delta^\perp$ can be expressed as a liner combination of the vectors in $S_{\Delta^\perp}$ or, more compactly, it can be expressed as $y = \sum_{i=2}^{N} y^{(i)}$, where $y^{(i)} = c^{(i)} \otimes v^{(i)}$ and where $c^{(i)} = (c_1^{(i)}, \ldots, c_n^{(i)})^T \in \mathbb{R}^n$ is a vector of coefficients.

Now, due to the orthogonality of $v^{(i)}$ and $v^{(j)}$, we have that for all $i \neq j$, ${y^{(j)}}^T M_1 y^{(i)} = 0$, and ${y^{(j)}}^T H_1 y^{(i)} = 0$, while remembering definitions (6) and (8), we have ${y^{(i)}}^T M_1 y^{(i)} =$

${c^{(i)}}^T M_1^{(i)} c^{(i)}$ and ${y^{(i)}}^T H_1 y^{(i)} = {c^{(i)}}^T H_1^{(i)} c^{(i)}$. So

$$y^T M_1 y = \sum_{i=2}^{N} {c^{(i)}}^T M_1^{(i)} c^{(i)} \tag{9}$$

and

$$y^T H_1 y = \sum_{i=2}^{N} {c^{(i)}}^T H_1^{(i)} c^{(i)}. \tag{10}$$

Now, guaranteeing that ${c^{(i)}}^T M_1^{(i)} c^{(i)} > 0$ and ${c^{(i)}}^T H_1^{(i)} c^{(i)} > 0$, for all $c^{(i)} \in \mathbb{R}^n - \{0\}$ and for all $i \in \mathcal{I}$, implies the strict positivity of (9) and (10), respectively. For this reason, the rest of this first part of the proof is devoted to showing the positive definiteness of matrices $M_1^{(i)}$ and $H_1^{(i)}$. Specifically, we first focus on proving the positivity of $M_1^{(i)}$ via an induction argument that exploits the recursive structure of the matrix itself. First of all, we can see that the trailing principal submatrix

$$M_{n-1}^{(i)} = \begin{bmatrix} 2\lambda_{n-1}^{(i)}\lambda_n^{(i)} & \lambda_{n-1}^{(i)} \\ \lambda_{n-1}^{(i)} & \lambda_n^{(i)} \end{bmatrix}$$

is positively defined. Indeed, Sylvester's criterion can be applied since $\lambda_n^{(i)} > 0$ and its determinant is positive due to the choice $\lambda_{n-1}^{(i)} < {\lambda_n^{(i)}}^2$ (initialization of Algorithm 1). So, trivially, we have that $\alpha_{n-1}^{(i)} > 0$ and, since $M_{n-1}^{(i)} = A_{n-1}^{(i)}$, the relation $z^T M_{n-1}^{(i)} z \geq z^T A_{n-1}^{(i)} z \geq \alpha_{n-1}^{(i)} z^T z$ holds for all $z \in \mathbb{R}^2$. Furthermore, $\gamma_{n-1} > 0$ trivially holds. For the induction argument, we suppose that the same relation holds for a generic $M_{n-k+1}^{(i)}$, with $k \geq 2$, namely

$$z^T M_{n-k+1}^{(i)} z \geq z^T A_{n-k+1}^{(i)} z \geq \alpha_{n-k+1}^{(i)} z^T z \quad \forall z \in \mathbb{R}^k \tag{11}$$

with $\alpha_{n-k+1}^{(i)} > 0$. We also suppose that $\gamma_{n-k}^{(i)} > 0$. With such an assumption, we study the quadratic form $\bar{z}_k^T M_{n-k} \bar{z}_k$, for all the vectors $\bar{z}_k \in \mathbb{R}^{k+1} - \{0\}$, and where we have defined $\bar{z}_k = (z_1, \ldots, z_{k+1})^T$. For convenience, we introduce the subvector $\bar{z}_{k-1}$ of the last k elements of $\bar{z}_k$, and so, in block form, we have $\bar{z}_k = [z_1 | \bar{z}_{k-1}^T]^T$. We obtain

$$\begin{aligned}\bar{z}_k^T M_{n-k} \bar{z}_k &= 2\lambda_{n-k}^{(i)}\lambda_{n-k+1}^{(i)} z_1^2 + \sum_{j=2}^{k} 4\lambda_{n-k}^{(i)}\lambda_{n-k+j}^{(i)} z_1 z_j \\ &\quad + 2\lambda_{n-k}^{(i)} z_1 z_{k+1} + \bar{z}_{k-1}^T M_{n-k+1}^{(i)} \bar{z}_{k-1}.\end{aligned}$$

Considering now $z_1 z_h = \min_{j=2,\ldots,k+1} z_1 z_j$, and remembering inequality (11), we obtain

$$\begin{aligned}\bar{z}_k^T M_{n-k} \bar{z}_k &\geq 2\lambda_{n-k}^{(i)}\lambda_{n-k+1}^{(i)} z_1^2 \\ &\quad + 2\left[1 + \sum_{j=2}^{k} 2\lambda_{n-k+j}^{(i)}\right] \lambda_{n-k}^{(i)} z_1 z_h + \alpha_{n-k+1}^{(i)} z_h^2 \\ &= 2\lambda_{n-k}^{(i)}\lambda_{n-k+1}^{(i)} z_1^2 + 2\gamma_{n-k}^{(i)}\lambda_{n-k}^{(i)} z_1 z_h \\ &\quad + \alpha_{n-k+1}^{(i)} z_h^2.\end{aligned}$$

Now, considering the definition of $A_{n-k}^{(i)}$, it can be immediately noticed that the above-mentioned quadratic expression can be written as $[z_i, z_h] A_{n-k}^{(i)} [z_i, z_h]^T$. So, its positivity is guaranteed if and only if the matrix $A_{n-k}^{(i)}$ is positively defined. Since $\alpha_{n-k+1}^{(i)} > 0$, and since condition (3b) in Algorithm 1 imposes the positivity of the determinant of $A_{n-k}^{(i)}$, applying again Sylvester's criterion, we conclude that $A_{n-k}^{(i)} > 0$. Iterating the reasoning for all $k = 2, \ldots, n-1$, we obtain $M_1^{(i)} > 0$.

