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Agreeing under Randomized Network Dynamics

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***Abstract*— In this paper, we study randomized consensus processing over general random graphs. At time step k, each node will follow the standard consensus algorithm, or stick to current state by a simple Bernoulli trial with success probability p_k. Connectivity-independent and arc-independent graphs are defined, respectively, to capture the fundamental independence of random graph processes with respect to a consensus convergence. Sufficient and/or necessary conditions are presented on the success probability sequence for the network to reach a global a.s. consensus under various conditions of the communication graphs. Particularly, for arc-independent graphs with simple self-confidence condition, we show that $\sum_k p_k = \infty$ is a sharp threshold corresponding to a consensus $0-1$ law, i.e., the consensus probability is 0 for almost all initial conditions if $\sum_k p_k$ converges, and jumps to 1 for all initial conditions if $\sum_k p_k$ diverges.**

Keywords: Consensus algorithms, Random graphs, Dynamics Randomization, Threshold

I. INTRODUCTION

In recent years, there has been considerable research effort on distributed algorithms for exchanging information, for estimating and for computing over a network of nodes, due to a variety of potential applications in sensor, peer-to-peer and wireless networks. Targeting design of simple decentralized algorithms for computation or estimation, where each node exchanges information only in a neighboring view, distributed averaging serves as a primitive toward more sophisticated information processing algorithms.

Deterministic consensus over time-invariant or time-varying graphs has been extensively studied, in which the problems were typically devoted on sufficient and/or connectivity conditions of the underlying communication graph for convergence, convergence rate and optimal convergence [16], [17], [20], [21], [18], [24], [15], [14], [23], [34]. On the other hand, the network where consensus algorithms are carried out may be randomized. In [25], the authors studied the linear consensus dynamics and almost sure convergence was shown when the communication graph was independent, identically distributed (i.i.d.) as an Erdös–Rényi random graph model. Then more general models were studied in [26], [27], [28], [29], [37], [32], [33], [35], [31].

In this paper, we study consensus algorithms with randomized decision-making. At time slot k, each agent independently decides to follow the averaging algorithm with probability p_k, and to stick to its current state with probability $1-p_k$. This randomized decision-making protocol may come from the random node failure in wireless networks [28], or come from nodes' preference in social networks [40]. The communication graph is assumed to be a general random digraph process independent with the agents' decision making process.

The rest of the paper is organized as follows. In Section II we recall some notations in graph theory. Section III presents the randomized algorithm and the main results on the impossibility and possibility for the algorithm to converge. Then in Section IV, the proof for the impossibility conclusions is given. In Sections V and VI, the convergence analysis for connectivity-independent and arc-independent graphs are proposed, respectively. Finally Section VII gives some concluding remarks.

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II. PRELIMINARIES

In this section, we introduce some notations on directed graphs. A (simple) directed graph, i.e., digraph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of nodes and an arc set $\mathcal{E}$, where each element $e = (i, j) \in \mathcal{E}$ is an ordered pair of two different nodes in $\mathcal{V}$ from node i to node j [4]. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_n v_n$ of nodes v_i and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, n$, the sequence is called a (directed) *path* with *length* n, and if $v_0 = v_n$ a (directed) *cycle*. A path with no repeated nodes is called a *simple path*. A digraph without cycles is said to be *acyclic*. A digraph $\mathcal{G}$ is called to be *bidirectional* if $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$.

A simple path from i to j is denoted as $i \to j$, and the length of $i \to j$ is denoted as $|i \to j|$. If there exists a path from node i to node j, then node j is said to be reachable from node i. Each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. $\mathcal{G}$ is said to be *strongly connected* if it contains path $i \to j$ and $j \to i$ for every pair of nodes i and j. $\mathcal{G}$ is said to be *quasi-strongly connected* if $\mathcal{G}$ has a center [6].

Additionally, if $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ have the same node set, the union of the two digraphs is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$.

III. PROBLEM DEFINITION AND MAIN RESULTS

A. Network Model

Consider a network with node set $\mathcal{V} = \{1, 2, \dots, n\}$. A (simple) directed graph (digraph), $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a node set $\mathcal{V}$ and an arc set $\mathcal{E}$, where each element $e = (i, j) \in \mathcal{E}$ is an ordered pair of two different nodes in $\mathcal{V}$ from node i to node j [4]. Then there are as many as $2^{n(n-1)}$ different digraphs with node set $\mathcal{V}$. We label these graphs from 1 to $2^{n(n-1)}$ by an arbitrary order. In the following, we will identify an integer in $[1, 2^{n(n-1)}]$ with the corresponding graph in this order. Denote $\Omega = \{1, \dots, 2^{n(n-1)}\}$ as the graph set.

The communication graph of the network over time, is model as a sequence of random variables, $\{\mathcal{G}_k(\omega) = (\mathcal{V}, \mathcal{E}_k(\omega))\}_{k=0}^{\infty}$, which take value in Ω. Where there is no possible confusion, we write $\mathcal{G}_k(\omega)$ as $\mathcal{G}_k$.

