SIAM J. Control Optim.
Vol. 51, No. 5, pp. 3673–3691

ROBUST CONSENSUS FOR CONTINUOUS-TIME MULTIAGENT DYNAMICS*

GUODONG SHI† and KARL HENRIK JOHANSSON†

Abstract. This paper investigates consensus problems for continuous-time multiagent systems with time-varying communication graphs subject to input noise. Based on input-to-state stability and integral input-to-state stability, robust consensus and integral robust consensus are defined with respect to L_∞ and L_1 norms of the noise function, respectively. Sufficient and/or necessary connectivity conditions are obtained for the system to reach robust consensus or integral robust consensus under mild assumptions. The results answer the question on how much interaction is required for a multiagent network to converge despite a certain amount of input disturbance. The ϵ-convergence time is obtained for the consensus computation on directed and K-bidirectional graphs.

Key words. multiagent systems, robust consensus, time-varying graphs, convergence rates

AMS subject classifications. 93C15, 93D25, 05C20

DOI. 10.1137/110841308

1. Introduction.

Coordination of multiagent networks has attracted a significant amount of attention in the past few years, due to its broad applications in various fields of science including physics, engineering, biology, ecology, and social science [42, 15, 25, 21, 9]. Distributed control using neighboring information flow has been shown to ensure collective tasks such as formation, flocking, rendezvous, and aggregation [24, 17, 18, 19, 30].

Central to multiagent coordination is the study of consensus, or state agreement, which requires that all agents achieve a common state. The idea of distributed consensus arose as early as 1980s in the classical work [41] for the study of distributed optimization methods. Since then consensus seeking has been extensively studied in the literature for both continuous-time and discrete-time models [15, 21, 38, 39, 3, 4, 28, 18, 45, 46, 49, 48], where node interactions are carried out over an underlying communication graph. The connectivity of this communication graph plays a key role in consensus analysis. The "joint connectivity," i.e., connectivity defined on the union graph over a time interval, and similar concepts are important in the analysis of consensus stability with time-dependent topology. Uniformly joint connectivity, which requests that the union graph be connected for all intervals longer than some positive constant, has been employed for consensus problems for discrete-time and continuous-time agent dynamics, as well as directed and undirected interconnection topologies [41, 15, 18, 13, 6]. In [15], the authors proved the consensus of a simplified Vicsek model under uniformly joint connectivity, followed by some more precise analysis in [3, 4, 28]. In [13] and [6], the jointly connected coordination was investigated for second-order agent dynamics. A nonlinear continuous-time

*Received by the editors July 18, 2011; accepted for publication (in revised form) June 27, 2013; published electronically September 26, 2013. A preliminary version of this paper was presented at the IEEE Conference on Decision and Control and European Control Conference in 2011. Compared to the previous version, in the current paper the results have been improved by relaxing the standing assumptions, and all the proofs are provided with details. This work has been supported in part by the Knut and Alice Wallenberg Foundation and the Swedish Research Council.
http://www.siam.org/journals/sicon/51-5/84130.html
†ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden (guodongs@kth.se, kallej@kth.se).

model was discussed in [18] with directed communications, in which convergence to a consensus is shown to be uniform for bounded initial conditions. The $[t, \infty)$-joint connectivity requires the union graph to be connected for infinitely many disjoint intervals in $[0, \infty]$ and was discussed in [21] for consensus seeking of discrete-time agents. This connectivity concept was then extended to continuous-time distributed control analysis for target set convergence and state agreement in [30].

Communication over networks is often unreliable and noisy. This inspired research on the robustness of consensus algorithms [43, 44, 22, 14, 16, 40]. In [27], robustness was discussed for average consensus algorithms. In [16], a Kalman filter consensus algorithm was shown to be input-to-state stable. Then in [43, 44], robust consensus was studied under directed communication graphs for discrete-time systems. For continuous-time multiagent systems, robustness of consensus was established by an H_2 bound for networks of single integrators with a fixed directed communication graph in [47]. In [40], robust consensus with diverse input delays and asymmetric interconnection perturbations was discussed for a second-order leader-follower model. Recently, an optimal synchronization protocol was studied for discrete-time double integrators subject to process noise [5].

Clearly, robustness of consensus algorithms subject to noise highly relies on the convergence rate for the algorithm in the absence of noise. Convergence rates for discrete-time models have been treated for both deterministic and randomized models [18, 4, 26, 23, 2], where the concept of ϵ-convergence time defined as the minimum time required for the network to reach a certain level of consensus captured by a parameter ϵ, served as a proper measure for the convergence speed. Bounds of ϵ-convergence time have been widely established in the literature for discrete-time dynamics [2, 4, 26], and recently a sharp bound was presented in [23] indicating that the convergence time is of order $O(n^2B)$, where n is the number of nodes in the network and B is a lower bound for the time interval in the definition of uniformly joint connectedness. Few results have been obtained on the convergence rates for continuous-time multiagent systems reaching a consensus on switching directed graphs. In [25], a convergence rate was established when the switching graphs are always kept strongly connected. In [20], it was proved that exponential consensus can be achieved if a node can be found such that it is the root of some union graphs on time intervals with a positive lower bound of length. In [18], for a generalized nonlinear variation of the continuous-time dynamics with a more restricted switching rule, the authors extended this result and showed that uniform asymptotic consensus can be achieved if and only if the union graph on every time interval with a positive lower bound of length admits a directed spanning tree. In all these results, explicit convergence rates were not obtained, so the ϵ-convergence time for continuous-time systems with switching directed graphs is still open. In [31], convergence rates were established explicitly for a leader-follower model with multiple moving leaders, but the analysis cannot be applied to general multiagent systems due to the special structure of leader-follower models and the assumptions on the switching graphs.

The primary aim of this paper is to establish consensus convergence for first-order, continuous-time multiagent systems with input noise for time-varying and directed communication graphs. Borrowing ideas from input-to-state stability (ISS) and integral input-to-state stability (iISS) [36, 37], we define robust consensus and integral robust consensus. We present explicit convergence bounds for the system with respect to L_∞ and L_1 norms of the input noise. Sufficient and necessary connectivity conditions are obtained for the system to reach robust consensus or integral robust consensus, respectively, for directed graphs and a class of K-bidirectional graphs. As a result of the robustness analysis, some upper bounds for the ϵ-convergence time are established. To the best of our knowledge, the results are the first to show that consensus is reached exponentially in t with uniformly jointly connected graphs, while "exponentially" in the times that the joint graph are connected with infinitely jointly connected graphs. Compared to the literature, our results are based on quite mild assumptions which allow infinite switches in bounded time intervals and unbounded weight functions for the arcs in the communication graph.

A preliminary version of this paper was presented in [33], where the weight functions were assumed to admit some positive lower and upper bounds. In the current work, inspired by [20], we can derive the same robust consensus results based on conditions relying only on the integral of these weight functions. In [35], the role persistent arcs play in consensus seeking was discussed on a fixed underlying graph, where interaction arcs whose weight functions have infinite integral over the entire time horizon are called persistent arcs. The current paper enjoys some similar idea and analysis techniques on weight function integrals as [35], but the results in the current work and [35] cannot cover each other because [35] considers fixed graphs with a crucial arc-balance condition. We also believe that our robust consensus results would be useful in various more advanced problems in multiagent systems since the noise term can be interpreted in many ways. Such examples include [34, 32], where in [34] the noise term corresponds to an event-triggering condition in distributed event-triggered consensus, and in [32] the noise term corresponds to a convex projection term in a distributed optimization problem.

The paper is organized as follows. In section 2, some preliminary concepts are introduced. We set up the system model and present our standing assumptions and the problem of interest in section 3. Then convergence analysis is carried out for directed and K-bidirectional graphs in sections 4 and 5, respectively. Finally, some concluding remarks are given in section 6.

2. Preliminaries.

Here we introduce some notation and theories on directed graphs and Dini derivatives.

2.1. Directed graphs.

A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, \dots, N\}$ of nodes and an arc set $\mathcal{E}$, where an element $e = (i, j) \in \mathcal{E}$ is an arc from node i and to j [11]. An alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_n v_n$ of nodes $v_i, i = 1, 2, \dots, n$, with arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E} \, \forall \ i$, is called a (directed) *path* with *length* n. A path with no repeated nodes is called a *simple path*. If there exists a path from node i to node j, then node j is said to be reachable from node i. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. A digraph $\mathcal{G}$ is said to be *strongly connected* if each node is reachable from the other for every two different nodes and *quasi-strongly connected* if $\mathcal{G}$ has a center [1, 18]. For graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if each arc $(i, j) \in \mathcal{E}$ is additionally associated with a weight $a_{ji} > 0$, we call $\mathcal{G}$ a weighted digraph, and we denote it as $\mathcal{G}_A = \big(\mathcal{V}, \mathcal{E}, A\big)$ with $A = [a_{ij}] \in \mathbb{R}^{N \times N}$. In this paper, we define the (generalized) *distance* from i to j, $d(i, j)$, as the length of a longest simple path from i to j if j is reachable from i, and the (generalized) *diameter* of $\mathcal{G}$ as $\max\{d(i, j) : i, j \in \mathcal{V}, \ j \text{ is reachable from } i\}$.