Analogous reasoning can be adopted to prove positive definiteness of $H_1^{(i)}$. Indeed, it is immediate to see that the trailing principal submatrix $H_n^{(i)} \in \mathbb{R}^{1\times 1}$ is positive since $H_n^{(i)} = \beta_{n-1}^{(i)} = {\lambda_n^{(i)}}^2 - \lambda_{n-1}^{(i)} > 0$, again for the initial choice $0 < \lambda_{n-1}^{(i)} < {\lambda_n^{(i)}}^2$. Obviously, the relation

$$z^T H_n^{(i)} z \geq \beta_{n-1}^{(i)} z^T z$$

holds for all $z \in \mathbb{R}$. As done for $M_{n-k}^{(i)}$, also for proving the positive definiteness of $H_{n-k}^{(i)}$, an induction argument will be used. To do so, we suppose

$$z^T H_{n-k+1}^{(i)} z \geq \beta_{n-k} z^T z \quad \forall z \in \mathbb{R}^k \tag{12}$$

with $\beta_{n-k} > 0$. Furthermore, from the iterative reasoning applied for proving that $M_1^{(i)} > 0$, we implicitly obtained that $\gamma_{n-k}^{(i)} > 0$, for all $k = 1, \ldots, n-1$, since $\lambda_{n-k}^{(i)} > 0$, for all $k = 1, \ldots, n-1$. Defining $\bar{z}_k$ as before, we can write the quadratic form $\bar{z}_k^T H_{n-k}^{(i)} \bar{z}_k$, for all $\bar{z}_k \in \mathbb{R}^{k+1} - \{0\}$, as

$$\begin{aligned}\bar{z}_k^T H_{n-k}^{(i)} \bar{z}_k &= \left[{\lambda_{n-k}^{(i)}}^2 - 2\lambda_{n-k-1}^{(i)}\lambda_{n-k+1}^{(i)}\right] z_1^2 \\ &\quad - \sum_{j=2}^{k} 2\lambda_{n-k-1}^{(i)}\lambda_{n-k+j} z_1 z_j \\ &\quad - \lambda_{n-k-1}^{(i)} z_1 z_{k+1} + \bar{z}_{k-1}^T H_{n-k+1}^{(i)} \bar{z}_{k-1}.\end{aligned}$$

Considering $z_1 z_h = \max_{j=2,\ldots,k+1} z_1 z_j$, and taking into account (12), we obtain the following inequality:

$$\begin{aligned}\bar{z}_k^T H_{n-k}^{(i)} \bar{z}_k &\geq \left[{\lambda_{n-k}^{(i)}}^2 - 2\lambda_{n-k-1}^{(i)}\lambda_{n-k+1}^{(i)}\right] z_1^2 \\ &\quad - \left[1 + \sum_{j=2}^{k} 2\lambda_{n-k+j}^{(i)}\right] \lambda_{n-k-1}^{(i)} z_1 z_h + \beta_{n-k}^{(i)} z_h^2 \\ &= \left[{\lambda_{n-k}^{(i)}}^2 - 2\lambda_{n-k-1}^{(i)}\lambda_{n-k+1}^{(i)}\right] z_1^2 \\ &\quad - \gamma_{n-k}^{(i)} z_1 z_h + \beta_{n-k}^{(i)} z_h^2.\end{aligned}$$

Observing that the above-mentioned quadratic form can be obtained from $[z_i, z_h] B_{n-k-1}^{(i)} [z_i, z_h]^T$, since $\beta_{n-k}^{(i)} > 0$ for the Sylvester's criterion, the positive definiteness of B_{n-k-1} is guaranteed by the positivity of its determinant. The latter condition is given by (3c) of Algorithm 1 evaluated at $k+1$. Repeating the reasoning for $k = 1, \ldots, n$, we obtain $z^T H_1^{(i)} z \geq \beta_0^{(i)} z^T z$, with $\beta_0^{(i)} > 0$, which guarantees the positive definiteness of $H_1^{(i)}$. It is also possible to further analyze the quadratic form

(10), as this will turn useful later in Step 2 of the proof. For all $y \in \Delta^{\perp} - \{0\}$, we have

$$y^T H_1 y = \sum_{i=2}^{N} c^{(i)^T} H_1^{(i)} c^{(i)} \geq \sum_{i=2}^{N} \beta_0^{(i)} c^{(i)^T} c^{(i)} \geq \bar{\beta} \sum_{i=2}^{N} c^{(i)^T} c^{(i)} \geq \bar{\beta} y^T y \quad (13)$$

where $\bar{\beta} = \min_{i=2,\ldots,N} \beta_0^{(i)}$ is a positive scalar and where we considered $y^{(i)^T} y^{(i)} = [c^{(i)^T} \otimes v^{(i)^T}][c^{(i)} \otimes v^{(i)}] = c^{(i)^T} c^{(i)}$, with $y^{(i)^T} y^{(j)} = 0$, for $i \neq j$.