We call node j a *neighbor* of i if there is an arc from j to i in graph $\mathcal{G}$, and each node is supposed to be a neighbor of itself. Denote the random set $\mathcal{N}_i(k) = \{j \in \mathcal{V} : \ (j, i) \in \mathcal{E}_k\} \cup \{i\}$ as the neighbor set of node i at time k. The agent dynamics is described as follows:

$$x_i(k+1) = \begin{cases} \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) x_j(k), & \text{with prob. } p_k \\ x_i(k), & \text{with prob. } 1 - p_k \end{cases} \tag{1}$$

where $0 \le p_k < 1$ and $a_{ij}(k)$ denotes the weight of arc (j, i). For $a_{ij}(k)$, we assume the following weights rule as our standing assumptions.

A1. For all i and k, we have $\sum\limits_{j \in \mathcal{N}_i(k)} a_{ij}(k) = 1$.

A2. There exists a constant $\eta > 0$ such that $\eta \le a_{ij}(k)$ for all i, j and k.

Denote

$$H(k) \doteq \max_{i=1,\dots,n} x_i(k), \quad h(k) \doteq \min_{i=1,\dots,n} x_i(k)$$

as the maximum and minimum states among all nodes, respectively, and define $\mathcal{H}(k) \doteq H(k) - h(k)$ as the consensus metric. Our interest is in the consensus convergence of the randomized consensus algorithm and in the (absolute) time it takes for the network to reach a consensus [31].

Definition 3.1: A global a.s. *consensus* of (1) is achieved if

$$\mathbf{P}(\lim_{k \to \infty} \mathcal{H}(k) = 0) = 1 \tag{2}$$

for any initial condition $x(0) = (x_1(0) \dots x_n(0))^T \in \mathbb{R}^n$. Moreover, for any $0 \le \epsilon < 1$, the ϵ-computation time is denoted by $T_{com}(\epsilon)$, and is defined as

$$T_{com}(\epsilon) \doteq \sup_{x(0)} \inf \Big\{ k : \ \ \mathbf{P}\Big(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \ge \epsilon \Big) \le \epsilon \Big\}. \tag{3}$$

B. Main Results

We first present an impossibility conclusion.

Theorem 3.1: If $\sum_{k=0}^{\infty} p_k < \infty$, then global a.s. consensus cannot be achieved for Algorithm (1). Moreover, a general lower bound for $T_{com}(\epsilon)$ can be given by

$$T_{com}(\epsilon) \ge \sup \Big\{ k : \ \ \sum_{i=0}^{k-1} \log(1 - p_i)^{-1} \le \frac{\log \epsilon^{-1}}{n} \Big\}.$$

Note that, Theorem 3.1 holds for all possible graph processes. Plus a simple self-confidence assumption, this impossibility claim can be improved as follows.

Theorem 3.2: Assume that $a_{ii}(k) \ge \gamma_0$ for all i and k, where $\gamma_0 > 1/2$ is a constant. If $\sum_{k=0}^{\infty} p_k < \infty$, then for almost all initial conditions, Algorithm (1) achieves consensus with probability 0.

In order to establish possibility answers to a global consensus, we need independence and connectivity of the graph processes.

Definition 3.2: Let $\{\mathcal{G}_k\}_0^{\infty}$ be a random graph process. Then $\{\mathcal{G}_k\}_0^{\infty}$ is called to be

(i) *connectivity-independent* if events $\mathcal{C}_k \doteq \big\{\mathcal{G}_k$ is quasi-strongly connected$\big\}$, $k = 0, 1, \dots$, are independent.

(ii) *arc-independent* if there exists a (nonempty) deterministic graph $\mathcal{G}^* = (\mathcal{V}, \mathcal{E}^*)$ such that events $\mathcal{A}_{k,\tau} \doteq \big\{(i_\tau, j_\tau) \in \mathcal{G}_k\big\}$, $(i_\tau, j_\tau) \in \mathcal{E}^*, k = 0, 1, \dots$, are independent. In this case $\mathcal{G}^*$ is called a basic graph of this random graph process.

Note that, connectivity-independence and arc-independence are actually different levels of *independence* for the sequence of random graphs $\mathcal{G}_0, \mathcal{G}_1, \dots$. This sequence is not necessarily independent to be either connectivity-independent or arc-independent. For instance, $\mathcal{G}_0, \mathcal{G}_1, \dots$ can be given by a Markov chain which is clearly not independent, but it can be connectivity-independence or arc-independence as long as the transition matrix is properly chosen.

The sufficiency results for consensus convergence are stated in the following, respectively, for connectivity-independent and arc-independent graphs.

Theorem 3.3: Suppose $\{\mathcal{G}_k\}_0^{\infty}$ is connectivity-independent and there exists a constant $0 < q < 1$ such that $\mathbf{P}\big(\mathcal{G}_k$ is quasi-strongly connected$\big) \ge q$ for all k. Assume in addition that $p_{k+1} \le p_k$. Then Algorithm (1) achieves a global a.s. consensus if $\sum_{s=0}^{\infty} p_k^{n-1} = \infty$. Moreover, an upper bound for $T_{com}(\epsilon)$ can be given by

$$T_{com}(\epsilon) \le \inf \Big\{ M : \ \sum_{i=1}^{M} \log \Big(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot p_{i(n-1)^2}^{n-1} \Big)^{-1} \ge \log \epsilon^{-2} \Big\} \times (n-1)^2. \tag{4}$$