2.2. Dini derivatives.

The upper Dini derivative of a function $h : (a, b) \to \mathbb{R}$ at t is defined as

$$D^+ h(t) = \limsup_{s \to 0^+} \frac{h(t+s) - h(t)}{s}.$$

The next result is useful for the calculation of Dini derivatives [8, 18].

LEMMA 2.1. *Let $V_i(t,x) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ $(i = 1,\dots,n)$ be C^1 and $V(t,x) = \max_{i=1,\dots,n} V_i(t,x)$. If $\mathcal{I}(t) = \{i \in \{1,2,\dots,n\} : V(t,x(t)) = V_i(t,x(t))\}$ is the set of indices where the maximum is reached at t, then $D^+V(t,x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t,x(t))$.*

Notation. For a vector $z = (z_1,\dots,z_N)^T$ in $\mathbb{R}^N$, $|z|$ denotes the maximum norm, i.e., $|z| \doteq \max_{i=1,\dots,N} |z_i|$. When $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ is a measurable function defined on $[0,+\infty)$, $\|z\|_\infty$ denotes the essential supremum of $\{|z(t)|, t \in [0,+\infty)\}$. Moreover, a function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is said to be a $\mathcal{K}$-class function if it is continuous, strictly increasing, and $\gamma(0) = 0$; a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a $\mathcal{KL}$-class function if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t) \to 0$ decreasingly as $t \to \infty$ for each fixed $s \geq 0$.

3. Problem definition.

3.1. Network, dynamics, and assumption. Consider a multiagent system with agent set $\mathcal{V} = \{1,\dots,N\}$, $N \geq 2$. Each agent (node) i holds a state $x_i(t) \in \mathbb{R}$. The initial time is $t_0 \geq 0$. The evolution of the node states follows

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(t)\big(x_j(t) - x_i(t)\big) + w_i(t), \quad i = 1,\dots,N, \tag{3.1}$$

where $a_{ij}(t) \geq 0$ is a function marking the strength of the information flow from j to i for $i,j \in \mathcal{V}$, and $w_i(t)$ is the input noise in node i's dynamics which may come from the information exchange with other nodes or simply measurement disturbance.

For $a_{ij}(t)$ and $w_i(t)$, $i,j \in \mathcal{V}$, we impose the following assumption, which will be our standing assumption throughout the rest of the paper.

Assumption. (i) For all $i,j \in \mathcal{V}$, $a_{ij}(t)$ and $w_i(t)$ are continuous functions on $[0,\infty)$ except for at most a set with measure zero; (ii) $a_{ii}(t) \equiv 0 \,\forall\, i$; (iii) there exists a constant $M_0 > 0$ such that $\int_{t_1}^{t_2} a_{ij}(s)ds \leq M_0|t_2 - t_1| \,\forall\, i,j \in \mathcal{V}$ and $0 \leq t_1 < t_2 < \infty$.

Under this assumption the set of discontinuity points for the right-hand side of (3.1) has measure zero. Therefore, the Caratheodory solutions of (3.1) exist for arbitrary initial conditions, and they are absolutely continuous functions that satisfy (3.1) for almost all t on the maximum interval of existence [10, 7]. In the following, each solution of (3.1) is considered in the sense of Caratheodory without explicit mention. Moreover, note that our assumption also allows the weight functions $a_{ij}(t)$ to be unbounded, which generalizes the model discussed in [20].

3.2. Arcs, graph, and connectivity. Naturally the node dynamics (3.1) corresponds to a time-varying, directed underlying communication graph, defined as follows.

DEFINITION 3.1. *The underlying communication graph of system* (3.1) *at time t is defined as weighted graph $\mathcal{G}_{A(t)} = \big(\mathcal{V},\mathcal{E}_t,A(t)\big)$, where $\forall i,j \in \mathcal{V}$, $(j,i) \in \mathcal{E}_t$ if and only if $a_{ij}(t) > 0$.*

This graph $\mathcal{G}_{A(t)}$ characterizes all the information exchange among the nodes and therefore plays a fundamental role in the evolution of the node states. In light of the definition of δ-graphs in [20], we introduce the following definition on the connectivity of $\mathcal{G}_{A(t)}$.

DEFINITION 3.2. *Let $\delta > 0$ be a given constant.*

(i) *An arc (i,j) is said to be a δ-arc of $\mathcal{G}_{A(t)}$ on time interval $[t_1,t_2)$ if $\int_{t_1}^{t_2} a_{ji}(t) \geq \delta$. A path is said to be a δ-path of $\mathcal{G}_{A(t)}$ on time interval $[t_1,t_2)$ if every arc is a δ-arc in this path.*

(ii) $\mathcal{G}_{A(t)}$ *is said to be uniformly quasi-strongly* δ*-connected if there exists a constant* $T > 0$ *such that for any* $t \geq 0$*, the* δ*-arcs of* $\mathcal{G}_{A(t)}$ *on time interval* $[t, t+T)$ *form a quasi-strongly connected graph on node set* $\mathcal{V}$*.*

(iii) $\mathcal{G}_{A(t)}$ *is said to be infinitely jointly quasi-strongly* δ*-connected if for any* $t \geq 0$*, the* δ*-arcs of* $\mathcal{G}_{A(t)}$ *on time interval* $[t, \infty)$ *form a quasi-strongly connected graph on node set* $\mathcal{V}$*.*

Remark 3.3. Consider the time-varying graph $\mathcal{G}_t = (\mathcal{V}, \mathcal{E}_t)$ ignoring the arc weights. Then we see that $\mathcal{G}_t$ is not necessarily piecewise constant since infinite switches are allowed for $a_{ij}(t), i, j \in \mathcal{V}$ in a bounded interval for the considered model.

3.3. The robust consensus problem. Denote $x(t) = (x_1(t), \dots, x_N(t))^T$. Consider system (3.1) with initial condition $x(t_0) = x^0 \in \mathbb{R}^N$. Let

$$\hbar(t) \doteq \max_{i \in \mathcal{V}} \{x_i(t)\}, \quad \ell(t) \doteq \min_{i \in \mathcal{V}} \{x_i(t)\}$$

be the maximum and minimum state value at time t, respectively. Denote $\mathcal{H}\big(x(t)\big) \doteq \hbar(t) - \ell(t)$ which serves as a metric of consensus for the considered system.

Introduce $\mathcal{F} \doteq \{z : \mathbb{R}_{\geq 0} \to \mathbb{R}^N : \|z\|_\infty < \infty$, and z continuous except for at most a set with measure zero$\}$. Inspired by the concepts of ISS and iISS [37, 36], we introduce the following definition.

Definition 3.4.

(i) *System* (3.1) *achieves global robust consensus if there exist a* $\mathcal{KL}$*-function* β *and a* $\mathcal{K}$*-function* γ *such that* $\forall\ w \in \mathcal{F}$*, initial time* $t_0 \geq 0$*, and initial state* $x(t_0) = x^0$*, it holds that*

$$\mathcal{H}\big(x(t)\big) \leq \beta\big(\mathcal{H}(x^0), t\big) + \gamma(\|w\|_\infty),\ t \geq 0. \tag{3.2}$$

(ii) *System* (3.1) *achieves global integral robust consensus if there exist a* $\mathcal{KL}$*-function* β *and a* $\mathcal{K}$*-function* γ *such that* $\forall\ w \in \mathcal{F}$*, initial time* $t_0 \geq 0$*, and initial state* $x(t_0) = x^0$*, it holds that*

$$\mathcal{H}\big(x(t)\big) \leq \beta\big(\mathcal{H}(x^0), t\big) + \int_0^t \gamma(|w(s)|)ds,\ t \geq 0. \tag{3.3}$$

We also introduce the following definition on consensus.

Definition 3.5.