Part 2: Lyapunov stability analysis. For convenience, we consider the error stack system of the following form:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ &\vdots \\ \dot{e}_n &= F(t,x) - \bar{f}(t,x) \cdot 1_N + \tilde{u}(t) \end{aligned} \quad (14)$$

where $\bar{f}(t,x) = 1/N \sum_{j=1}^{N} f(t, x^{(j)})$ and with $\tilde{u}(t) = -l \sum_{k=1}^{n} L_k e_k(t)$, where L_k, with $k = 1,\ldots,n$, are given in the theorem statement. Remembering the definition of matrix M_1 in (5) with $k = n-1$, we can also rewrite it in the following block form:

$$M_1 = \begin{bmatrix} M_\vartheta & M_\varsigma \\ M_\varsigma^T & L_n \end{bmatrix}$$

with $M_\vartheta \in \mathbb{R}^{(n-1)N \times (n-1)N}$ leading principal submatrix. For the error system (14), we can finally consider the quadratic candidate Lyapunov function[2] $V(e,n) = 1/2 e^T \widetilde{M} e$, where $\widetilde{M} \in \mathbb{R}^{nN \times nN}$ is defined from M_1 by considering as leading principal submatrix lM_ϑ, whereas all other submatrices are the same as in M_1, i.e.,

$$\widetilde{M} = \begin{bmatrix} lM_\vartheta & M_\varsigma \\ M_\varsigma^T & L_n \end{bmatrix}. \quad (15)$$

It easy to see that the quadratic form is a valid candidate Lyapunov function for proving synchronization since $y^T \widetilde{M} y = 0$ for all $y \in \Delta$, whereas $y^T \widetilde{M} y > 0$ for all $y \in \Delta^{\perp} - \{0\}$. The first property follows immediately from the definition, whereas the latter can be shown partitioning the generic y as $y = [y_\vartheta^T, y_\varsigma^T]^T$ and considering $y^T \widetilde{M} y = y^T M_1 y + (l-1) y_\vartheta^T M_\vartheta y_\vartheta$. The positivity is so proved remembering that M_1 is positive definite on $\Delta^{\perp} - \{0\}$, as shown in Part 1, while M_ϑ is its leading principal minor and is, therefore, positive. Considering the time derivative of $V(e,n)$, we obtain

$$\dot{V}(e,n) = e^T \widetilde{M} \dot{e} = e^T \widetilde{M} \Phi(t,x) + e^T \widetilde{M} \Xi(e) \quad (16)$$

with $\Phi(t,x) = [0_N^T, \ldots, 0_N^T, F^T(t,x) - \bar{f}(t,x) \cdot 1_N^T]^T$ and $\Xi(e) = [e_2^T, \ldots, e_n^T, -(\sum_{k=1}^{n} L_k e_k(t))^T]^T$. We now analyze separately the two terms in (16). For the first one, we have

$$\begin{aligned} e^T \widetilde{M} \Phi(t,x) &= \sum_{k=1}^{n} e_k^T L_k \left[F(t,x) - \bar{f}(t,x) \cdot 1_N\right] \\ &= \sum_{k=1}^{n} \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{kij} \left[e_k^{(i)} - e_k^{(j)}\right] \left[f(t,x^{(i)}) - f(t,x^{(j)})\right] \\ &= \sum_{k=1}^{n} \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{kij} \left[x_k^{(i)} - x_k^{(j)}\right] \left[f(t,x^{(i)}) - f(t,x^{(j)})\right] \end{aligned}$$

from which, using the weak-Lipschitz property

$$e^T \widetilde{M} \Phi(t,x) \leq \sum_{k=1}^{n} \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{kij} w \left[x^{(i)} - x^{(j)}\right]^T \left[x^{(i)} - x^{(j)}\right] \quad (17)$$

$$\begin{aligned} &= \sum_{k=1}^{n} w \sum_{h=1}^{n} \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{kij} \left[e_h^{(i)} - e_h^{(j)}\right]^2 \\ &= \sum_{k=1}^{n} w \sum_{h=1}^{n} e_h^T L_k e_h = \sum_{k=1}^{n} w e^T (I_n \otimes L_k) e \\ &= w e^T \left(I_n \otimes \sum_{k=1}^{n} L_k\right) e = w e^T \left(I_n \otimes \bar{L}\right) e. \end{aligned} \quad (18)$$

For the analysis of the second term in (16), we first write matrix H_1 in a block form analogous to M_1, namely

$$H_1 = \begin{bmatrix} H_\vartheta & H_\varsigma \\ H_\varsigma^T & L_n^2 - L_{n-1} \end{bmatrix}.$$

From the aforementioned matrix, we define $\widetilde{H} \in \mathbb{R}^{nN \times nN}$ as

$$\widetilde{H} = \begin{bmatrix} lH_\vartheta & H_\varsigma \\ H_\varsigma^T & lL_n^2 - L_{n-1} \end{bmatrix}.$$

Now, performing suitable algebraic manipulations, we can show that

$$e^T \widetilde{M} \Xi(e) = -e^T \widetilde{H} e. \quad (19)$$

To do so, we take advantage of the recursive structure of the matrices M_1 and H_1, respectively, obtained nesting (5) and (7) up to index $k = n-1$. Remembering that $L_{n+1} = 1/2 \cdot I_N$ and $L_0 = O_N$, we have that $\widetilde{M} = \widetilde{M}_1$, with $\widetilde{M}_1$ defined nesting up to $k = n-1$ the following:

$$\widetilde{M}_{n-k} = \begin{bmatrix} \widetilde{M}_{\varphi,n-k} & \widetilde{M}_{\psi,n-k} \\ \widetilde{M}_{\psi,n-k}^T & \widetilde{M}_{n-k+1} \end{bmatrix}$$

[2]The explicit dependence on n of the Lyapunov function points out that matrix M_1, from which $\widetilde{M} \in \mathbb{R}^{nN \times nN}$ is derived, has a specific structure depending on the system order n considered.

where $\widetilde{M}_{\varphi,n-k} = 2lL_{n-k}L_{n-k+1}$ and $\widetilde{M}_{\psi,n-k} = [2lL_{n-k}L_{n-k+2}, \ldots, 2lL_{n-k}L_n, L_{n-k}]$, and where as terminal condition of the recursion, we define $\widetilde{M}_n = L_n$.