Theorem 3.4: Suppose $\{\mathcal{G}_k\}_0^{\infty}$ is arc-independent with a quasi-strongly connected basic graph, and there exists a constant $0 < \theta_0 < 1$ such that $\mathbf{P}\big((i, j) \in \mathcal{E}_k\big) \ge \theta_0$ for all k and $(i, j) \in \mathcal{E}^*$. Then Algorithm (1) achieves a global a.s. consensus if and only if $\sum_{k=0}^{\infty} p_k = \infty$. In this case, we have

$$T_{com}(\epsilon) \le \inf \Big\{ k : \sum_{i=0}^{k-1} \big(1 - (1 - p_i)^n\big) \ge \frac{(n-1)}{\log A} \log \big(A\epsilon^2/n\big) \Big\} \tag{5}$$

where $A = 1 - \big(\frac{\eta\theta_0}{n}\big)^{(n-1)|\mathcal{E}^*|}$ with $|\mathcal{E}^*|$ as the number of elements in $\mathcal{E}^*$.

Connectivity is a global property for a graph, and it indeed does not rely on any specific arc. We believe that the convergence condition given in Theorem 3.3 is quite tight since the probability that all the links function in Algorithm (1) at time k is p_k^n, and connectivity can be lost easily by losing any single link. Moreover, the convergence conditions given in Theorems 3.3 and 3.4 are consistent with the widely-used *decreasing gain* condition in the study of stochastic approximations on various adaptive algorithms [3].

Combing Theorems 3.2 and 3.4, we see that $\sum_{k=0}^{\infty} p_k = \infty$ is a sharp *threshold* for Algorithm (1) to reach consensus with arc-independet graphs and self-confidence assumption (see Fig. 1). In other words, a similar $0-1$ law is established for consensus dynamics on random graphs as the classical random graph theory [5].

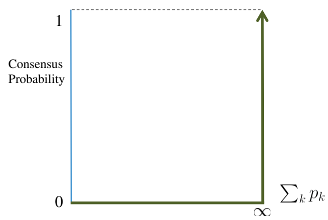

Fig. 1. Consensus appears *suddenly* for arc-independent graphs with $a_{ii}(k) \geq \gamma_0$.

IV. Impossibility Analysis

This section focuses on the proof of Theorems 3.1 and 3.2. The following lemma is well-known.

Lemma 4.1: Suppose $0 \leq b_k < 1$ for all k. Then $\sum_{k=0}^{\infty} b_k = \infty$ if and only if $\prod_{k=0}^{\infty}(1-b_k) = 0$.

A. Proof of Theorem 3.1

From algorithm (1), if $\sum_{k=0}^{\infty} b_k < \infty$, we have

$$\mathbf{P}\Big(x_i(k+1) = x_i(k),\ k = 0, 1, \dots\Big) \geq \prod_{k=0}^{\infty}(1-p_k) \doteq r_0,$$

where $0 < r_0 < 1$ is a well-defined constant according to Lemma 4.1. Then it is straightforward to see that the impossibility claim of Theorem 3.1 holds.

Next, we define a scalar random variable $\varpi(k)$, by that $\varpi(k) = \mathcal{H}(k+1)/\mathcal{H}(k)$ when $\mathcal{H}(k) > 0$, and $\varpi(k) = 1$ when $\mathcal{H}(k) = 0$. Obviously, $h(k)$ is non-decreasing, and $H(k)$ is non-increasing. Thus, it always holds that $\varpi(k) \leq 1$. We see from the considered algorithm that

$$\mathbf{P}\Big(\varpi(k) = 1\Big) \geq (1-p_k)^n. \tag{6}$$

As a result, we obtain

$$\begin{aligned}\mathbf{P}\Big(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon\Big) &\geq \mathbf{P}\Big(\varpi(j) = 1,\ j = 0, \dots, k-1\Big) \\ &\geq \prod_{j=0}^{k-1}(1-p_j)^n, \end{aligned}\tag{7}$$

and then the lower bound for the ϵ-computation given in Theorem 3.1 can be easily obtained. The proof of Theorem 3.1 is completed.

B. Proof of Theorem 3.2

In order to prove Theorem 3.2, we need the following lemma.

Lemma 4.2: Assume that $a_{ii}(k) \geq \gamma_0 > 1/2$ for all i and k. Then

$$\mathcal{H}(k+1) \geq \big(2\gamma_0 - 1\big)\mathcal{H}(k)$$

for all k.

Proof. Suppose $x_m(k) = h(k)$ for some $m \in \mathcal{V}$. Then we have

$$\begin{aligned}\sum_{j\in\mathcal{N}_m(k)} a_{mj}(k)x_j(k) &\leq a_{mm}(k)h(k) + \big(1-a_{mm}(k)\big)H(k) \\ &\leq \gamma_0 h(k) + \big(1-\gamma_0\big)H(k),\end{aligned}$$

which implies

$$h(k+1) \leq \gamma_0 h(k) + \big(1-\gamma_0\big)H(k). \tag{8}$$

A symmetric argument leads to

$$H(k+1) \geq \big(1-\gamma_0\big)h(k) + \gamma_0 H(k). \tag{9}$$

Based on (8) and (9), we obtain

$$\begin{aligned}\mathcal{H}(k+1) &= H(k+1) - h(k+1) \\ &\geq \big(1-\gamma_0\big)h(k) + \gamma_0 H(k) \\ &\quad - \big[\gamma_0 h(k) + \big(1-\gamma_0\big)H(k)\big] \\ &\geq \big(2\gamma_0 - 1\big)\mathcal{H}(k).\end{aligned}\tag{10}$$

The desired conclusion follows. □

Noting the fact that Lemma 4.2 holds for all possible communication graphs, we see that

$$\mathbf{P}\Big(2\gamma_0 - 1 \leq \varpi(k) \leq 1\Big) = 1 \tag{11}$$

and

$$\begin{aligned}\mathbf{P}\Big(\varpi(k) < 1\Big) &\leq \mathbf{P}\Big(\text{at least one node takes averaging}\Big) \\ &= 1 - (1-p_k)^n\end{aligned}\tag{12}$$

where $\varpi(k)$ follows the definition in the proof of Theorem 3.1.