(i) *System* (3.1) *achieves global consensus if for any initial condition* $x(t_0) = x^0 \in \mathbb{R}^N$*, we have* $\lim_{t\to\infty} \mathcal{H}\big(x(t)\big) = 0$*.*

(ii) *Assume that* $\mathcal{F}_0 \subseteq \mathcal{F}$*. System* (3.1) *achieves global asymptotic consensus with respect to* $\mathcal{F}_0$ *if* $\forall x^0 \in \mathbb{R}^N$*,* $\forall w \in \mathcal{F}_0$*,* $\forall \varepsilon > 0$*,* $\forall c > 0$*,* $\exists T > 0$ *such that* $\forall t_0 \geq 0$*,*

$$\mathcal{H}(x^0) \leq c \quad \Longrightarrow \quad \mathcal{H}\big(x(t)\big) \leq \varepsilon\ \forall t \geq t_0 + T.$$

In [31], a set tracking problem is studied for multiagent systems guided by multiple leaders. However, the convergence results in [31] cannot be applied to the system discussed in this paper. In the leader-follower model, the leader can always be treated as a center node and therefore the network has a very special topology. The main difficulty in this paper lies in that the center node may be different for different time intervals and that its dynamics is influenced by other nodes. As will be shown in the following discussions, the symmetry in the structure of $\mathcal{H}(t)$ plays a key role in the convergence analysis. Hence, the contribution of this paper is far beyond [31].

We conclude this section with a few remarks. System (3.1) is a basic model for continuous-time distributed consensus, and it serves as a basic method for cooperative control of multiagent systems [25, 30, 31, 19, 29, 15]. The consensus convergence of system (3.1) or its nonlinear variations for the noiseless case has been extensively studied in existing works [25, 20, 28, 30, 18]. Compared to these previous works, the model considered in this paper is quite general for the reasons that we do not impose piecewise continuous switching for node dynamics with or without dwell time, nor the bounded arc weights of the information flow. Moreover, the consensus metric $\mathcal{H}(x(t))$ is also used in [20, 18], which turns out to be a suitable measure for consensus under directed and switching node interactions. In fact, the idea of introducing $\mathcal{H}(x(t))$ are consistent with the analysis of Markov chains in classical works [12]. The target of the paper is to investigate the role of the underlying communication graph and the presence of noise in robust consensus. The analysis essentially relies on a careful characterization for the convergence rates of system (3.1) in the absence of noise.

4. Directed graphs. For directed communication graphs, we present the following results.

THEOREM 4.1. *System* (3.1) *achieves global robust consensus if and only if there exists a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected.*

THEOREM 4.2. *System* (3.1) *achieves global integral robust consensus if there exists a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected.*

It has been shown in [36] that ISS implies iISS. Now combining Theorems 4.1 and 4.2, we see that robust consensus implies a uniformly quasi-strongly δ-connected communication graph, which further implies integral robust consensus. Thus, robust consensus and integral robust consensus are consistent with the ISS and iISS properties. Moreover, it is worth pointing out that Theorem 4.2 is not conservative since simple examples can show that uniformly jointly quasi-strong δ-connectivity is not necessary for integral robust consensus.

In this section, we first establish two technical lemmas, followed by the proofs of Theorems 4.1 and 4.2.

4.1. Key lemmas. We first establish the following lemma indicating that the Dini derivative of $\hbar(t)$ is bounded above by $|w(t)|$, and the Dini derivative of $\ell(t)$ is bounded below by $-|w(t)|$.

LEMMA 4.3. *For all $t \geq t_0 \geq 0$, we have*

$$D^+\hbar(t)| \leq |w(t)|; \quad D^+\ell(t) \geq -|w(t)|.$$

Proof. We prove $D^+\hbar(t) \leq |w(t)|$. The other part can be proved similarly.

Let $\mathcal{I}(t)$ represent the set containing all the agents that reach the maximum in the definition of $\hbar(t)$ at time t, i.e., $\mathcal{I}(t) = \{i \in \mathcal{V} |\ x_i(t) = \hbar(t)\}$. Then according to Lemma 2.1, we obtain

$$\begin{aligned} D^+\hbar(t) &= \max_{i\in\mathcal{I}(t)} \dot{x}_i(t) \\ &= \max_{i\in\mathcal{I}(t)} \left[\sum_{j\in N_i(\sigma(t))} a_{ij}(t)\big(x_j(t) - x_i(t)\big) + w_i(t) \right] \leq \max_{i\in\mathcal{I}(t)} w_i(t) \leq |w(t)|, \end{aligned}$$

which completes the proof. $\square$

Suppose there exists a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected. Define a set-valued function $f : \mathbb{Z}^+ \to 2^{\{1,\dots,N\}}$, where $2^{\{1,\dots,N\}}$ represents the power set containing all the subsets of $\{1,\dots,N\}$:

$$f(s) = \big\{ j : \text{there is a } \delta\text{-path from } j \text{ to every other nodes over } [(s-1)T, sT) \big\},$$

where $s = 1, 2, \dots$. In other words, $f(s)$ is a set consisting of all nodes that are center nodes for the graph of δ-arcs on time interval $[(s-1)T, sT)$. The following lemma holds.

LEMMA 4.4. *Suppose there exists a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected. Then for any $t = 1, 2, \dots$ and any integer $D > 0$, there exists $k_0 \in \{1, 2, \dots, N\}$ such that $k_0 \in f(s)$ at least D times for $s = t, t+1, \dots, t + (d_0 - 1)N$.*

Proof. Suppose $k \in f(s)$ for less than D times (i.e., less than or equal to $D-1$) during $[t, t + (D-1)N]$ $\forall\, k \in \{1, 2, \dots, N\}$. Then, the total number of the elements of all the preimages of f on interval $s \in [t, t + (D-1)N]$ is no larger than $(D-1)N$. However, on the other hand, there are at least $(D-1)N + 1$ elements (counting times for the same node) belonging to $f(s)$ during $s \in [t, t + (d_0 - 1)N]$ since $f(\varsigma) \neq \emptyset$ $\forall\, \varsigma = 1, 2, \dots$. Then we get a contradiction and the conclusion is proved. ∎

4.2. Proof of Theorem 4.1.

4.2.1. Necessity. Suppose there is no $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected. Then $\forall \delta > 0, \forall T_* > 0$, there exists $t_* \geq 0$ such that the graph $\mathcal{G}^* = (\mathcal{V}, \mathcal{E}^*)$, with $\mathcal{E}^*$ defined by $\mathcal{E}^* = \big\{(i,j) : \int_{t_*}^{t_*+T_*} a_{ji}(t)dt \geq \delta\big\}$ containing all the δ-arcs on time interval $[t_*, t_* + T_*)$, is not quasi-strongly connected.

Consequently, there exist two distinct nodes i and j such that $\bar{\mathcal{V}}_1 \cap \bar{\mathcal{V}}_2 = \emptyset$, where $\bar{\mathcal{V}}_1 = \{\text{nodes from which } i \text{ is reachable in } \mathcal{G}^*\}$ and $\bar{\mathcal{V}}_2 = \{\text{nodes from which } j \text{ is reachable in } \mathcal{G}^*\}$. Let $w_i(t) \equiv 0$ for $i \in \bar{\mathcal{V}}_1$ and $w_i(t) \equiv 1$ for $i \in \mathcal{V} \setminus \bar{\mathcal{V}}_1$ when $t \in [t_*, t_* + T_*]$. Let the initial time be t_* with $x_i(t_*) = 0$ $\forall i \in \mathcal{V}$ so that $\mathcal{H}(x(t_*)) = 0$.

Now define

$$\ell_*(t) \doteq \max_{i \in \bar{\mathcal{V}}_1} \{x_i(t)\}, \quad \hbar_*(t) \doteq \min_{i \in \bar{\mathcal{V}}_2} \{x_i(t)\},$$

and then $H_*(t) = \hbar_*(t) - \ell_*(t)$. Following Lemma 4.3, it is easy to obtain that $\forall\, t \in [t_*, t_* + T_*)$,

$$\ell(t) \geq 0; \quad \hbar(t) \leq T_*. \tag{4.1}$$

We denote $\Theta(t) = \sum_{(j,i) \notin \mathcal{E}^*} a_{ij}(t)$. With (4.1) and according to the definition of $\bar{\mathcal{V}}_1$ and $\bar{\mathcal{V}}_2$, a analysis similar to the proof of Lemma 4.3 leads to

$$D^+ \hbar_*(t) \geq 1 - \Theta(t)\hbar_*(t), \quad D^+ \ell_*(t) \leq \Theta(t)\big(T_* - \ell_*(t)\big), \tag{4.2}$$

for $t \in [t_*, t_* + T_*)$, which yields

$$D^+ H_*(t) \geq 1 - \Theta(t)\big(H_*(t) - T_*\big), \quad t \in [t_*, t_* + T_*). \tag{4.3}$$

This implies

$$
\begin{aligned}
H_*(t_*+T_*) &\geq \int_{t_*}^{t_*+T_*} e^{-\int_t^{t_*+T_*}\Theta(s)ds}dt - \Big(1-e^{-\int_{t_*}^{t_*+T_*}\Theta(s)ds}\Big)T_* \\
&\geq e^{-\int_{t_*}^{t_*+T_*}\Theta(s)ds}T_* - \Big(1-e^{-\int_{t_*}^{t_*+T_*}\Theta(s)ds}\Big)T_* \\
&\geq (2e^{-E_0\delta}-1)T_*
\end{aligned}
\tag{4.4}
$$

from Grönwall's inequality, where E_0 is an integer denoting the number of all possible arcs on node set $\mathcal{V}$, and the last inequality holds from the definition of $\mathcal{E}^*$ and $\Theta(t)$.