Analogously, we have $\widetilde{H} = \widetilde{H}_1$, with $\widetilde{H}_1$ defined nesting up to $k = n-1$, the following equation:

$$\widetilde{H}_{n-k} = \begin{bmatrix} \widetilde{H}_{\varphi,n-k} & \widetilde{H}_{\psi,n-k} \\ \widetilde{H}^T_{\psi,n-k} & \widetilde{H}_{n-k+1} \end{bmatrix}$$

where $\widetilde{H}_{\varphi,n-k} = lL^2_{n-k} - 2lL_{n-k-1}L_{n-k+1}$, $\widetilde{H}_{\psi,n-k} = [-lL_{n-k-1}L_{n-k+2}, \ldots, -lL_{n-k-1}L_n, -\frac{1}{2}L_{n-k-1}]$, and with $\widetilde{H}_n = lL^2_n - L_{n-1}$.

Relation (19) can be proved by focusing on a generic trail principal submatrix $\widetilde{H}_{n-k}$ of $\widetilde{H}_1$. In particular, we restrict our attention on the first row and column of submatrix $\widetilde{H}_{n-k}$. The associated terms will be involved in the bilinear terms $e_i^T \eta_{ij} e_j$ with $i = n-k$ and $j = n-k, \ldots, n$ and with $i = n-k, \ldots, n$ and $j = n-k$, where with η_{ij}, we have here denoted the i,jth entry of matrix $\widetilde{H}_1$, that is, $\widetilde{H}_1 = [\eta_{ij}]$.

From the definition of matrix $\widetilde{H}_1$, it is easy to see that the terms in $e^T \widetilde{H}_1 \Xi(e)$ corresponding to the bilinear terms $e_i^T \eta_{ij} e_j$ considered are given by

$$\begin{aligned} - \sum_{\substack{i=n-k, j=n-k,\ldots,n \\ i=n-k,\ldots,n, j=n-k}} e_i^T \eta_{ij} e_j = {} & 2le^T_{n-k} L_{n-k-1} L_{n-k+1} e^T_{n-k} \\ & + \sum_{j=n-k+1}^{n} 2le^T_{n-k} L_{n-k} L_j e_j \\ & - \sum_{j=n-k}^{n} le^T_{n-k} L_{n-k} L_j e_j \\ & - \sum_{j=n-k+1}^{n} le^T_i L_i L_{n-k} e_{n-k} \\ & + \sum_{i=n-k+1}^{n-1} 2le^T_i L_{n-k-1} L_{i+1} e_{n-k} \\ & + e^T_n L_{n-k-1} e_{n-k}, \end{aligned}$$

from which we obtain

$$\begin{aligned} - \sum_{\substack{i=n-k, j=n-k,\ldots,n \\ i=n-k,\ldots,n, j=n-k}} e_i^T \eta_{ij} e_j = {} & 2e^T_{n-k} L_{n-k-1} L_{n-k+1} e^T_{n-k} \\ & - e^T_{n-k} L^2_{n-k} e_{n-k} \\ & + \frac{1}{2} \sum_{i=n-k+1}^{n-1} \left[e^T_i L_{n-k-1} L_{i+1} e_{n-k} \right. \\ & \left. + e^T_{n-k} L_{n-k-1} L_{i+1} e_i \right] \\ & + \frac{1}{2} e^T_n L_{n-k-1} e_{n-k} \\ & + \frac{1}{2} e^T_{n-k} L_{n-k-1} e_n. \end{aligned}$$

Repeating the same reasoning for all $k \in \{0, \ldots, n-1\}$, we finally have (19). Writing $e = [e^T_\vartheta, e^T_\varsigma]^T$, we have $e^T \widetilde{H} e = e^T H_1 e + (l-1) e^T_\vartheta H_\vartheta e_\vartheta + (l-1) e^T_\varsigma L^2_n e_\varsigma$ and so, remembering (13), the following inequality holds:

$$e^T \widetilde{H} e \geq \bar{\beta} e^T e + (l-1)\tilde{\beta} e^T e \quad \forall e \in \Delta^\perp - \{0\}. \tag{20}$$

Combining (18) and (20), from (16), the following inequality holds:

$$\begin{aligned} \dot{V}(e,n) & \leq we^T \left(I_n \otimes \bar{L}\right) e - \bar{\beta} e^T e - (l-1)\tilde{\beta} e^T e \\ & \leq w\bar{\lambda}_{\max} e^T e - \bar{\beta} e^T e - (l-1)\tilde{\beta} e^T e. \end{aligned}$$

Imposing $w\bar{\lambda}_{\max} - \bar{\beta} - (l-1)\tilde{\beta} < 0$, condition (4) is obtained which guarantees, together with $l > 1$, a negative quadratic upper bound for $\dot{V}(e,n)$ and so the synchronization of the agents to the same trajectory.

Remark 3: It is worth noticing that the relevant case of consensus of double [30], [32] and higher order [31] integrators is included in the previous analysis as a particular case when $f(t, x^{(i)}) = 0$, and can be studied following exactly the same way of constructing the quadratic Lyapunov function $V(e,2) = 1/2 e^T \widetilde{M} e$, with $\widetilde{M}$ given in (15). Specifically, for the consensus of double integrators, the matrix $\widetilde{M}$ can be easily shown to be given by

$$\begin{bmatrix} 2lL_1L_2 & L_1 \\ L_1 & L_2 \end{bmatrix}.$$

Notice also that in our study, we directly consider in Algorithm 1 and Algorithm 2 at least a second-order degree, that is, $n \geq 2$, for the interconnected agents. In principle, a first-order case could still be studied observing that the $n \times n$ block matrix M_1, and so also matrix $\widetilde{M}$, grows in size according to the degree n of the agents from the bottom-right corner L_n, thus resulting in the specific recursive structure we highlighted. The case of $n = 1$ would so result in the bottom-right corner only, thus having $\widetilde{M} = lL$, which gives a well-known Lyapunov function for studying the classical problem of consensus for single integrators [44]. Also, when considering the higher order consensus problem and a $(\mathcal{G}, n)$-collection is chosen, the controller shows an analogous structure to the one described in [31]. However, in that paper, a different criterion based on Kharitonov' s theorem is provided in order to select the feedback coefficients l_{n-k}.