Next, by Lemma 4.1, it is not hard to find

$$\begin{aligned}\sum_{k=0}^{\infty} p_k < \infty &\iff \prod_{k=0}^{\infty}(1-p_k) > 0 \\ &\iff \prod_{k=0}^{\infty}(1-p_k)^n > 0 \\ &\iff \sum_{k=0}^{\infty}\Big(1-(1-p_k)^n\Big) < \infty,\end{aligned}\tag{13}$$

where the last equivalence is obtained by taking $b_k = 1 - (1-p_k)^n$ in Lemma 4.1.

Therefore, if $\sum_{k=0}^{\infty} p_k < \infty$, applying the First Borel-Cantelli Lemma [2] on (12), it follows immediately that

$$\mathbf{P}\Big(\varpi(k) < 1 \text{ for infinitely many } k\Big) = 0. \tag{14}$$

Furthermore, based on (11), we eventually have

$$\begin{aligned}&\mathbf{P}\Big(\lim_{k\to\infty}\mathcal{H}(k) = 0 \text{ for } \mathcal{H}(0) > 0\Big) \\ &\leq \mathbf{P}\Big(\varpi(k) < 1 \text{ for infinitely many } k\Big) \\ &= 0.\end{aligned}\tag{15}$$

Since $\{x(0) : \mathcal{H}(0) = 0\}$ has zero measure in $\mathbb{R}^n$, Theorem 3.2 follows and this ends the proof. □

V. CONNECTIVITY-INDEPENDENT GRAPHS

In this section, we present the convergence analysis for connectivity-independent graphs. We are going to study some more general cases relying on the joint graphs only.

Joint connectivity has been widely studied in the literature on consensus seeking [16], [17]. The joint graph of $\mathcal{G}_k$ on time interval $[k_1, k_2]$ for $0 \le k_1 \le k_2 \le +\infty$, is denoted by

$$\mathcal{G}_{[k_1,k_2]} = \Big(\mathcal{V}, \bigcup_{k\in[k_1,k_2]} \mathcal{E}_k\Big).$$

Then we introduce the following connectivity definition.

Definition 5.1: $\{\mathcal{G}_k\}_0^\infty$ is said to be

(i) *stochastically uniformly quasi-strongly connected*, if there exist two constants $B \ge 1$ and $0 < q < 1$ such that $\{\mathcal{G}_{[mB,(m+1)B-1]}\}_{m=0}^\infty$ is connectivity-independent and for all m, we have

$$\mathbf{P}\Big(\mathcal{G}_{[mB,(m+1)B-1]} \text{ is quasi-strongly connected}\Big) \ge q.$$

(ii) *stochastically infinitely quasi-strongly connected*, if there exist a sequence $0 = c_0 < \cdots < c_m < \dots$ and a constant $0 < q < 1$ such that $\{\mathcal{G}_{[c_m,c_{m+1})}\}_{m=0}^\infty$ is connectivity-independent and for all m, we have

$$\mathbf{P}\Big(\mathcal{G}_{[c_m,c_{m+1})} \text{ is quasi-strongly connected}\Big) \ge q.$$

Roughly speaking, uniform (or infinite) joint-connections are defined on the union graphs in bounded (or boundless) time intervals.

A. Uniformly Joint Graphs

The following result is for consensus seeking on stochastically uniformly quasi-strongly connected graphs.

Proposition 5.1: Suppose $\{\mathcal{G}_k\}_0^\infty$ is stochastically uniformly quasi-strongly connected. Algorithm (1) achieves a global consensus almost surely if $\sum_{s=0}^\infty \bar{p}_s = \infty$, where

$$\bar{p}_s = \inf_{\alpha_1,\dots,\alpha_{n-1}} \{\prod_{l=1}^{n-1} p_{\alpha_l} | \\ s(n-1)^2B \le \alpha_1 < \cdots < \alpha_{n-1} < (s+1)(n-1)^2B\}.$$

The proof is based on the following lemma.

Lemma 5.1: Assume that $\mathcal{G}_k$ is stochastically uniformly quasi-strongly connected. Then for any $s = 0,1,\dots$, the probability that there exists a node $i_0 \in \mathcal{V}$ such that i_0 is a center for at least $n-1$ graphs within $\mathcal{G}_{[\tau B,(\tau+1)B-1]}, \tau = s(n-1)^2,\dots,(s+1)(n-1)^2-1$ is no less than $q^{(n-1)^2}$.