Now noticing that δ and T_* can be arbitrary positive numbers in (4.4), we see that $\mathcal{H}(x(t_*+T_*))$ cannot be bounded above by $\gamma(1)$ for any fixed $\mathcal{K}$-function γ since apparently $\mathcal{H}(x(t_*+T_*)) \geq H_*(t_*+T_*)$. Hence, global robust consensus cannot be achieved and this completes the proof for the necessity statement of Theorem 4.1.

4.2.2. Sufficiency. Suppose there is a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected. Let d_0 be the (generalized) diameter of $\mathcal{G}_\infty^\delta$, where $\mathcal{G}_\infty^\delta$ is the graph containing all the δ-arcs present over the time intervals $[(s-1)T, sT)$, $s = 1, 2, \dots$. Let the initial time be $t_0 = 0$ for simplicity. The analysis of $\mathcal{H}\big(x(t)\big)$ will be carried out on time intervals $t \in [sK_0, (s+1)K_0]$ for $s = 0, 1, 2, \dots$, where $K_0 = [(d_0-1)N+1]T$.

Based on Lemma 4.3, we see that $\forall\ t \in [sK_0, (s+1)K_0]$,

$$
\hbar(t) \leq \hbar(sK_0) + \|w\|_\infty K_0; \quad \ell(t) \geq \ell(sK_0) - \|w\|_\infty K_0. \tag{4.5}
$$

We divide the rest of the proof into three steps, in which convergence bounds are established over the network node by node on time intervals $[sK_0, (s+1)K_0], s = 0, 1, \dots$.

Step 1. According to Lemma 4.4, we can choose $i_0 \in \mathcal{V}$ such that there is a δ-path from i_0 to every other nodes over each of the d_0 time intervals $[j_mT, (j_m+1)T), m = 1, 2, \dots, d_0$, with $[j_mT, (j_m+1)T) \subseteq [sK_0, (s+1)K_0]$ for every m. Assume that

$$
x_{i_0}(sK_0) \leq \frac{1}{2}\ell(sK_0) + \frac{1}{2}\hbar(sK_0). \tag{4.6}
$$

In this step, we bound $x_{i_0}(t)$ on time interval $[sK_0, (s+1)K_0]$.

Denote $\mathcal{Y}_i(t) = \sum_{j=1}^N a_{ij}(t)$ for every node $i \in \mathcal{V}$. With (4.5), we have

$$
\frac{d}{dt}x_{i_0}(t) \leq -\mathcal{Y}_{i_0}(t)\big(x_{i_0}(t) - \hbar(sK_0) - K_0\|w\|_\infty\big) + |w(t)|, \quad t \in [sK_0, (s+1)K_0], \tag{4.7}
$$

which implies

$$
\begin{aligned}
x_{i_0}(t) &\leq \Big[1 - e^{-\int_{sK_0}^t \mathcal{Y}_{i_0}(\tau)d\tau}\Big](\hbar(sK_0) + K_0\|w\|_\infty) \\
&\quad + e^{-\int_{sK_0}^t \mathcal{Y}_{i_0}(\tau)d\tau}x_{i_0}(sK_0) + K_0\|w\|_\infty \\
&\leq \frac{1}{2}e^{-\int_{sK_0}^t \mathcal{Y}_{i_0}(\tau)d\tau}\ell(sK_0) + \Big[1 - \frac{1}{2}e^{-\int_{sK_0}^t \mathcal{Y}_{i_0}(\tau)d\tau}\Big]\hbar(sK_0) + 2K_0\|w\|_\infty \\
&\leq \xi_0\ell(sK_0) + (1-\xi_0)\hbar(sK_0) + 2K_0\|w\|_\infty, \quad t \in [sK_0, (s+1)K_0],
\end{aligned}
\tag{4.8}
$$

where $\xi_0 = e^{-(N-1)K_0M_0}/2$ with M_0 defined in our standing assumption. Here the first inequality of (4.8) follows from Grönwall's inequality, and the last one holds based on the simple fact that $\ell(t) \le \hbar(t)\ \forall\ t$. We denote $\chi \doteq e^{-(N-1)K_0M_0}$.

Step 2. Since there is a δ-path from k_0 to every other node over time interval $[j_1\hat{T}, (j_1+1)\hat{T})$, we can define a set

$$\mathcal{V}_1 \doteq \big\{j : \text{there exists a } \delta\text{-arc from } k_0 \text{ to } j \text{ for } \mathcal{G}_{A(t)} \text{ on interval } [j_1T, (j_1+1)T)\big\}.$$

In this step, we will establish an upper bound for $x_{i_1}(t), i_1 \in \mathcal{V}_1$.

Take $i_1 \in \mathcal{V}_1$ and define $\hat{\mathcal{Y}}_{i_1}(t) = \mathcal{Y}_{i_1}(t) - a_{i_1i_0}(t)$. Now denoting $B_1 \doteq \hbar(sK_0) + K_0\|w\|_\infty$ and $B_2 = \xi_0\ell(sK_0) + (1-\xi_0)\hbar(sK_0) + 2K_0\|w\|_\infty$, for $t \in [j_1T, (j_1+1)T))$, we have

$$\frac{d}{dt}x_{i_1}(t) \le \hat{\mathcal{Y}}_{i_1}(t)\big(B_1 - x_{i_1}(t)\big) + a_{i_1i_0}(t)\big(B_2 - x_{i_1}(t)\big) + w_{i_1}(t). \tag{4.9}$$

Using Grönwall's inequality, we thus obtain

$$\begin{aligned} &x_{i_1}\big((j_1+1)T\big) \\ &\quad\le e^{-\int_{j_1T}^{(j_1+1)T}\mathcal{Y}_{i_1}(t)dt}x_{i_1}(j_1T) + B_1\int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}\mathcal{Y}_{i_1}(\tau)d\tau}\hat{\mathcal{Y}}_{i_1}(t)dt \\ &\qquad + B_2\int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}\mathcal{Y}_{i_1}(\tau)d\tau}a_{i_1i_0}(t)dt + K_0\|w\|_\infty \\ &\quad\le \left(\xi_0\int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}\mathcal{Y}_{i_1}(\tau)d\tau}a_{i_1i_0}(t)dt\right)\ell(sK_0) \\ &\qquad + \left(1-\xi_0\int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}\mathcal{Y}_{i_1}(\tau)d\tau}a_{i_1i_0}(t)dt\right)\hbar(sK_0) + 3K_0\|w\|_\infty, \end{aligned} \tag{4.10}$$

where the second inequality follows from the fact that $x_{i_1}(j_1T) \le \hbar(sK_0) + K_0\|w\|_\infty$ and some simple algebra based on

$$\int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}\mathcal{Y}_{i_1}(\tau)d\tau}\mathcal{Y}_{i_1}(t)dt = 1 - e^{-\int_{j_1T}^{(j_1+1)T}\mathcal{Y}_{i_1}(t)dt}.$$

Furthermore, noticing that

$$\begin{aligned} &\int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}\mathcal{Y}_{i_1}(\tau)d\tau}a_{i_1i_0}(t)dt \\ &\quad= \int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}\hat{\mathcal{Y}}_{i_1}(\tau)d\tau}\cdot e^{-\int_t^{(j_1+1)T}a_{i_1i_0}(\tau)d\tau}a_{i_1i_0}(t)dt \\ &\quad\ge e^{-\int_{j_1T}^{(j_1+1)T}\hat{\mathcal{Y}}_{i_1}(\tau)d\tau}\int_{j_1T}^{(j_1+1)T} e^{-\int_t^{(j_1+1)T}a_{i_1i_0}(\tau)d\tau}a_{i_1i_0}(t)dt \\ &\quad\ge e^{-(N-2)M_0T}\Big(1 - e^{-\int_{j_1T}^{(j_1+1)T}a_{i_1i_0}(t)dt}\Big) \\ &\quad\ge e^{-(N-2)M_0T}\big(1-e^{-\delta}\big) \\ &\quad\doteq \zeta, \end{aligned} \tag{4.11}$$

we conclude from (4.10) that

$$x_{i_1}\big((j_1+1)T\big) \le \zeta\xi_0\ell(sK_0) + (1-\zeta\xi_0)\hbar(sK_0) + 3K_0\|w\|_\infty. \tag{4.12}$$

Applying inequality (4.7) on $x_{i_1}(t)$ for $t \in [(j_1+1)T, (s+1)K_0]$, it turns out that

$$x_{i_1}(t) \le \xi_1\ell(sK_0) + (1-\xi_1)\hbar(sK_0) + 4K_0\|w\|_\infty, \quad t \in [(j_1+1)T, (s+1)K_0], \tag{4.13}$$

$\forall\ i_1 \in \mathcal{V}_1$, where $\xi_1 = \zeta\chi^2/2$.