Furthermore, together with the case of consensus of integrators, the relevant case of the synchronization of linear systems can be also addressed with our framework, as will be shown later in Corollary 1.

B. Synchronization With PI^hD^{n-1} Controllers

The analysis conducted in Section V-A control action is used to achieve free synchronization, is now extended to the case where an integral control action of any arbitrary degree $h \geq 1$, with $h \in \mathbb{N}$, is also considered.

In further detail, considering a generic integrable function $\eta(\cdot) : \mathbb{R} \mapsto \mathbb{R}^n$, we define its integral of degree $h \in \mathbb{N}$ with the following notation:

$$\int_0^{t,(h)} \eta(\tau)d\tau := \int_0^t \int_0^{\tau,(h-1)} \eta(\tau')d\tau' d\tau \qquad \text{if } h > 1$$

while in case $h = 1$, we simply have

$$\int_0^{t,(1)} \eta(\tau)d\tau := \int_0^t \eta(\tau)d\tau.$$

We now give the following theorem.

Theorem 2: Let us consider N dynamical agents in the companion form and suppose that $f(t, x^{(i)})$ is weak-Lipschitz with constant w. Then, the free synchronization problem is solvable with a $PI^h D^{n-1}$ controller of arbitrary degree $h \geq 1$ of the following form:

$$\tilde{u}^{(i)}(t) = l \sum_{k=1}^{n} \sum_{j=1}^{N} l_{PD,kij}(x_k^{(j)}(t) - x_k^{(i)}(t)) + l \sum_{m=1}^{h} \sum_{j=1}^{N} l_{I,mij} \int_0^{t,(m)} (x_1^{(j)}(\tau) - x_1^{(i)}(\tau))d\tau$$

with $i = 1, \ldots, N$. Furthermore, the gain l and the matrices $L_{PD,k} = [l_{PD,kij}]$, with $k = 1, \ldots, n$, and $L_{I,m} = [l_{I,mij}]$, with $m = 1, \ldots, h$, can be selected analogously to Theorem 1 considering the following position:

$$L_{I,h-\vartheta+1} = L_\vartheta, \qquad \vartheta = 1, \ldots, h \tag{21}$$

$$L_{PD,\vartheta-h} = L_\vartheta, \qquad \vartheta = h+1, \ldots, h+n \tag{22}$$

with $\{L_1, \ldots, L_{h+n}\}$ being a $(N, h+n)$-*collection* (or a $(\mathcal{G}, h+n)$-*collection* with $\mathcal{G}$ any connected graph).

Proof: The proof of Theorem 2 is given in [45, Appendix]. ■

Remark 4: The previous result extends Theorem 1, allowing an additional integral control action of any degree. The benefits of integral control actions in polynomial-type disturbance rejection are well known in the literature. Therefore, such an additional degree of freedom can be usefully exploited for this aim, as shown in the numerical examples section.

VI. Synchronization Under Canonical Transformation

The results stated in Section V can be extended to the relevant class of dynamical systems admitting a canonical control transformation. Roughly speaking, for general nonlinear systems, it suffices to find a nonlinear state transformation $z(t) = T(x(t))$ and apply the $PI^h D^{n-1}$ control law of Theorem 2 to such a transformed state. The computation of this nonlinear transformation under suitable involutivity conditions of the nonlinear vector field is a well-known result in nonlinear control and can be found in [16]. Also, when the special case of linear systems is considered, the canonical control transformation can be found in [46] and represents a fundamental result in the control theory.

In this section, we first analyze the general case of nonlinear systems, and later the case of linear systems as a separate result. Notice that for the sake of simplicity in the notation, we will consider only time-independent systems. However, analogous results hold for the case of time-dependent systems.

Theorem 3: Let us consider a connected graph $\mathcal{G}$ and a multiagent system of nonlinear dynamical agents of the following form:

$$\dot{x}^{(i)} = f\left(x^{(i)}\right) + g\left(x^{(i)}\right)u^{(i)}, \qquad i = 1, \ldots, N \tag{23}$$

with $x^{(i)} \in \mathbb{R}^n$ and $u^{(i)} \in \mathbb{R}$. Suppose that, for all $x^{(i)} \in \mathbb{R}^n$, the following conditions hold.

1) The vectors $\{g, \mathrm{ad}_f g, \ldots, \mathrm{ad}_f^{n-1} g\}$ are linearly independent.
2) The set $\{g, \mathrm{ad}_f g, \ldots, \mathrm{ad}_f^{n-2} g\}$ is involutive.
3) The function $\mathcal{L}_f^n(T^{-1}(\xi))$, with $\xi \in \mathbb{R}^n$, is weak-Lipschitz with constant w.