Proof. Since $\mathcal{G}_k$ is stochastically uniformly quasi-strongly connected, the probability that each graph $\mathcal{G}_{[\tau B,(\tau+1)B-1]}$ for $\tau = s(n-1)^2,\dots,(s+1)(n-1)^2-1$, has a center is no less than $q^{(n-1)^2}$. We count a time whenever there is a center node in $\mathcal{G}_{[\tau B,(\tau+1)B-1]}, \tau = s(n-1)^2,\dots,(s+1)(n-1)^2-1$. These $(n-1)^2$ graphs will lead to at least $(n-1)^2$ counts. However, the total number of the nodes is n. Thus, at least one node is counted more than $(n-2)$ times. The conclusion follows. □.

The main result on randomized consensus for SUQSC graphs is stated as follows.

Proof. Denote

$$h(k) = \min_{i=1,\dots,n} x_i(k); \quad H(k) = \max_{i=1,\dots,n} x_i(k).$$

Obviously, we have $h(k)$ is non-decreasing, while $H(k)$ is non-increasing. Then a global almost sure consensus is achieved for (1) if and only if $\mathbf{P}\{\lim_{k\to+\infty} S(k) = 0\} = 1$, where $S(k) = H(k)-h(k)$. Denote $k_s = s(n-1)^2B$ for $s = 0,1,\dots$. Let i_0 be the center node defined in Lemma 5.1 such that the probability that i_0 is a center of $\mathcal{G}_{[\tau_j B,(\tau_j+1)B-1]}$ for $j = 1,\dots,n-1$ with $k_s \le \tau_j B \le k_{s+1}-1$ is no less than $q^{(n-1)^2}$.

Assume that $x_{i_0}(k_s) \le \frac{1}{2}h(k_s) + \frac{1}{2}H(k_s)$. With the weights rule, we see that

$$\sum_{j\in\mathcal{N}_{i_0}(k_s)} a_{i_0 j}(k_s)x_j(k_s) \le \frac{\eta}{2}h(k_s) + (1-\frac{\eta}{2})H(k_s). \quad (16)$$

Thus, with $\eta < 1$, we obtain

$$x_{i_0}(k_s+1) \le \frac{\eta}{2}h(k_s) + (1-\frac{\eta}{2})H(k_s). \quad (17)$$

Continuing the same estimations, we know that for any $\varrho = 0,1,\dots$,

$$x_{i_0}(k_s+\varrho) \le \frac{\eta^\varrho}{2}h(k_s) + (1-\frac{\eta^\varrho}{2})H(k_s). \quad (18)$$

When i_0 is a center of $\mathcal{G}_{[\tau_1 B,(\tau_1+1)B-1]}$, there will be a node $i_1 \in \mathcal{V}$ different with i_0 and a time instance $\hat{k}_1 \in [\tau_1 B, (\tau_1+1)B-1]$ such that $(i_0,i_1) \in \mathcal{E}_{\hat{k}_1}$. Denote $\hat{k}_1 = k_s + \varsigma$ with $\tau_1 B - k_s \le \varsigma \le \tau_1 B - k_s + B - 1$. If i_1 takes the average option at time step $\hat{k}_1+1$, with (18), we obtain

$$\mathbf{P}\{x_{i_l}(k_s+\varrho) \le \frac{\eta^\varrho}{2}h(k_s) + (1-\frac{\eta^\varrho}{2})H(k_s), \\ l = 0,1; \varrho = (\tau_1+1)B - k_s,\dots\} \ge p_{\hat{k}_1} q^{(n-1)^2}.$$

We proceed the analysis on time interval $[\tau_2 B, (\tau_2+1)B-1]$. When i_0 is a center of $\mathcal{G}_{[\tau_2 B,(\tau_2+1)B-1]}$, there will be a node $i_2 \in \mathcal{V}$ different with i_0 and i_1 and a time instance $\hat{k}_2 \in [\tau_2 B, (\tau_2+1)B-1]$ such that either $(i_0,i_2) \in \mathcal{E}_{\hat{k}_2}$ or $(i_0,i_2) \in \mathcal{E}_{\hat{k}_2}$. By similar analysis, we obtain that

$$\mathbf{P}\{x_{i_l}(k_s+\varrho) \le \frac{\eta^\varrho}{2}h(k_s) + (1-\frac{\eta^\varrho}{2})H(k_s),\ l = 0,1,2; \\ \varrho = (\tau_2+1)B - k_s,\dots\} \ge p_{\hat{k}_1} p_{\hat{k}_2} q^{(n-1)^2}.$$

Repeating the estimations on time intervals $[\tau_j B, (\tau_j+1)B-1]$ for $j = 3,\dots,n-1$, $\hat{k}_3,\dots,\hat{k}_{n-1}$ can be defined respectively; and bounds for $i_3,\dots,i_{n-1}$ can be similarly given by

$$\mathbf{P}\{x_{i_l}(k_s+\varrho) \le \frac{\eta^\varrho}{2}h(k_s) + (1-\frac{\eta^\varrho}{2})H(k_s),\ l = 0,\dots, \\ n-1; \varrho = (\tau_{n-1}+1)B - k_s,\dots\} \ge \prod_{l=1}^{n-1} p_{\hat{k}_l} q^{(n-1)^2},$$

which implies

$$\mathbf{P}\{S(k_{s+1}) \le (1-\frac{\eta^{(n-1)^2}}{2})S(k_s)\} \ge \bar{p}_s q^{(n-1)^2}. \quad (19)$$

Moreover, similar analysis will show that (19) also holds for the other case with $x_{i_0}(k_s) > \frac{1}{2}h(k_s) + \frac{1}{2}H(k_s)$ by estimating the lower bound of $h(k_{s+1})$.