Step 3. Continuing the analysis on time interval $[j_2\hat{T}, (j_2+1)\hat{T})$, we can similarly define

$$\mathcal{V}_2 \doteq \big\{j : \text{there exists a } \delta\text{-arc from } \mathcal{V}_1 \text{ to } j \text{ for } \mathcal{G}_{A(t)} \text{ on interval } [j_2T, (j_2+1)T)\big\}.$$

Repeating the analysis in Step 2, we have

$$x_{i_2}(t) \le \xi_2\ell(sK_0) + (1-\xi_2)\hbar(sK_0) + 8K_0\|w\|_\infty, \quad t \in [(j_2+1)T, (s+1)K_0], \tag{4.14}$$

$\forall\ i_2 \in \mathcal{V}_2$, where $\xi_2 = \zeta^2\chi^3/2$.

Recall that d_0 is the (generalized) diameter of $\mathcal{G}\big([0,+\infty)\big)$. We can proceed with the analysis on time intervals $[j_m\hat{T}, (j_{m+1})\hat{T})$ for $m = 3,\dots,d_0$, and $\mathcal{V}_3,\dots,\mathcal{V}_z$ can be defined for some $z \le d_0$, respectively, until we obtain $\mathcal{V} = \bigcup_{i=1}^{z}\mathcal{V}_i$. Moreover, we have

$$x_i\big((s+1)K_0\big) \le \xi_{d_0}\ell(sK_0) + (1-\xi_{d_0})\hbar(sK_0) + 4d_0K_0\|w\|_\infty, \quad i = 1,\dots,N, \tag{4.15}$$

where

$$\xi_{d_0} = \zeta^{d_0}\chi^{d_0+1}/2. \tag{4.16}$$

This leads to

$$\begin{aligned}\mathcal{H}\big(x((s+1)K_0)\big)& \\ &\le \xi_{d_0}\ell(sK_0) + (1-\xi_{d_0})\hbar(sK_0) + 4d_0K_0\|w\|_\infty - (\ell(sK_0) - \|w\|_\infty K_0)\\ &= (1-\xi_{d_0})\mathcal{H}\big(x(sK_0)\big) + (4d_0+1)K_0\|w\|_\infty.\end{aligned} \tag{4.17}$$

For the opposite case of (4.6) with $x_{i_0}(sK_0) > \frac{1}{2}\ell(sK_0) + \frac{1}{2}\hbar(sK_0)$, we see that (4.17) also holds using a symmetric argument by investigating the lower bound for $\ell\big((s+1)K_0\big)$.

Since s is arbitrarily chosen in (4.17), we have

$$\begin{aligned}\mathcal{H}\big(x(nK_0)\big) &\le (1-\xi_{d_0})^n\mathcal{H}(x^0) + \sum_{j=0}^{n-1}(1-\xi_{d_0})^j(4d_0+1)K_0\|w\|_\infty\\ &\le (1-\xi_{d_0})^n\mathcal{H}(x^0) + \frac{(4d_0+1)K_0}{\xi_{d_0}}\cdot\|w\|_\infty\end{aligned}$$

for any $n = 0,1,2,\dots$. From (4.5), we also know

$$\mathcal{H}\big(x(t)\big) \le \mathcal{H}\big(x(nK_0)\big) + 2K_0\|w\|_\infty,\ t \in [nK_0, (n+1)K_0). \tag{4.18}$$

The desired robust consensus inequality is therefore obtained by

$$\beta\big(\mathcal{H}(x^0), t\big) = (1-\xi_{d_0})^{\lfloor \frac{t}{K_0}\rfloor}\mathcal{H}(x^0), \quad \gamma(\|w\|_\infty) = \left(2 + \frac{4d_0+1}{\xi_{d_0}}\right)K_0\cdot\|w\|_\infty, \tag{4.19}$$

where $\lfloor\frac{t}{K_0}\rfloor$ denotes the largest integer no greater than $\frac{t}{K_0}$. The proof is completed.

4.3. Convergence time. Let us present some discussions on the convergence rate of system (3.1) in the absence of noise. We introduce the following definition.

DEFINITION 4.5. *Suppose $w \equiv 0$. An exponential consensus is achieved for system* (3.1) *if there exist two constants $C, \alpha > 0$ such that $\mathcal{H}\big(x(t)\big) \le Ce^{-\alpha(t-t_*)}\mathcal{H}(x^0)$ for $t \ge t_*$ given initial condition $x(t_*) = x^0$.*

A direct corollary follows from Theorem 4.1.

COROLLARY 4.6. *Suppose $w \equiv 0$. System* (3.1) *achieves an exponential consensus if and only if there exists a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected.*

The sufficiency part of Corollary 4.6 holds directly from the robust consensus inequality, and the necessity claim holds essentially from the linear node dynamics and can be proved by a simple variation of the necessity proof of Theorem 4.1. The details of the proof are therefore omitted.

We can use the concept of ϵ-convergence time to present a more precise characterization to the convergence rate of system (4.1). Bounds on ϵ-convergence time have been extensively established in the literature for discrete-time dynamics [2, 4, 26, 23]. Now let us introduce the following definition of convergence time as the corresponding continuous-time version.

DEFINITION 4.7. *Suppose $w(t) \equiv 0$. The ϵ-convergence time of system* (3.1) *is defined as*

$$\mathrm{T}_N(\epsilon) = \sup_{x^0\in\mathbb{R}^N,\ \mathcal{H}(x^0)\neq 0} \min\left\{t : \frac{\mathcal{H}\big(x(t)\big)}{\mathcal{H}(x^0)} \le \epsilon\right\}. \tag{4.20}$$

From (4.19), if $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected, we have

$$\mathcal{H}\big(x(t)\big) \le (1-\xi_{d_0})^{\lfloor \frac{t}{K_0}\rfloor}\mathcal{H}(x^0) \le (1-\xi_{d_0})^{\frac{t}{K_0}-1}\mathcal{H}(x^0) = \frac{1}{1-\xi_{d_0}}e^{-\lambda_0 t}\mathcal{H}(x^0), \tag{4.21}$$

where $\lambda_0 = \frac{1}{K_0}\log\frac{1}{1-\xi_{d_0}}$. Hence, simple computation leads to an upper bound for the ϵ-convergence time as follows:

$$\mathrm{T}_N(\epsilon) \le \frac{\log\big((1-\xi_{d_0})\epsilon\big)^{-1}}{\lambda_0} = O\left(K_0\Big[\log\big(1-\xi_{d_0}\big)^{-1}\Big]^{-1}\right)\log\epsilon^{-1}, \tag{4.22}$$

where by definition $a_N = O(b_N)$ means that $\lim_{N\to\infty}\frac{a_N}{b_N}$ is a nonzero constant.

Remark 4.8. The sufficiency statement of Corollary 4.6 is consistent with the result given in [20]. Compared to [20], our results are based on relaxed conditions, both on the weight functions as boundedness is no longer critical and on the generalized connectivity conditions as the graphs formed by δ-arcs on different time intervals no longer need to share a common center node.

Remark 4.9. Compared to the results for discrete-time consensus dynamics with uniformly jointly strongly connected graphs [26, 23], the convergence time given in (4.22) is relatively conservative. We believe that there exist sharper bounds for the convergence time. However, there might be some fundamental difference for the convergence time between strong connectivity and quasi-strong connectivity as well as between discrete-time dynamics and continuous-time dynamics.

4.4. L_∞-vanishing noise. Consider a set defined by

$$\mathcal{F}_1 \doteq \left\{z\in\mathcal{F} : \lim_{t\to\infty} z(t) = 0\right\},$$

and let $\mathcal{F}_1^0 \subseteq \mathcal{F}_1$ be a subset with $\lim_{t\to\infty} \sup_{z\in\mathcal{F}_1^0} |z(t)| = 0$. Then the following conclusion holds based on our robust consensus analysis.