Here, $T(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a suitable diffeomorphism. Then, the free synchronization problem for the multiagent system is solvable with distributed $PI^h D^{n-1}$ controllers, with $h \geq 0$, of the following form[3]:

$$u^{(i)}(t) = \frac{1}{\mathcal{L}_g \mathcal{L}_f^{n-1}(x^{(i)})} l \sum_{k=1}^{n} \sum_{j=1}^{N} l_{PD,kij} \left[T_k\left(x^{(j)}(t)\right) - T_k\left(x^{(i)}(t)\right)\right] + l \sum_{m=1}^{h} \sum_{j=1}^{N} l_{I,mij} \int_0^{t,(m)} \left[T_1\left(x^{(j)}(\tau)\right) - T_1\left(x^{(i)}(\tau)\right)\right] d\tau$$

for all $i = 1, \ldots, N$, and with the gains $l, l_{PD,kij}$, and $l_{I,mij}$ selected according to Theorem 2 and with $T_k(\cdot)$ being the kth element of $T(\cdot)$.

Proof: The proof of Theorem 3 is given in [45, Appendix].

Corollary 1: Let us consider a connected graph $\mathcal{G}$ and a multiagent system of linear dynamical agents of the form $\dot{x}^{(i)} = Ax^{(i)} + bu^{(i)}$, with $x^{(i)} \in \mathbb{R}^n$ and $u^{(i)} \in \mathbb{R}$. If the pair (A, b) is controllable, then there exists a full-rank matrix T such that the free synchronization problem for the multiagent system is solvable with distributed $PI^h D^{n-1}$ controllers of the following form:

$$u^{(i)}(t) = l \sum_{k=1}^{n} \sum_{j=1}^{N} l_{PD,kij} \left[T_k x^{(j)}(t) - T_k x^{(i)}(t)\right] + l \sum_{m=1}^{h} \sum_{j=1}^{N} l_{I,mij} \int_0^{t,(m)} \left[T_1 x^{(j)}(\tau) - T_1 x^{(i)}(\tau)\right] d\tau$$

where the gains $l, l_{PD,kij}$, and $l_{I,mij}$ are selected according to Theorem 2 and with T_k being the kth row of matrix T.

Proof: The proof of Corollary 1 is given in [45, Appendix]. ■

[3]With a slight abuse of notation with $h = 0$, we mean here that no integral action is considered.

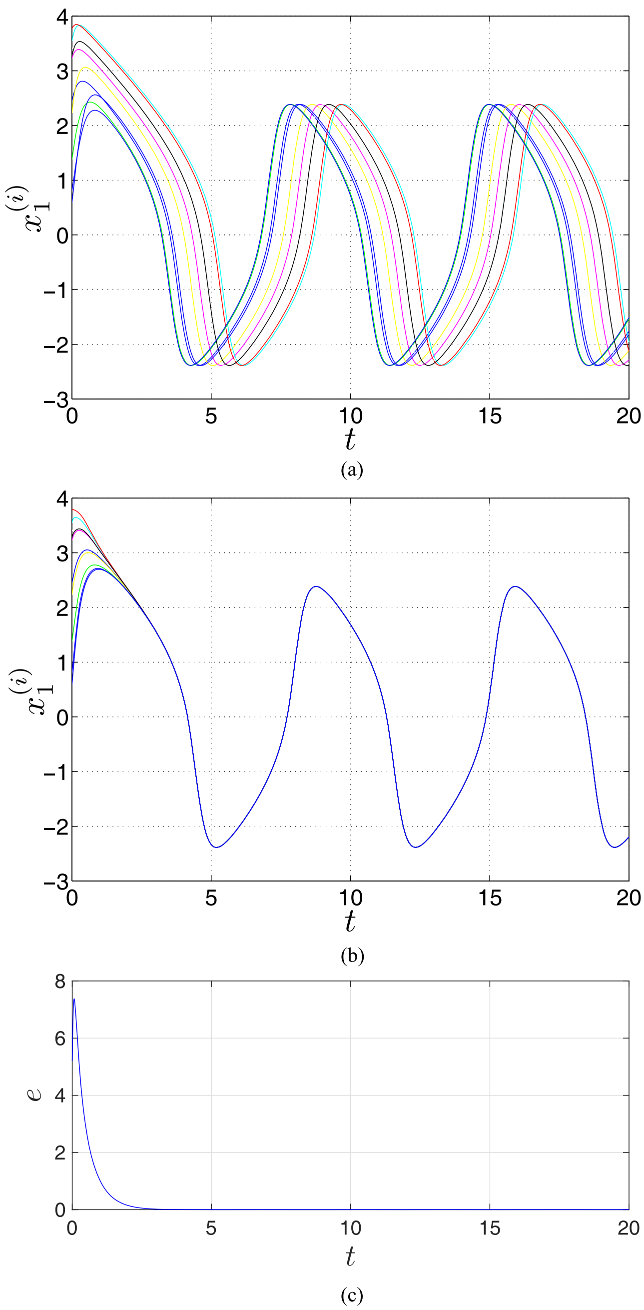

Fig. 1. Time evolution of the state components $x_1^{(i)}$ for the network of Van der Pol oscillators: (a) uncoupled case; (b) coupled case; and (c) synchronization error e.