With (19), we have

$$\mathbf{E}S(k_{s+1}) \le (1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{p}_s)\mathbf{E}S(k_s), \quad (20)$$

which implies

$$\mathbf{E}S(k_{M+1}) \le \prod_{s=0}^{M}(1- \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{p}_s)S(0), \quad M \ge 1 \quad (21)$$

because $\{\mathcal{G}_{[mB,(m+1)B-1]}\}_{m=0}^{\infty}$ is connectivity-independent. Thus, according to Lemma 4.1, if $\sum_{s=0}^{\infty} \bar{p}_s = \infty$, we have $\prod_{s=0}^{\infty}(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{p}_s) = 0$. Consequently, we obtain

$$\lim_{M\to\infty} \mathbf{E}S(k_M) = 0. \quad (22)$$

Because $S(k)$ is non-increasing, (22) immediately yields

$$\lim_{k\to\infty} \mathbf{E}S(k) = 0. \quad (23)$$

Using Fatou's lemma, we further obtain

$$0 \le \mathbf{E} \lim_{k\to\infty} S(k) \le \lim_{k\to\infty} \mathbf{E}S(k) = 0. \quad (24)$$

Therefore, we have $\mathbf{P}\{\lim_{k\to+\infty} S(k) = 0\} = 1$. The desired conclusion follows. □

Suppose $p_{k+1} \le p_k$ for all k. Then it is not hard to see that $\sum_{s=0}^{\infty} \bar{p}_s = \infty$ if and only if $\sum_{k=0}^{\infty} p_k^{n-1} = \infty$. Therefore, the following corollary holds immediately from Proposition 5.1.

Corollary 5.1: Suppose $\{\mathcal{G}_k\}_0^{\infty}$ is stochastically uniformly quasi-strongly connected and $p_{k+1} \le p_k$ for all k. Then algorithm (1) achieves a global a.s. consensus if $\sum_{k=0}^{\infty} p_k^{n-1} = \infty$.

Now we see that Theorem 3.3 holds as a special case of Corollary 5.1 with $B = 1$ in the joint connectivity definition.

B. Bidirectional Connections

Similar to Proposition 5.1, the following conclusion can be obtained for bidirectional graphs.

Proposition 5.2: Suppose $\mathbf{P}\{\mathcal{G}_k$ is bidirectional, $k = 1, 2 \dots\} = 1$. Suppose $\{\mathcal{G}_k\}_0^{\infty}$ is stochastically infinitely connected. Then (1) achieves a global a.s. consensus if $\sum_{s=0}^{\infty} \hat{p}_s = \infty$ with

$$\hat{p}_s = \inf\Big\{ \prod_{l=1}^{n-1} p_{\alpha_l} : \\ c_{s(n-1)} \le \alpha_1 < \dots < \alpha_{n-1} < c_{(s+1)(n-1)} \Big\}.$$

and also

$$T_{com}(\epsilon) \le \inf\Big\{ c_{s(n-1)} : \quad \sum_{i=0}^{s-1} \log\Big(1 - (q\eta)^{(n-1)} \cdot \hat{p}_i\Big)^{-1} \\ \ge \log \epsilon^{-2} \Big\}.$$

C. Acyclic Graphs

Here comes our main result for acyclic graphs.

Proposition 5.3: Assume that $\mathbf{P}\big(\mathcal{G}_{[0,\infty)}$ is acyclic$\big) = 1$ and $\{\mathcal{G}_k\}_0^{\infty}$ is stochastically infinitely quasi-strongly connected. Algorithm (1) achieves a global consensus almost surely if $\sum_{s=0}^{\infty} \tilde{p}_s = \infty$ with $\tilde{p}_s = \inf_{c_s \le \alpha < c_{s+1}} p_{\alpha}$, $s = 0, 1, \dots$.

Proposition 5.3 leads to the following conclusion with non-increasing decision probabilities immediately.

Corollary 5.2: Assume that $\mathbf{P}\big(\mathcal{G}_{[0,\infty)}$ is acyclic$\big) = 1$.

(i) Suppose $\{\mathcal{G}_k\}_0^{\infty}$ is stochastically infinitely quasi-strongly connected and $p_{k+1} \le p_k$ for all k. Then Algorithm (1) achieves a global a.s. consensus if $\sum_{m=0}^{\infty} p_{c_m} = \infty$.

(ii) Suppose either $\{\mathcal{G}_k\}_0^{\infty}$ is stochastically uniformly quasi-strongly connected with $B = 1$ or $p_{k+1} \le p_k$ for all k. Then Algorithm (1) achieves a global a.s. consensus if and only if $\sum_{k=0}^{\infty} p_k = \infty$.

VI. Arc-independent Graphs

In this section, we turn to the convergence analysis for the arc-independent graph processes. Different from previous discussions, we will prove Theorem 3.4 using a stochastic matrix argument.