PROPOSITION 4.10.

(i) *System* (3.1) *achieves global consensus for any $w \in \mathcal{F}_1$ if there exists a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected.*

(ii) *System* (3.1) *achieves global asymptotic consensus with respect to $\mathcal{F}_1^0$ if and only if there exists a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected.*

Proof.

(i) Suppose β and γ are defined as (4.19). Let $w_0 \in \mathcal{F}_1$ be a fixed function. Then, $\forall \varepsilon > 0$, $\exists T(\varepsilon) > 0$ such that $|w_0(t)| < \gamma^{-1}(\varepsilon)$ $\forall t \geq T(\varepsilon)$. Thus, applying Theorem 4.1 to system (3.1) with $t_0 = T(\varepsilon)$, we obtain

$$\mathcal{H}\big(x(t)\big) \leq \beta\Big(\mathcal{H}\big(x(T(\varepsilon))\big), t - T(\varepsilon)\Big) + \varepsilon. \tag{4.23}$$

Since ε can be arbitrarily small, the global consensus follows immediately by taking $t \to \infty$ in (4.23).

(ii) (Sufficiency.) Suppose β and γ are defined as (4.19). Then $\forall \varepsilon > 0$, $\exists \tilde{T}(\varepsilon) > 0$ such that $|w(t)| \leq \gamma^{-1}(\frac{\varepsilon}{2})\, \forall t \geq \tilde{T}(\varepsilon) \forall w \in \mathcal{F}_1^0$. Denote

$$\omega^* = \sup_{t\in[t_0,\tilde{T}]} \big\{ \sup_{z\in\mathcal{F}_1^0} |z(t)| \big\}.$$

There will be two cases:

- When $t_0 < \tilde{T}(\varepsilon)$, one has $\forall t \geq t_0$,

$$\begin{aligned}
\mathcal{H}\big(x(t)\big) &\leq \beta\Big(\mathcal{H}\big(x(\tilde{T}(\varepsilon))\big), t - \tilde{T}(\varepsilon)\Big) + \frac{\varepsilon}{2} \\
&\leq \beta\Big(\beta\big(\mathcal{H}(x^0) + \gamma(\omega^*), \tilde{T}(\varepsilon) - t_0\big), t - \tilde{T}(\varepsilon)\Big) + \frac{\varepsilon}{2} \\
&\leq \beta\Big(\beta\big(\mathcal{H}(x^0) + \gamma(\omega^*), 0\big), t - \tilde{T}(\varepsilon)\Big) + \frac{\varepsilon}{2}.
\end{aligned} \tag{4.24}$$

  Furthermore, $\forall c > 0$, $\exists T_1(c, \tilde{T}(\varepsilon)) > 0$ such that

$$\beta\Big(\beta\big(c + \gamma(\omega^*), 0\big), t - \tilde{T}(\varepsilon)\Big) \leq \frac{\varepsilon}{2}\ \forall t \geq T_1.$$

- When $t_0 \geq \tilde{T}(\varepsilon)$, one has $\forall t \geq t_0$,

$$\mathcal{H}\big(x(t)\big) \leq \beta(\mathcal{H}(x^0), t - t_0) + \frac{\varepsilon}{2}. \tag{4.25}$$

  Then $\forall c > 0$, $\exists T_2(c) > 0$ such that $\beta(\mathcal{H}(x^0), t - t_0) \leq \frac{\varepsilon}{2}$ $\forall t \geq T_2$.

Taking $T = \max\{T_1, T_2\}$, we obtain

$$\mathcal{H}(x^0) \leq c \Rightarrow \mathcal{H}\big(x(t)\big) \leq \varepsilon\ \forall t \geq t_0 + T, \forall w \in \mathcal{F}_1^0.$$

Hence the sufficient part is proved.

(Necessity.) Suppose we cannot find a constant $\delta > 0$ such that the underlying communication graph $\mathcal{G}_{A(t)}$ is uniformly quasi-strongly δ-connected. First $\forall \varepsilon > 0, \forall T_* > 0$, $\exists W > 0$, such that $|w(t)| \leq \frac{\varepsilon}{2T_*}$ $\forall t \geq W$. Then we define $\bar{\mathcal{V}}_1$ and $\bar{\mathcal{V}}_2$ in the same way as the necessity proof of Theorem 4.1 and let the initial condition be

$t_0 = t_*$ with $x_i(t_*) = 0\ \forall i \in \bar{\mathcal{V}}_1$ and $x_i(t_*) = c\ \forall i \notin \bar{\mathcal{V}}_1$ for some $c > 0$. Repeating the same argument as the proof of the necessity of Theorem 4.1 leads to that a global asymptotic consensus cannot be achieved. ∎

Remark 4.11. Proposition 4.10(ii) is consistent with the main result in [18], where asymptotic consensus was discussed for a nonlinear variation of system (3.1) under directed switching graphs with dwell time.

4.5. Proof of Theorem 4.2. The proof follows the same line as the proof of Theorem 4.1. We will bound $\mathcal{H}\big(x(t)\big)$ on time intervals $t \in [sK_0, (s+1)K_0]$ for $s = 0, 1, \dots$. Denote $\eta_s = \int_{sK_0}^{(s+1)K_0} |w(t)|dt$. Then based on Lemma 4.3, for any $t \in [sK_0, (s+1)K_0]$, we have

$$x_i(t) \in [\ell(s\hat{T}) - \eta_s, \hbar(s\hat{T}) + \eta_s],\ i = 1, \dots, N. \tag{4.26}$$

Suppose k_0 is a node as defined in the proof of Theorem 4.1. Provided, without loss of generality, that $x_{k_0}(sK_0) \le \frac{1}{2}\ell(sK_0) + \frac{1}{2}\hbar(sK_0)$ and as that

$$\frac{d}{dt}x_{k_0}(t) \le -\mathcal{Y}_{k_0}(t)\big(x_{k_0}(t) - \hbar(sK_0) - \eta_s\big) + |w(t)|, \quad t \in [sK_0, (s+1)K_0], \tag{4.27}$$

which implies

$$\begin{aligned} x_{k_0}(t) &\le \Big[1 - e^{-\int_{sK_0}^{t} \mathcal{Y}_{k_0}(\tau)d\tau}\Big](\hbar(sK_0) + \eta_s) + e^{-\int_{sK_0}^{t} \mathcal{Y}_{k_0}(\tau)d\tau} x_{k_0}(sK_0) \\ &\quad + \int_{sK_0}^{t} e^{-\int_{sK_0}^{z} \mathcal{Y}_{k_0}(\tau)d\tau} |w(z)|dz \\ &\le \xi_0 \ell(sK_0) + (1 - \xi_0)\hbar(sK_0) + 2\eta_s, \quad t \in [sK_0, (s+1)K_0], \end{aligned} \tag{4.28}$$

where the second inequality follows from the simple fact that $0 < e^{-\int_{sK_0}^{t} \mathcal{Y}_{k_0}(\tau)d\tau} \le 1$.

Therefore, similar to the proof of Theorem 4.1, the analysis can be carried on for the node sets $\mathcal{V}_1, \mathcal{V}_2, \dots$, and we can eventually arrive at

$$\mathcal{H}\big(x((s+1)K_0)\big) \le (1 - \xi_{d_0})\mathcal{H}\big(x(sK_0)\big) + (4d_0 + 1)\eta_s. \tag{4.29}$$

Consequently, for any $n = 0, 1, 2, \dots$, it holds that

$$\begin{aligned} \mathcal{H}\big(x(nK_0)\big) &\le (1 - \xi_{d_0})^n \mathcal{H}(x^0) + (4d_0 + 1)\sum_{j=0}^{n-1}(1 - \xi_{d_0})^{n-1-j}\eta_j \\ &\le (1 - \xi_{d_0})^n \mathcal{H}(x^0) + (4d_0 + 1)\sum_{j=0}^{n-1}\eta_j. \end{aligned} \tag{4.30}$$

Thus, together with the observation that

$$\mathcal{H}\big(x(t)\big) \le \mathcal{H}\big(x(nK_0)\big) + \int_{nK_0}^{t} |w(\tau)|d\tau,\ t \in [nK_0, (n+1)K_0), \tag{4.31}$$

the following integral robust consensus inequality is obtained:

$$\mathcal{H}\big(x(t)\big) \le (1 - \xi_{d_0})^{\lfloor \frac{t}{K_0} \rfloor}\mathcal{H}(x^0) + (4d_0 + 1)\int_0^t |w(\tau)|d\tau. \tag{4.32}$$

This completes the proof.

5. *K*-bidirectional graphs. In this section, we consider bidirectional node interactions. We introduce the following definition.