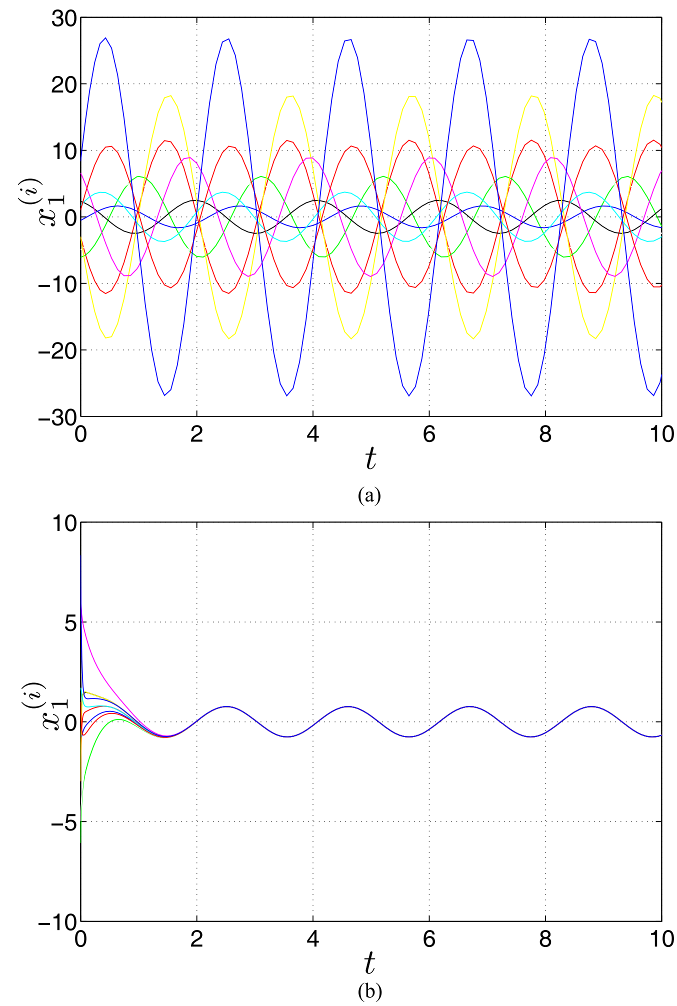

Fig. 2. Time evolution of the state components $x_1^{(i)}$ for the network of linear oscillators with PD controllers and no disturbances: (a) uncoupled case; and (b) coupled case.

Remark 5: Notice that from the aforementioned result, the controllability hypothesis suffices to guarantee the synchronizability of the agents, as already shown in a different way in [6]. However, it is worth noticing that the approach presented here naturally allows us to explicitly consider integral control actions for possible disturbances rejections.

VII. Numerical Example

In this section, we show the effectiveness of our results on two numerical examples. Specifically, the synchronization of nonlinear and linear oscillators with possible disturbances will be achieved via the coupling selection illustrated in Section IV.

A. Synchronization of Van der Pol Oscillators

We consider a network of ten identical Van der Pol oscillators whose model is given by the following relation:

$$\begin{aligned} \dot{x}_1^{(i)} &= x_2^{(i)} \\ \dot{x}_2^{(i)} &= -x_1^{(i)} + \mu\left(1 - |x_1^{(i)}|\right) x_2^{(i)} + u^{(i)}. \end{aligned} \tag{24}$$

For our example, we choose the parameter $\mu = 2.5$ and initial conditions randomly assigned in the interval $[0, 5]$ both for $x_1^{(i)}$ and $x_2^{(i)}$, and for all of the systems in the network. We validate Theorem 1 via creating a connected random graph $\mathcal{G}$ that sets the distributed control for the ten systems and a $(\mathcal{G}, 2)$-collection over such graph (notice that in this case $n = 2$).

Fig. 1(a) shows the first state component of the networked systems when no coupling is considered, while the effect of the coupling of the assigned $(\mathcal{G}, 2)$-collection allows the network

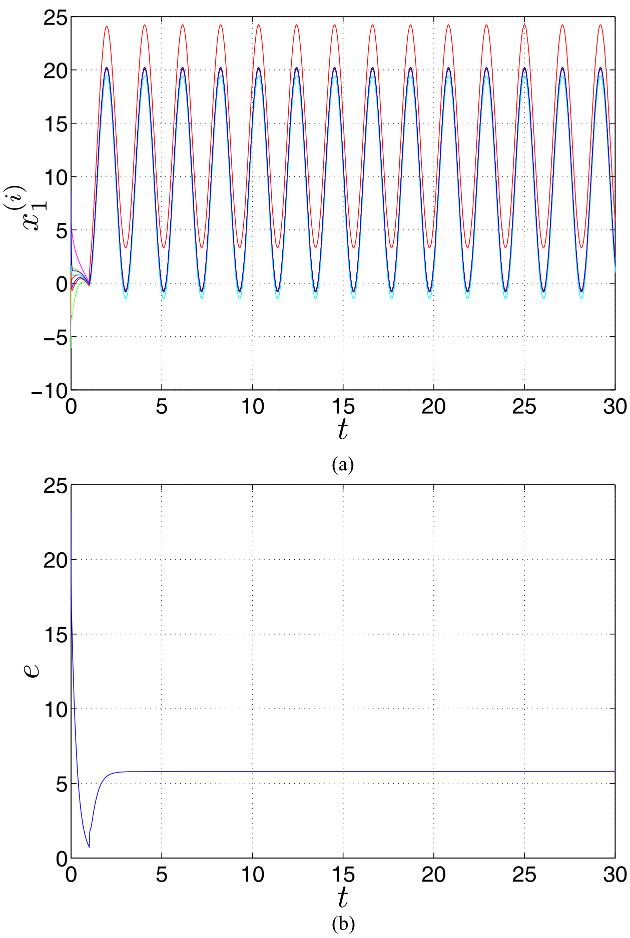

Fig. 3. Time evolution of the network of linear oscillators with PD controllers and heterogeneous disturbances: (a) state components $x_1^{(i)}$; and (b) global synchronization error e.

to synchronize over a common manifold [see Fig. 1(b)]. The synchronization error is depicted in Fig. 1(c).

B. Synchronization of Linear Oscillators

We now consider the synchronization of ten interconnected linear oscillators

$$\dot{x}^{(i)} = \begin{pmatrix} 4 & 5 \\ -5 & -4 \end{pmatrix} x^{(i)} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u^{(i)}$$

using a PD controller and a PID controller according to Corollary 1. Specifically, as done for the previous numerical example, we validate a distribute PD controller via generating a connected random graph $\mathcal{G}$ for the overall system and a related $(\mathcal{G}, 2)$-collection. It is easy to see that the system considered is controllable, and so, we use the distributed controller given in Corollary 1, where the transformation matrix T can be shown to be

$$T = \begin{pmatrix} 0.0556 & -0.0556 \\ 0.5 & 0.5 \end{pmatrix}.$$

From Fig. 2, it is possible to see the time evolution of the first state component for both cases of uncoupled [see Fig. 2(a)] and coupled [see Fig. 2(b)] networks, starting from randomly distributed initial conditions in the interval $[-10, 10]$ for both state components.