Let $e_i = (0 \dots 1 \dots \ 0)^T$ be an $n \times 1$ unit vector with the ith component equal to 1. Denote $r_i(k) = (r_{i1} \dots r_{in})^T$ as an $n \times 1$ unit vector with $r_{ij}(k) = a_{ij}(k)$ if $j \in \mathcal{N}_i(k)$, and $r_{ij}(k) = 0$ otherwise for $j = 1, \dots, n$. Let $W(k) = (w_1(k) \dots w_n(k))^T \in \mathbb{R}^{n\times n}$ be a random matrix with

$$w_i(k) = \begin{cases} r_i(k), & \text{with probability } p_k \\ e_i, & \text{with probability } 1 - p_k \end{cases} \quad (25)$$

for $i = 1, \dots, n$. Algorithm (1) is transformed into a compact form:

$$x(k+1) = W(k)x(k). \quad (26)$$

A. Key Lemmas

A finite square matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times n}$ is called *stochastic* if $m_{ij} \ge 0$ for all i, j and $\sum_j m_{ij} = 1$ for all i. For a stochastic matrix M, introduce

$$\delta(M) = \max_j \max_{\alpha,\beta} |m_{\alpha j} - m_{\beta j}| \quad (27)$$

and

$$\lambda(M) = 1 - \min_{\alpha,\beta} \sum_j \min\{m_{\alpha j}, m_{\beta j}\}. \quad (28)$$

If $\lambda(M) < 1$ we call M a *scrambling* matrix. The following lemma can be found in [10].

Lemma 6.1: For any k ($k \ge 1$) stochastic matrices $M_1, \dots, M_k$,

$$\delta(M_1 M_2 \dots M_k) \le \prod_{i=1}^{k} \lambda(M_i). \quad (29)$$

We can associate a unique digraph $\mathcal{G}_M = \{\mathcal{V}, \mathcal{E}_M\}$ with node set $\mathcal{V} = \{1, \dots, n\}$ to a stochastic matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times n}$ in the way that $(j, i) \in \mathcal{E}_M$ if and only if $m_{ij} > 0$, and vice versa.

We first establish several lemmas. The following lemma is given on the induced graphs of products of stochastic matrices.

Lemma 6.2: For any k ($k \geq 1$) stochastic matrices $M_1, \dots, M_k$ with positive diagonal elements, we have $\left(\bigcup_{i=1}^{k} \mathcal{G}_{M_i}\right) \subseteq \mathcal{G}_{M_1 \dots M_k}$.

Proof. We prove the case for $k = 2$, and the conclusion will follow by induction for other cases.

Denote $\bar{a}_{ij}$, $\hat{a}_{ij}$ and a^*_{ij} as the ij-entries of M_1, M_2 and M_1M_2, respectively. Note that, we have

$$a^*_{i_1 i_2} = \sum_{j=1}^{n} \bar{a}_{i_1 j}\hat{a}_{j i_2} \geq \bar{a}_{i_1 i_2}\hat{a}_{i_2 i_2} + \bar{a}_{i_1 i_1}\hat{a}_{i_1 i_2}. \quad (30)$$

Then the conclusion follows immediately since $\bar{a}_{i_1 i_1}, \hat{a}_{i_2 i_2} > 0$. □

Another lemma holds for determining whether a product of several stochastic matrices is a scrambling matrix.

Lemma 6.3: Let $M_1, \dots, M_{n-1}$ be $n-1$ stochastic matrices with positive diagonal elements. Assume that $\mathcal{G}_{M_\tau}, \tau = 1, \dots, n-1$ are all quasi-strongly connected sharing a common center. Then $M_{n-1} \dots M_1$ is a scrambling matrix.

Next, we define a sequence of random variable related to the nodes' decision making. We will call a node i *succeeds* at time k if it chooses to take the averaging part. Denote

$$\Psi_k = \begin{cases} 1, & \text{if at least one node succeeds at time } k; \\ 0, & \text{otherwise.} \end{cases} \quad (31)$$

Then, we have $\Psi_k = 1$ with probability $1 - (1 - p_k)^n$ and $\Psi_k = 0$ with probability $(1 - p_k)^n$. Moreover, $\Psi_0, \Psi_1, \dots$ are independent. We give another lemma on Ψ_k.

Lemma 6.4: $\mathbf{P}\{\Psi_k = 1$ for infinitely many $k\}$=1 if and only if $\sum_{k=0}^{\infty} = \infty$.

Proof. We have

$$\prod_{k=T}^{\infty}(1 - p_k) = 0 \ \Leftrightarrow \ \prod_{k=T}^{\infty}(1 - p_k)^n = 0 \quad (32)$$

for any $T \geq 0$. Then Lemmas 4.1 leads to the conclusion immediately. □

B. Proof of Theorem 3.4

We only need to prove the sufficiency part. Noting the fact that

$$1 - ny \leq (1 - y)^n, \ \ y \in [0, 1], n \geq 1,$$

we obtain

$$1 - (1 - p_k)^n \leq np_k, \ k = 0, \dots.$$

Thus, one has

$$\begin{aligned} &\mathbf{P}\{\text{node } i \text{ succeeds at time } k | \Phi_k = 1\} \\ &= \frac{p_k}{1 - (1 - p_k)^n} \geq \frac{p_k}{np_k} = \frac{1}{n} \end{aligned} \quad (33)$$

for all $i = 1, \dots, n$ and $k = 0, \dots$.