DEFINITION 5.1. *The underlying communication graph $\mathcal{G}_{A(t)}$ is K-bidirectional if there exists a constant $K \geq 1$ such that*

$$K^{-1}a_{ij}(t) \leq a_{ji}(t) \leq Ka_{ij}(t)$$

$\forall\ i, j \in \mathcal{V}$ *and* $t \geq 0$.

Intuitively, K-bidirectional graphs mean that the information flow between two nodes should be balanced from one node to the other by a bounded proportion K. Note that a 1-bidirectional graph corresponds to conventional bidirectional graphs [15, 25]. For K-bidirectional graphs, we present the following result.

THEOREM 5.2. *Assume that $\mathcal{G}_{A(t)}$ is K-bidirectional. System* (3.1) *achieves global integral robust consensus if and only if there exists a constant $\delta > 0$ such that $\mathcal{G}_{A(t)}$ is infinitely jointly quasi-strongly δ-connected.*

We introduce a partition, $0 = T_0 < T_1 < T_2 < \dots$, for the time-axis.

Let $T_0 = 0$. Then $T_k, k = 1, 2, \dots$, can be defined by induction as

$$T_k = \inf\{t \geq T_{k-1} :\ \text{the } \delta\text{-arcs of } \mathcal{G}_{A(t)} \text{ on time interval } [T_{k-1}, t) \\ \text{form a quasi-strongly connected graph on } \mathcal{V}\}. \tag{5.1}$$

Note that that when there exists a constant $\delta > 0$ such that $\mathcal{G}_{\sigma(t)}$ is infinitely jointly quasi-strongly δ-connected, T_k is finite $\forall\ k = 1, 2, \dots$.

We can thus define

$$J(t) = \max\{k : t > T_k\}.$$

Then $J(t)$ characterizes how many jointly δ-connected graphs can be found during time interval $[0, t)$.

5.1. Proof of Theorem 5.2.

5.1.1. Necessity. Suppose $\mathcal{G}_{A(t)}$ is not infinitely jointly quasi-strongly δ-connected for any $\delta > 0$. Then $\forall\delta > 0$, $\exists t_* > 0$ such that the graph $\mathcal{G}_* = (\mathcal{V}, \mathcal{E}_*)$, with $\mathcal{E}_*$ defined by $\mathcal{E}_* = \big\{(i, j) : \int_{t_*}^{\infty} a_{ji}(t)dt \geq \delta\big\}$ containing all the δ-arcs on time interval $[t_*, \infty)$, is not quasi-strongly connected.

Consequently, there exist two distinct nodes i and j such that $\hat{\mathcal{V}}_1 \cap \hat{\mathcal{V}}_2 = \emptyset$, where $\hat{\mathcal{V}}_1 = \{$nodes from which i is reachable in $\mathcal{G}_*\}$ and $\hat{\mathcal{V}}_2 = \{$nodes from which j is reachable in $\mathcal{G}_*\}$. Let $w_i(t) \equiv 0$ for all $i \in \hat{\mathcal{V}}$. Let the initial time be t_*. Take $x_i(t_*) = 0, i \in \hat{\mathcal{V}}_1$, and $x_i(t_*) = 1, i \notin \hat{\mathcal{V}}_1$, so that $\mathcal{H}(x(t_*)) = 1$.

Similar to the necessity proof of Theorem 4.1, we define

$$l(t) \doteq \max_{i\in\hat{\mathcal{V}}_1}\{x_i(t)\}, \quad L(t) \doteq \min_{i\in\hat{\mathcal{V}}_2}\{x_i(t)\},$$

and then $H = L(t) - l(t)$. Lemma 4.3 ensures that $\forall\ t \in [t_*, \infty)$, we have $\ell(t) \geq 0;\ \ \hbar(t) \leq 1$.

Denoting $\bar{\Theta}(t) = \sum_{(j,i)\notin\mathcal{E}_*} a_{ij}(t)$, a similar analysis as the proof of Lemma 4.3 gives us

$$D^+H(t) \geq -\bar{\Theta}(t)\big(H(t) + 1\big), \quad t \in [t_*, \infty). \tag{5.2}$$

This implies

$$H(t) \geq 2e^{-\int_{t_*}^{t} \bar{\Theta}(s)ds} - 1 \geq 2e^{-E_0\delta} - 1, \quad t \in [t_*, \infty). \tag{5.3}$$

Noticing that δ can be arbitrarily small in (5.3), we see that $\mathcal{H}(x(t))$ cannot be bounded above by $\beta(1,t)$ for any fixed $\mathcal{KL}$-function β. Therefore, global integral robust consensus cannot be achieved. The necessity statement of Theorem 5.2 holds.

5.1.2. Sufficiency. The proof relies on the time-axis partition defined previously. Suppose there exists a constant $\delta > 0$ such that $\mathcal{G}_{A(t)}$ is infinitely jointly quasi-strongly δ-connected. Let $0 = T_0 < T_1 < T_2 < \cdots$ be the sequence of time instants given by (5.1). Denote $\varpi_0 = \int_{T_0}^{T_{N-1}} |w(t)|dt$. Then based on Lemma 4.3, we have

$$\hbar(t) \leq \hbar(T_0) + \varpi_0; \quad \ell(t) \geq \ell(T_0) - \varpi_0 \tag{5.4}$$

$\forall\ T_0 \leq t \leq T_{N-1}$.

We divide the rest of the proof into four steps.

Step 1. We first define an instant $\bar{t}_1$ by

$$\bar{t}_1 \doteq \inf\left\{t \geq T_0 : \exists i_0, i_1 \in \mathcal{V} \text{ such that } \int_{T_0}^{t} a_{i_0 i_1}(s)ds \geq \delta\right\}.$$

Then we have $\bar{t}_1 \leq T_1$ according to the definition of T_1. Without loss of generality, we assume that

$$x_{i_0}(T_0) \leq \frac{1}{2}\ell(T_0) + \frac{1}{2}\hbar(T_0). \tag{5.5}$$

Noticing that $\int_{T_0}^{t} \mathcal{Y}_{i_0}(\tau)d\tau \leq (N-1)\delta$, $t \in [T_0, \bar{t}_1]$, the inequality

$$\frac{d}{dt}x_{i_0}(t) \leq -\mathcal{Y}_{i_0}(t)\big(x_{i_0}(t) - \hbar(T_0) - \varpi_0\big) + |w(t)|, \quad t \in [T_0, \bar{t}_1], \tag{5.6}$$

implies

(5.7)
$$\begin{aligned} x_{i_0}(t) &\leq \left[1 - e^{-\int_{T_0}^{t} \mathcal{Y}_{i_0}(\tau)d\tau}\right](\hbar(T_0) + \varpi_0) + e^{-\int_{T_0}^{t} \mathcal{Y}_{i_0}(\tau)d\tau} x_{i_0}(T_0) + \int_{T_0}^{t} |w(t)|dt \\ &\leq \frac{1}{2}e^{-\int_{T_0}^{t} \mathcal{Y}_{i_0}(\tau)d\tau}\ell(T_0) + \left[1 - \frac{1}{2}e^{-\int_{T_0}^{t} \mathcal{Y}_{i_0}(\tau)d\tau}\right]\hbar(T_0) + 2\varpi_0 \\ &\leq m_0\ell(T_0) + (1-m_0)\hbar(T_0) + 2\varpi_0 \end{aligned}$$

$\forall\ t \in [T_0, \bar{t}_1]$, where $m_0 = \eta/2$ with $\eta \doteq e^{-(N-1)\delta}$.