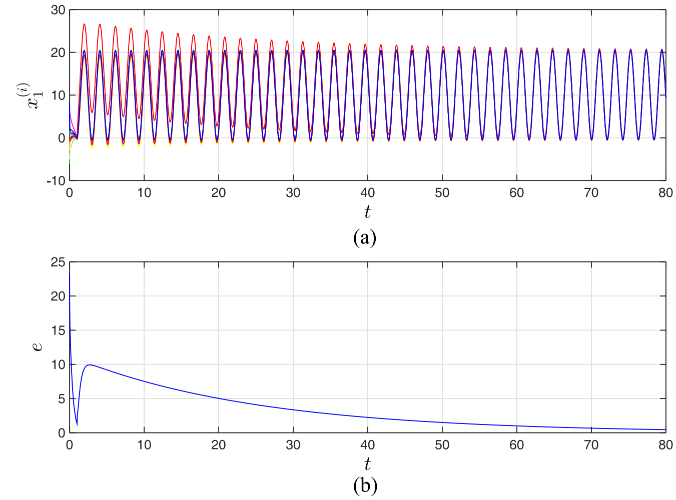

Fig. 4. Time evolution of the network of linear oscillators with PID controllers and heterogeneous disturbances: (a) state components $x_1^{(i)}$; and (b) global synchronization error e.

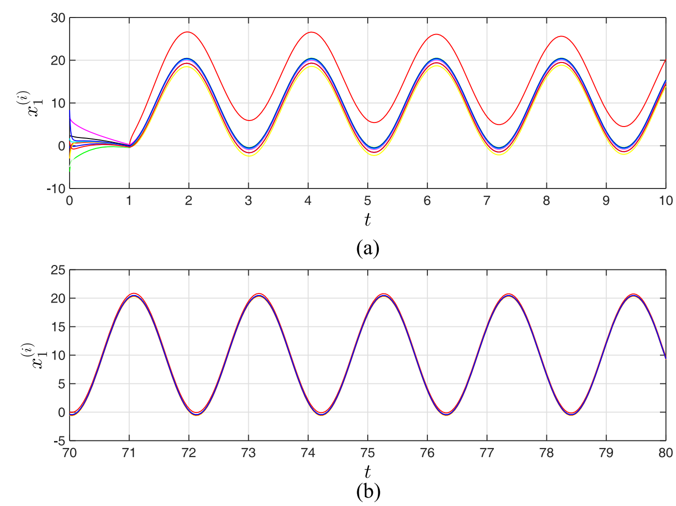

Fig. 5. Time evolution of the network of linear oscillators with PID controllers and heterogeneous disturbances. Zoom of state components $x_1^{(i)}$: (a) beginning of the simulation horizon; and (b) end of the simulation horizon.

In order to validate the effectiveness of the distributed integral action, we add a step disturbance on a system in the network. In Fig. 3(a), the first state component is again shown. It is possible to see that the synchronization is no longer achieved. The residual global synchronization error reaches a constant value in the limit when $t \to +\infty$, which is equal to $e_\infty = 5.8$, as depicted in Fig. 3(b).

In order to reject the disturbance, we then consider a distributed PID controller, coupling the network via a $(\mathcal{G}, 3)$-collection (notice that we have now $n = 2$ and $h = 1$). As clearly emerges from Fig. 4(a), the integral control action is able to reject constant heterogeneous disturbances, thus leading the network to synchronization. In Fig. 5(a) and (b), the same evolution is given, zooming for a time span of 10 s at the beginning and at the end of the simulation horizon, respectively. As can be witnessed, and different from what happens in Fig. 3(a), all of the nodes converge to the same oscillatory orbit.

Fig. 4(b) shows the asymptotic convergence to zero of the global synchronization error associated with such a PID scheme, in comparison with the case shown in Fig. 3(b) where no integral action is considered.

VIII. Discussion and Future Work

In this paper, we addressed the problem of higher order free synchronization for nonlinear systems. Via an iterative procedure, we proved the existence of a class of feedback matrices, able to guarantee distributed state synchronization over any connected graph topology. The framework is related to any system order and easily embeds a possible distributed integral action of any order. The case of higher order consensus is naturally embedded in our results as a particular case. Furthermore, the methodology can also be extended to those linear and nonlinear systems admitting a (local) canonical transformation. In particular, for the specific case of linear systems, the synchronization with distributed $PI^h D^{n-1}$ controllers is guaranteed under the mild hypothesis of controllability of the agent's dynamics.

The presence of a distributed integral control action allows us to attenuate possible distributed heterogeneous disturbances affecting the agents and, as shown in the numerical simulations, greatly improves the convergence performances.

Future work will address, in detail, the analysis of the robust synchronization of agents with parameters' mismatch and subjected to heterogeneous noises/disturbances as well as the case of a directed/pinned network.

A future direction of investigation is to recast the methodology adopted in this paper to the discrete-time case. At the current stage, such an extension is not trivial since the entire analysis (definitions of matrices M_1 and H_1, Algorithm 1 and Algorithm 2) is conducted for the continuous-time case. Therefore, the discrete-time case requires further studies.

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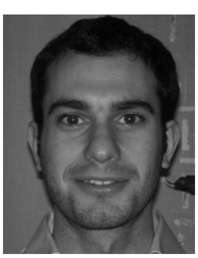

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