According to Lemma 6.4, we can define the (Bernoulli) sequence of Φ_k,

$$\zeta_1 < \dots < \zeta_m < \zeta_{m+1} < \dots,$$

with probability one such that ζ_m is the mth time which $\Phi_k = 1$ for $m = 1, 2, \dots$.

Denote $\theta_0 = \min_{(i,j) \in \mathcal{E}^*} \theta_{ij}$. With (33), for any $(i, j \in \mathcal{E}^*)$, we have

$$\mathbf{P}\{(i, j) \in \mathcal{G}_{W(\zeta_m)}\} \geq \frac{\theta_0}{n}. \quad (34)$$

Therefore, denoting $H_1 = W(\zeta_{|\mathcal{E}^*|}) \dots W(\zeta_2)W(\zeta_1)$, where $|\mathcal{E}^*|$ represents the number of elements in $\mathcal{E}^*$, (34) leads to

$$\mathbf{P}\{(i_\tau, j_\tau) \in \mathcal{G}_{W(\zeta_\tau)}, \ \tau = 1, \dots, |\mathcal{E}^*|\} \geq (\frac{\theta_0}{n})^{|\mathcal{E}^*|}, \quad (35)$$

where (i_τ, j_τ) denotes an elements in $\mathcal{E}^*$. As a result, we see from Lemma 6.2 that

$$\mathbf{P}\{\mathcal{G}^* \subseteq \mathcal{G}_{H_1}\} \geq \mathbf{P}\{\mathcal{G}^* \subseteq \bigcup_{\tau=1}^{|\mathcal{E}^*|} \mathcal{G}_{W(\zeta_\tau)}\} \geq (\frac{\theta_0}{n})^{|\mathcal{E}^*|}. \quad (36)$$

Similarly, we define $H_s = W(\zeta_{s|\mathcal{E}^*|}) \dots W(\zeta_{(s-1)|\mathcal{E}^*|+1})$ for $s = 2, 3, \dots$, and

$$\mathbf{P}\{\mathcal{G}^* \subseteq \mathcal{G}_{H_s}\} \geq (\frac{\theta_0}{n})^{|\mathcal{E}^*|}. \quad (37)$$

can also be obtained for all s.

Next, because $\mathcal{G}^*$ is QSC, applying Lemma 6.3 on $H_1, \dots, H_{n-1}$ yields

$$\mathbf{P}\{\lambda(H_{n-1} \dots H_1) < 1\} \geq (\frac{\theta_0}{n})^{(n-1)|\mathcal{E}^*|}. \quad (38)$$

Moreover, $H_{n-1} \dots H_1$ represents a product of $(n-1)|\mathcal{E}^*|$ stochastic matrices, each of which satisfies the weights rule A0. Therefore, it is not hard to see that for each nonzero entry, h_{ij} of $H_{n-1} \dots H_1$, we have

$$h_{ij} \geq \eta^{(n-1)|\mathcal{E}^*|}, \quad (39)$$

which implies

$$\mathbf{P}\{\lambda(H_{n-1} \dots H_1) < 1 - \eta^{(n-1)|\mathcal{E}^*|}\} \geq (\frac{\theta_0}{n})^{(n-1)|\mathcal{E}^*|}. \quad (40)$$

Denoting $G_\tau = H_{\tau(n-1)} \dots H_{(\tau-1)(n-1)+1}, \tau = 1, 2, \dots,$ we have

$$\mathbf{P}\{\lambda(G_\tau) < 1 - \eta^{(n-1)|\mathcal{E}^*|}\} \geq (\frac{\theta_0}{n})^{(n-1)|\mathcal{E}^*|} \quad (41)$$

for all $\tau = 1, 2, \dots$. Thus,

$$\mathbf{P}\{\lambda(G_\tau) < 1 - \eta^{(n-1)|\mathcal{E}^*|} \text{ for infinitely many } \tau\} = 1,$$

which yields

$$\mathbf{P}\{\lim_{m \to \infty} \delta(\prod_{\tau=1}^{m} G_\tau) \leq \lim_{m \to \infty} \prod_{\tau=1}^{m} \lambda(G_\tau) = 0\} = 1 \quad (42)$$

from Lemma 6.1. Thus, we finally obtain

$$\mathbf{P}\{\lim_{k \to \infty} \delta(W(k) \dots W(0)) = 0\} = 1$$

because $W(k)$ is the identical matrix for any $k \notin \{\zeta_1, \zeta_2, \dots\}$. This completes the proof. □

VII. Conclusions

This paper investigated standard consensus algorithms coupled with randomized individual node decision-making over stochastically time-varying graphs. Each node determined its dynamics by a sequence of Bernoulli trials with time-varying probabilities. We introduced connectivity-independence and arc-independence for random graph processes. An impossibility theorem showed that an a.s. consensus could not be achieved unless the sum of the success probability sequence diverges. Then a serial of sufficiency conditions were given for the network to reach a global a.s. consensus under different connectivity assumptions. Particularly, when either the graph was arc-independent or overall acyclic, the sum of the success probability sequence diverging was a sharp threshold condition for consensus under a simple self-confidence assumption. In other words, consensus appeared from probability zero to one as the sum of the probability sequence goes to infinity. Consistent with classical random graph theory, this so-called $0-1$ law was first established in the literature for dynamics on random graphs.

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