Step 2. We establish a bound for $x_{i_1}(\bar{t}_1)$ in this step. According to the definition of the K-bidirectional graph, we have $\int_{T_0}^{\bar{t}_1} a_{i_1 i_0}(t)dt \geq \delta/K$. Similar to (4.9), for $t \in [T_0, \bar{t}_1)$, we have

$$\begin{aligned} \frac{d}{dt}x_{i_1}(t) \leq{}& \hat{\mathcal{Y}}_{i_1}(t)\big(\hbar(T_0) + \varpi_0 - x_{i_1}(t)\big) \\ &+ a_{i_1 i_0}(t)\big(m_0\ell(T_0) + (1-m_0)\hbar(T_0) + 2\varpi_0 - x_{i_1}(t)\big) + w_{i_1}(t), \end{aligned} \tag{5.8}$$

which gives

$$x_{i_1}(\bar{t}_1) \le \left(m_0 \int_{T_0}^{\bar{t}_1} e^{-\int_t^{\bar{t}_1} \mathcal{Y}_{i_1}(\tau)d\tau} a_{i_1 i_0}(t)dt \right) \ell(T_0) + \left(1 - m_0 \int_{T_0}^{\bar{t}_1} e^{-\int_t^{\bar{t}_1} \mathcal{Y}_{i_1}(\tau)d\tau} a_{i_1 i_0}(t)dt \right) \hbar(T_0) + 4\varpi_0. \tag{5.9}$$

Observing that

$$\begin{aligned} \int_{T_0}^{\bar{t}_1} e^{-\int_t^{\bar{t}_1} \mathcal{Y}_{i_1}(\tau)d\tau} a_{i_1 i_0}(t)dt &\ge e^{-\int_{T_0}^{\bar{t}_1} \hat{\mathcal{Y}}_{i_1}(\tau)d\tau} \int_{T_0}^{\bar{t}_1} e^{-\int_t^{\bar{t}_1} a_{i_1 i_0}(\tau)d\tau} a_{i_1 i_0}(t)dt \\ &\ge e^{-(N-2)\delta}\big(1 - e^{-\int_{T_0}^{\bar{t}_1} a_{i_1 i_0}(t)dt}\big) \\ &\ge e^{-(N-2)\delta}\big(1 - e^{-\delta/K}\big), \end{aligned} \tag{5.10}$$

we conclude from (5.7) that

$$x_{i_1}(\bar{t}_1) \le m_1 \ell(T_0) + (1 - m_1)\hbar(T_0) + 4\varpi_0, \tag{5.11}$$

where $m_1 = \eta\vartheta/2$ with $\vartheta = e^{-(N-2)\delta}(1 - e^{-\delta/K})$. Noticing $m_1 < m_0$, the right-hand side of inequality (5.11) is also an upper bound for $x_{i_0}(\bar{t}_1)$.

Step 3. Continuing, we define $\bar{t}_2$ by

$$\bar{t}_2 \doteq \inf\Big\{ t \ge \bar{t}_1 : \exists i_2 \in \mathcal{V} \text{ such that } \max\big\{ \textstyle\int_{\bar{t}_1}^t a_{i_0 i_2}(s)ds, \int_{\bar{t}_1}^t a_{i_1 i_2}(s)ds \big\} \ge \delta \Big\}.$$

Obviously $\bar{t}_2 \le T_2$ according to the definition of T_2. Now we define

$$P(t) \doteq \max\{i_0(t), i_1(t)\}; \quad \mathcal{Y}_{i_0,i_1}(t) \doteq \sum_{j=1}^N \big(a_{i_0 j}(t) + a_{i_1 j}(t)\big).$$

Then the inequality

$$D^+ P(t) \le -\mathcal{Y}_{i_0,i_1}(t)\big(P(t) - \hbar(T_0) - \varpi_0\big) + |w(t)|, \quad t \in [\bar{t}_1, \bar{t}_2], \tag{5.12}$$

leads to

$$P(t) \le \eta^2 m_1 \ell(T_0) + (1 - \eta^2 m_1)\hbar(T_0) + 5\varpi_0, \quad t \in [\bar{t}_1, \bar{t}_2]. \tag{5.13}$$

Next, similar to (5.8), an upper bound for $x_{i_2}(\bar{t}_2)$ can be presented as

$$x_{i_2}(\bar{t}_2) \le m_2 \ell(T_0) + (1 - m_2)\hbar(T_0) + 6\varpi_0, \tag{5.14}$$

where $m_2 = \eta^3\vartheta^2/2$.

Step 4. Proceeding this analysis, $\bar{t}_3, \dots, \bar{t}_{N-1}$ can be found with $\bar{t}_{N-1} \le T_{N-1}$, respectively, and we eventually have

$$x_i(\bar{t}_{N-1}) \le \frac{\eta^{N(N-1)/2}\vartheta^{N-1}}{2} \ell(T_0) + \left(1 - \frac{\eta^{N(N-1)/2}\vartheta^{N-1}}{2}\right) \hbar(T_0) + 2N\varpi_0, \tag{5.15}$$

which implies

$$\mathcal{H}\big(x(T_{N-1})\big) \le \left(1 - \frac{\eta^{N(N-1)/2}\vartheta^{N-1}}{2}\right) \mathcal{H}\big(x(T_0)\big) + 2(N+1)\varpi_0. \tag{5.16}$$

Since (5.16) holds independent with the initial condition, we can further conclude that

$$\mathcal{H}\big(x(T_{n(N-1)})\big) \le \left(1-\frac{\eta^{N(N-1)/2}\vartheta^{N-1}}{2}\right)^{n}\mathcal{H}(x^0) + 2(N+1)\sum_{j=0}^{n-1}\left(1-\frac{\eta^{N(N-1)/2}\vartheta^{N-1}}{2}\right)^{n-1-j}\varpi_j \tag{5.17}$$

$\forall\ n=0,1,2,\dots,$ where $\varpi_j \doteq \int_{T_{j(N-1)}}^{T_{(j+1)(N-1)}} |w(t)|dt$.

Therefore, the desired integral robust consensus inequality can be obtained by

$$\mathcal{H}\big(x(t)\big) \le \left(1-\frac{\eta^{N(N-1)/2}\vartheta^{N-1}}{2}\right)^{\lfloor \frac{J(t)}{N-1}\rfloor}\mathcal{H}(x^0) + 2(N+1)\int_0^t |w(\tau)|d\tau. \tag{5.18}$$

The proof is completed.

5.2. Convergence time. Suppose $w(t)\equiv 0$. Then (5.18) leads to

$$\mathcal{H}\big(x(t)\big) \le (1-c_*)^{\lfloor \frac{J(t)}{N-1}\rfloor}\mathcal{H}(x^0) \le \big(1-c_*\big)^{-1} e^{J(t)\frac{\log(1-c_*)}{N-1}}\mathcal{H}(x^0), \tag{5.19}$$

where $c_* \doteq \frac{\eta^{N(N-1)/2}\vartheta^{N-1}}{2}$.

From (5.19), when the communication graph $\mathcal{G}_{A(t)}$ is K-bidirectional and infinitely jointly quasi-strongly δ-connected, system (3.1) with bidirectional communications and in the absence of noise will reach a consensus exponentially with respect to $J(t)$, i.e., the number of δ-connected graphs. Furthermore, an upper bound for the ϵ-convergence time $\mathrm{T}_N(\epsilon)$ is obtained by

$$\begin{aligned}\mathrm{T}_N(\epsilon) &\le \inf\left\{t : J(t) \le \frac{N-1}{\log(1-c_*)}\log\big(\epsilon(1-\eta_*)\big)\right\}\\ &\le \inf J^{-1}\left(\left\lceil O\Big((N-1)\Big[\log(1-c_*)^{-1}\Big]^{-1}\Big)\log\epsilon^{-1}\right\rceil\right),\end{aligned} \tag{5.20}$$

where $J^{-1}(z)=\{t : J(t)=z\}$ and $\lceil z\rceil$ denotes the smallest integer no smaller than z.

5.3. L_1-vanishing noise. Consider the following set:

$$\mathcal{F}_2 \doteq \left\{z\in\mathcal{F} :\ \int_0^\infty |z(t)|dt<\infty\right\}.$$

Let $\mathcal{F}_2^0 \subseteq \mathcal{F}_2$ be a subset of $\mathcal{F}_2$ with $\int_0^\infty \sup_{z\in\mathcal{F}_2^0}|z(t)|dt<\infty$. The following conclusion holds.

PROPOSITION 5.3.

(i) *System* (3.1) *achieves global asymptotic consensus with respect to* $\mathcal{F}_2^0$ *if and only if there exists a constant* $\delta>0$ *such that the underlying communication graph* $\mathcal{G}_{A(t)}$ *is uniformly quasi-strongly* δ*-connected.*

(ii) *Assume that* $\mathcal{G}_{A(t)}$ *is* K*-bidirectional. Then system* (3.1) *achieves a global consensus* $\forall\ w\in\mathcal{F}_2$ *if and only if there exists a constant* $\delta>0$ *such that* $\mathcal{G}_{A(t)}$ *is infinitely quasi-strongly* δ*-connected.*

This proposition follows straightforwardly from the integral robust consensus property shown in Theorems 4.2 and 5.2. The proof is therefore omitted.

6. Conclusions. This paper focused on the robustness of continuous-time consensus algorithms. We provided a precise answer to how much connectivity is required for the network to agree asymptotically despite the input noise. The idea of ISS and iISS inspired us to our definitions of robust consensus and integral robust consensus. We showed that uniformly joint connectivity is critical with respect to robust consensus for general directed graphs; infinitely joint connectivity is critical with respect to integral robust consensus for K-bidirectional graphs. Upper bounds for the ϵ-convergence time were obtained as a result from the robustness analysis.

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